311:09 **Final Exam** **12/15/94**

There are 10 problems. Do all of the problems. You may use notes on one sheet of $8\frac{1}{2}''$ by $11''$ paper. Good luck!

(2) 0. Write your name on your blue book.

(15) 1. Suppose that (a_n) is a convergent sequence with $\lim a_n = A$, and that (b_n) is a convergent sequence with $\lim b_n = B$. Define $c_n := a_n + b_n$ for all $n \in \mathbb{N}$. Prove that the sequence (c_n) converges, and that $\lim c_n$ is $A + B$.

(15) 2. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a bounded function: that is, there is $M > 0$ so that $|g(x)| \leq M$ for all $x \in \mathbb{R}$. If $f(x) := x^2 g(x)$ prove that f is differentiable at 0. What is $f'(0)$?

(15) 3. Prove that if a sequence converges, then it is bounded.

(48) 4. For each of the following statements, prove (if true) or give a counterexample (if false).

a) Suppose (x_n) is a convergent sequence, and the limit of the sequence is x. If each x_n is irrational, then x is irrational.

b) Suppose (x_n) is a convergent sequence, and the limit of the sequence is x. If each x_n is non-negative (that is, greater than or equal to 0), then x is non-negative.

c) Suppose (x_n) is a convergent sequence, and the limit of the sequence is x. If x is non-negative, then eventually each x_n is non-negative (that is, there is $N \in \mathbb{N}$ so that x_n is non-negative for $n \geq N$).

d) Suppose that f is a non-negative integrable function on $[a, b]$. If $\displaystyle\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

e) A differentiable function is continuous.

f) A one-one increasing function defined on an interval is continuous.

(15) 5. If f is integrable on $[0, 1]$, define $g : [-1, 1] \to \mathbb{R}$ by $g(x) := \begin{cases} 0, & \text{if } x < 0 \\ f(x), & \text{if } x \geq 0 \end{cases}$.

Prove that g is integrable on $[-1, 1]$ and that $\displaystyle\int_{-1}^1 g = \int_0^1 f$.

(15) 6. Suppose $a > -1$. Prove that if $n \in \mathbb{N}$ then $(1 + a)^n \geq 1 + na$.

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(15) 7. If $T := \left\{ \frac{1}{n} + \frac{1}{m^2} : n, m \in \mathbb{N} \right\}$, find $\inf T$ and $\sup T$. Prove that your answers are correct.

(15) 8. Prove that $\left((-1)^n\right)$ does not converge. (Hint: use of the Cauchy criterion in your answer will probably help.)

(15) 9. Suppose a is a fixed number in $(0,1)$. Define the sequence (x_n) by

$$x_1 = a, \text{ and } x_{n+1} = {x_n}^2 \text{ for } n > 1$$

Prove that (x_n) converges and find $\lim x_n$, explaining your result.

(30) 10. In this problem, suppose that $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) := \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases} .$$

a) Is F continuous at 0? Prove your assertion.

b) Is F differentiable at 0? Prove your assertion.

c) Is F integrable on $[0,1]$? Prove your assertion.

Thank you. Have a good holiday!