311:09 **Exam #2: Questions <u>and</u> Answers** **11/27/94**

(36) 1. For each of the following statements, prove (if true) or give a counterexample (if false). This is probably the *most* difficult question on the exam. Deciding **True/False** is uncomfortably like reality: complicated statements are presented to you, you must try to make sense of them, and then justify your conclusions. In each part, I gave 1 point for the correct decision, and allocated 5 points for the justification.

a) Suppose (x_n) is a convergent sequence, and the limit of the sequence is x. If each x_n is rational, then x is rational.

False There are many ways to create an appropriate example.

i) The Density Theorem asserts that every interval of positive length contains a rational number. Thus the interval $\left(\sqrt{2}-\frac{1}{n}, \sqrt{2}+\frac{1}{n}\right)$ contains a rational number x_n for every positive number, n. And we know that $\left|\sqrt{2}-x_n\right| < \frac{1}{n}$ for all $n \in \mathbb{N}$. So $\lim x_n = \sqrt{2}$ and we have created a sequence of rational numbers which converges to an irrational number.

ii) In class and in the textbook a sequence (x_n) was discussed early in the course. The sequence was used to prove that the rationals were not complete. It was a strictly increasing sequence of positive rationals whose squares approached 2. Thus this sequence is a nice example showing that statement a) is false.

iii) (Suggested by a student) The well-known number e (you know, the one that "begins" $2.71828\ldots$) is irrational. But e is the limit of the sequence of partial sums of the infinite series $\sum_{j=0}^{\infty} \frac{1}{j!}$. Each partial sum is $\sum_{j=0}^{n} \frac{1}{j!}$, a finite sum of rational numbers, and thus clearly rational. So again we've exhibited a sequence of rationals converging to an irrational number.

b) Suppose (x_n) is a convergent sequence, and the limit of the sequence is x. If each x_n is positive, then x is positive.

False If $x_n = \frac{1}{n}$ then each x_n is positive, $\lim x_n$ exists and is 0, and 0 is not positive.

c) Suppose (x_n) is a convergent sequence, and the limit of the sequence is x. If each x_n is an integer, then x is an integer.

True It is hard to think of examples of such sequences, because in some sense there aren't very many. Each such sequence is eventually constant. But this must be proved. Here are two ways.

i) Since the sequence (x_n) converges, it must be Cauchy. Take $\varepsilon = \frac{1}{2}$. Then there is $K \in \mathbb{N}$ so that for n and m greater than K, $|x_n - x_m| < \frac{1}{2}$. But distinct integers are a distance at least 1 apart, so all the terms of the sequence after x_K must coincide. Thus the sequence (x_n) is ultimately constant, and the limit, x, coincides with one of those terms, so it also must be an integer.

ii) This approach (by contradiction) was suggested by a student: what if x were* *not* an integer? Then the distance from x to the integers would be positive. That is, $\inf\{|n-x| : n \in \mathbb{N}\}$ would be > 0 (this would, of course, contradict the convergence hypothesis). If the distance were $= 0$, there would be two integers within, say, $\frac{1}{2}$ of x (remember, x is not an integer here, so there must be lots of integers close to x or else the inf could not be 0). The distance between those two integers (by using the $\triangle \leq$) would have to be < 1 which is impossible. This contradiction establishes that x must be an integer itself.
In this case proofs i) and ii) are mirrors of each other: different ways of phrasing the same argument.

d) Suppose (x_n) is a convergent sequence, and the limit of the sequence is x. If $f : \mathbb{R} \to \mathbb{R}$ is any function, then the sequence $(f(x_n))$ converges, and its limit is $f(x)$.

False Perhaps this question and e) below are close to entrapment *but* I hoped that people would be rescued by the prominent appearance of the word "continuous" in f). That is, they'd say to themselves, "Oh, darn it! I implicitly *assumed* continuity in my answers to d) and e) before. I'd better go back and revise them." In fact, I *really* hoped that people would have learned to read questions in this course carefully by now, and would have answered them correctly and precisely during their first attempt**. In any case, examples are easy to get. Choose (x_n) to be $\left(\frac{1}{n}\right)$ and let f be defined by $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$.
Then $f(x_n) = 1$ for all n and so $\lim f(x_n) = 1$ but $f(x) = f(\lim x_n) = f(0) = 0$.
There are clearly many examples possible. Please note that one answer given by some students used the same sequence given above and used the function $f(x) = \frac{1}{x}$ (written just like that!). The correct assertion was made that the sequence $(f(x_n))$ did not converge (supported by the unbounded nature of that sequence). I gave 4 points out of 5 for this answer (remember, 1 point was given for the answer **False** already). Note that the function f is not specified at 0, and the domain in the statement is all of $\mathbb{R}$.

e) Suppose I is a closed and bounded interval, and $f : I \to \mathbb{R}$ is a function with $f(x) > 0$ for all $x \in I$. Then $\inf\{f(x) : x \in I\}$ is positive.

