**640:251:19-20** **9/5/96**

# Problems, Week 1

*I'll tell you one problem whose solution should be written up for next week's meeting. Your written solution should be of high-quality, with complete explanations given including any useful diagrams or equations. While I encourage you to discuss the problems with other students and with me, your written work must be your own explanation of the problem.*

1. Suppose that $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ are given vectors in the plane. In this problem we ask: is it true that *every* vector $\mathbf{c} = \langle c_1, c_2 \rangle$ in the plane can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for some scalars $s$ and $t$?

(a) Suppose that $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$. Show that the answer is "yes" and find $s$ and $t$. Do the same if $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$.

(b) Suppose that $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $b = 2\mathbf{i} + 2\mathbf{j}$. Show that the answer is "no" by finding a vector $\mathbf{c}$ which cannot be written in the desired form.

(c) Parts (a) and (b) should suggest a geometric condition on $\mathbf{a}$ and $\mathbf{b}$ determining whether the answer is "yes" or "no." Formulate such a condition and show geometrically why it is correct.

(d) Try to solve the equation $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for $s$ and $t$. You should find an algebraic condition on $a_1$, $a_2$, $b_1$, and $b_2$ determining whether this is possible for all choices of $c_1$ and $c_2$, i.e., whether the answer to the original question is "yes" of "no." Relate your answer to part (c).

2. Let $A$, $B$, $C$, and $D$ be four points in space, considered as the vertices of a quadrilateral, and let $W$, $X$, $Y$, and $Z$ be the midpoints of the lines $AB$, $BC$, $CD$, and $DA$, respectively. We want to show that the lines $WY$ and $XZ$ bisect each other.

(a) Let $U$ be the midpoint of $WY$. Express $\overrightarrow{AW}$, $\overrightarrow{AY}$, $\overrightarrow{WY}$, and finally $\overrightarrow{AU}$ in terms of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$.

(b) Let $V$ be the midpoint of $XZ$. Express $\overrightarrow{AV}$ in terms of $\overrightarrow{AD}$, $\overrightarrow{DC}$, $\overrightarrow{CB}$.

(c) Show from (a) and (b) that $\overrightarrow{AU} = \overrightarrow{AV}$. Why does this complete the problem?

(Hint: What is the relation between $\overrightarrow{CD}$ and $\overrightarrow{DC}$? Between $\overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{AD} + \overrightarrow{DC}$?)

3. (a) Suppose that $\mathbf{a}$ and $\mathbf{b}$ are three-dimensional vectors and that $|\mathbf{a}| = 10$ and $|\mathbf{b}| = 8$. How big could $|\mathbf{a} + \mathbf{b}|$ be? How small? Give a geometric argument as to why your answer is correct.

(b) Suppose that in (a) we know only that $8 \le |\mathbf{a}| \le 12$ and $7 \le |\mathbf{b}| \le 9$. Answer the same questions.