False This recalls, of course, one of the five million or so examples of workshop #9. The glaring omission of the word "continuous" here is most important. There are many examples. If $f(x) = \begin{cases} x, & \text{if } x > 0 \\ 1, & \text{if } x = 0 \end{cases}$ and if the domain is $I = [0, 1]$ (a closed, bounded interval), then the inf of f on I is 0 and all of f's values are positive.

f) Suppose I is a closed and bounded interval, and $f : I \to \mathbb{R}$ is a continuous function with $f(x) > 0$ for all $x \in I$. Then $\inf\{f(x) : x \in I\}$ is positive.

True Finally a true and non-pathological statement! Such a function on such a domain *attains* its minimum. Thus there is $x_* \in I$ with $f(x_x) \leq f(x)$ for all $x \in I$. So $f(x_*) = \inf\{f(x) : x \in I\}$ but all values of f are positive, and therefore the inf is positive.

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* The subjunctive tense, I guess.

** Not only didn't the word "continuous" appear here, but also the word "giraffe" wasn't here. Was it expected?

(16) 2. a) Prove that $F(x)=\frac{1}{x}$ is uniformly continuous on the domain $[1,\infty)$.

A function f is *uniformly continuous* on a domain A if, given any $\varepsilon>0$, there is a $\delta>0$ so that

If x and y are in A and $|x-y|<\delta$, then $|f(x)-f(y)|<\varepsilon$.

The quantifiers (*any*, there is *a*) are especially important in that statement. Usually the first approach would be to find some connection between $|f(x)-f(y)|$ and $|x-y|$. This is an exceptionally easy example because that connection is easy to find. By the way, this example is discussed in the text (see several paragraphs in section 5.4, as well as problem 1 in that section). Thus:

$$|f(x)-f(y)|=\left|\frac{1}{x}-\frac{1}{y}\right|=\left|\frac{y-x}{xy}\right|=\frac{|y-x|}{xy}$$

since x and y are positive. We also know that $x\geq 1$ and $y\geq 1$, and therefore $\frac{1}{xy}\leq 1$. Thus if x and y are in $[1,\infty)$ we see that

$$|f(x)-f(y)|\leq|x-y|.$$

Given $\varepsilon>0$, if we take $\delta=\varepsilon$ then the implication "If $|x-y|<\delta$ then $|f(x)-f(y)|<\varepsilon$" surely follows for all $x,y\in[1,\infty)$.

b) Prove that $F(x)=\frac{1}{x}$ is not uniformly continuous on the domain $(0,1]$.

For the most direct answer to this question, you must cope with the *negation* of the definition of uniform continuity. What is it? You must prove that there is at least one $\varepsilon_0>0$ so that <u>no</u> δ "works". By this I mean that for any $\delta>0$ you must show that there are elements x and y in $(0,1]$ with $|x-y|<\delta$ and $|f(x)-f(y)|\geq\varepsilon_0$. We discussed negating this statement in class extensively. By using the Archimedean property we in fact need only provide two sequences (x_n) and (y_n) satisfying $|x_n-y_n|<\frac{1}{n}$ with $|f(x_n)-f(y_n)|\geq\varepsilon_0$. There are natural candidates for sequences to try: $x_n:=\frac{1}{n}$ and $y_n:=\frac{1}{n+1}$. Then direct computation reveals:

$$|x_n-y_n|=\left|\frac{1}{n}-\frac{1}{n+1}\right|=\frac{1}{n(n+1)}<\frac{1}{n}$$

and

$$|f(x_n)-f(y_n)|=\left|\frac{1}{\frac{1}{n}}-\frac{1}{\frac{1}{n+1}}\right|=1$$

So we are done if we realize that one valid choice of ε_0 is 1.

Several students tried to use a theorem which is stated in the book but which we did not have time to discuss in class, the so-called Continuous Extension Theorem. A statement of the theorem adapted to this situation is:

$f(x) = \dfrac{1}{x}$ is uniformly continuous on $(0,1]$ if and only if f has *some* continuous extension to $[0,1]$.

The catch (if there is one) to using this result is the need to demonstrate that f has *no* continuous extension (and not just remarking that the formula $\dfrac{1}{x}$ has no meaning when $x = 0$). So: if f had a continuous extension to $[0,1]$ then $\lim_{x\to 0} f(x)$ would exist (it would equal $f(0)$, of course). But that limit does not exist, since $\dfrac{1}{x}$ is unbounded in any neighborhood of 0.

A number of students tried to prove the result asked for in b) by asserting that the algebra in a) still applies (which it certainly does) and then asserting (essentially) that there is *no* constant $M > 0$ for which the inequality $|f(x) - f(y)| \le M|x - y|$ holds for all $x, y \in (0,1]$ (this uses the fact that for such x's and y's, the factor $\dfrac{1}{xy}$ is unbounded). This is also certainly true. Notice, please, that such an estimate (called a Lipschitz estimate) is sufficient for uniform continuity: that is, if the estimate is true on a domain, then the function is uniformly continuous on that domain. But the estimate is not necessary because a function need not satisfy a Lipschitz estimate to be uniformly continuous. For example, consider $\sqrt{x}$ on the domain $[0,1]$. Since this is a continuous function on a closed and bounded interval, it is uniformly continuous (this result will be used most importantly in about 10 days to deduce that such functions are integrable). But $|\sqrt{x} - \sqrt{y}| \le M|x - y|$ can't be true for all $x, y \in [0,1]$. To see this, take $y = 0$ and divide by a non-zero x and watch the inequality become $\dfrac{1}{\sqrt{x}} \le M$ for all x with $0 < x < 1$, a statement which is certainly not true for any fixed M.

(12) 3. Suppose that f and g are functions whose domain is all of $\mathbb{R}$. Prove directly from the definition of continuity that if f and g are both continuous at x_0 then $f + g$ is continuous at x_0.

This is a very straightforward problem. Since f is continuous at x_0 there is $\delta_1 > 0$ so that if $|x - x_0| < \delta_1$ then $|f(x) - f(x_0)| < \dfrac{\varepsilon}{2}$. Similarly, since g is continuous at x_0 there is $\delta_2 > 0$ so that if $|x - x_0| < \delta_2$ then $|g(x) - g(x_0)| < \dfrac{\varepsilon}{2}$. Then take $\delta = \min(\delta_1, \delta_2)$. If $|x - x_0| < \delta$, both of the above estimates are valid. And we know

$$|(f(x) + g(x)) - (f(x_0) + g(x_0))| = |(f(x) - f(x_0)) + (g(x) - g(x_0))| \overset{\triangle}{\le}$$

$$|(f(x) - f(x_0))| + |(g(x) - g(x_0))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so we are done. Note that the use of the triangle inequality is necessary, as is the specification of δ which makes x's satisfying *both* of the continuity hypotheses.

(12) 4. a) Find a positive number b so that if $|x-1|<b$ then $|x^2-1|<\frac{1}{100}$. Prove your assertion.

Let's assume $b<1$. Then $|x-1|<1$ so that $-1<x-1<1$ and so that $0<x<2$ and so that $1<x+1<3$ and so that $|x+1|<3$. That means we can conclude that $|x^2-1|=|(x-1)(x+1)|=|x-1||x+1|<3|x-1|$. If we *now* assume $|x-1|<\frac{1}{1\,000}$ we see (since $\frac{1}{1\,000}<1$) that $|x^2-1|<\frac{3}{1\,000}$ and we're done since $\frac{3}{1\,000}$ is surely less than $\frac{1}{100}$.

There are many possible correct b's answering this question. I did **not** ask for a "best possible" b and found attempts to produce such somewhat incomprehensible. I did **not** ask for a list of all possible b's and found attempts to produce such a list or to exhibit characteristics of such b's generally incomprehensible. I also would not give full credit to just an endpoint estimate: that is, someone who asserted something like: "Hey, $|x-1|<\frac{1}{100\,000}$ *surely* works because $|x^2-1|\leq\left|\left(1+\frac{1}{100\,000}\right)^2-1\right|<\frac{1}{100}$ clearly." I wanted the logic to convince me that the stated claim (for *all* eligible x's!) was correct.

b) Give an example of an unbounded sequence that has a convergent subsequence.

There are many possible correct answers to this question. For example, the sequence (a_n) whose even terms are all 0's and whose odd terms obey $a_n=n$ is such a sequence. Then the even subsequence is the constant (hence convergent) sequence of 0's and the odd subsequence is emphatically unbounded.

(12) 5. In this problem suppose that (s_n) and (t_n) are <u>both</u> increasing sequences and that for all $n\in\mathbb{N}$, $s_n\leq t_n$.

a) Prove that if (t_n) converges, then (s_n) converges.

The Monotone Convergence Theorem can be used to prove this result easily. Since (t_n) converges and (t_n) is increasing, (t_n) is bounded above: that is, there is an $M\in\mathbb{R}$ so that $t_n\leq M$ for all n. But $s_n\leq t_n\leq M$ so M is an upper bound for the increasing sequence (s_n) and therefore (s_n) converges.

The Monotone Convergence Theorem is thus used twice. I wanted and needed some explicit citation of such a result. Without the structure of an increasing sequence, very few conclusions can be drawn. It is also quite necessary to declare that the whole sequence (s_n) has an upper bound, not just a few elements.

b) Prove that there are examples of sequences satisfying the hypotheses of this problem so that (s_n) converges and (t_n) does not converge.

There are many examples. (s_n) could be the sequence which is all 0's. It surely converges to 0. Such a sequence is increasing since our definition allows an *increasing* sequence not to be strictly increasing. The sequence defined by $t_n=n$ is also increasing but is unbounded (Archimedean property!), and therefore does not converge. And $s_n\leq t_n$ for all n. So we have produced an example.

(12) 6. Suppose $f : \mathbb{R} \to \mathbb{R}$, and $\lim_{x\to 0} f(x) = L$. Let $g(x) = f(2x)$. Prove that $\lim_{x\to 0} g(x) = L$.

There seem to be two straightforward ways to approach this problem: either using an $\varepsilon - \delta$ argument or with sequences. Let me show you both (and both were actually used by students).

i) Since $\lim_{x\to 0} f(x) = L$ we know: given $\varepsilon > 0$, there is $\nu > 0$ so that if $0 < |w| < \nu$ then $|f(w) - L| < \varepsilon$. (I believe that using non-routine letters probably makes this easier to write and to understand.) Now we need to prove: given $\varepsilon > 0$, there is $\rho > 0$ so that if $0 < |u| < \rho$ then $|g(u) - L| < \varepsilon$. But the last inequality is actually the same as $|f(2u) - L| < \varepsilon$. And this will certainly be true if $0 < |2u| < \nu$ which is the same as $0 < |u| < \frac{\nu}{2}$. Thus if we take ρ to be $\frac{\nu}{2}$ we'll be done.

ii) Since $\lim_{x\to 0} f(x) = L$ we know: given any non-zero sequence, (x_n), if $\lim x_n = 0$ then $\lim f(x_n) = L$. But if $\lim x_n = 0$ we know that $\lim 2x_n$ exists and is 0. Thus we know that $\lim f(2x_n)$ exists and is L since we knew the preceding f-limit for *any* sequence with non-zero elements whose limit is 0. But now we know: if (x_n) is *any* sequence with non-zero elements whose limit is 0 then $\lim g(x_n)$ exists and is L. But we proved that complicated statement was equivalent to the assertion $\lim_{x\to 0} g(x)$ exists and is L. (Whew! I wrote this out at length in an effort to make the logic of the problem clear. I hope I did.)

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Many of these problems were homework problems, past exam problems, workshop problems, and/or discussed in class. I had hoped that this would not be a exam with low grades. The statistics of the grades are as follows:

Mean: 57.3529
Standard Deviation: 19.0477
Median: 57
Range of grades: $16 \le \mathit{Numerical\ grade} \le 88$.

Letter grades correspond to numerical grades using this function:

$$\text{Letter Grade}\,(\mathit{Numerical\ grade}) = \begin{cases} \textbf{F}, & \text{if } \ 0 \le \mathit{Numerical\ grade} < 50 \\ \textbf{D}, & \text{if } 50 \le \mathit{Numerical\ grade} < 55 \\ \textbf{C}, & \text{if } 55 \le \mathit{Numerical\ grade} < 60 \\ \textbf{C+}, & \text{if } 60 \le \mathit{Numerical\ grade} < 65 \\ \textbf{B}, & \text{if } 65 \le \mathit{Numerical\ grade} < 75 \\ \textbf{B+}, & \text{if } 75 \le \mathit{Numerical\ grade} < 80 \\ \textbf{A}, & \text{if } 80 \le \mathit{Numerical\ grade} \le 100. \end{cases}$$