1. The text asserts and proves various formulas by induction. For example, on p. 19 the following assertion is proved:
if $n \in \mathbb{N}$ then

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Try to discover and then *prove* a formula for the sum of the first n odd squares. That is, find and prove a formula for

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2.$$

Hint: if $\mathcal{F}(n)$ denotes the sum of the first n odd squares, then $\mathcal{F}(4) - \mathcal{F}(3) = 7^2$. What is $\mathcal{F}(n+1) - \mathcal{F}(n)$ algebraically? Is it a polynomial? Of what degree? What happens to the degrees of polynomials when you take differences at successive integers? That is, what is $(n+1)^2 - n^2$? What is $(n+1)^3 - n^3$? Can you link all this together and make a guess at what $\mathcal{F}$ might be?

2. $n!$ grows very fast.

a) Prove that $n^3 \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large. Write a careful proof of this fact; be sure to be precise about what "all ... sufficiently large" means.*

b) Prove that $n^7 \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large. Write a careful proof of this fact; be sure to be precise about what "all ... sufficiently large" means.*

c) Suppose $k \in \mathbb{N}$ is fixed. Prove that $n^k \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large. Write a careful proof of this fact; be sure to be precise about what "all ... sufficiently large" means.*

3. $n!$ grows very fast.

a) Prove that $3^n \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large. Write a careful proof of this fact; be sure to be precise about what "all ... sufficiently large" means.*

b) Prove that $7^n \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large. Write a careful proof of this fact; be sure to be precise about what "all ... sufficiently large" means.*

c) Suppose $k \in \mathbb{N}$ is fixed. Prove that $k^n \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large. Write a careful proof of this fact; be sure to be precise about what "all ... sufficiently large" means.*

4. Compare the rates of growth of n^k and k^n for $k \in \mathbb{N}$ fixed and $n \in \mathbb{N}$. Which grows faster? Make your assertion precise and prove it. Suggestion: try some special cases first, like $k = 3$ and $k = 7$.

---

* Be sure to include a specific n_0 so that the assertion is true for all $n \geq n_0$. You are <u>not</u> asked to make a "best possible" assertion and find the smallest valid n_0 — just to find some n_0 that works!

Comments on #1

Problem 1 is too hard and somewhat pointless — students ended up solving a system of 4 linear equations in 4 unknowns and never got to the point of the induction proof. The inequalities (problem 2) turn out to be difficult. Parts a and b of problem 2 were done. The transition to an "abstract" k in part c was too hard for the students. It would have been good to mediate a transition with a section proposing the problem with $k = 127$ (for example). This would be a number too large to find n_0 by calculator. Then a good hint would be useful, and maybe general k could have been dealt with. But students found the process of doing these problems difficult and long. *No* group did part c in problem 2 — it was just too difficult.

Students do not write things down. They refuse to get their hands "dirty" with numbers or with algebra. They would talk without writing, without really explaining, and would not manipulate formulas. They aren't trying to communicate effectively with other people in their group, to convince and educate them. Having a student give a solution to 2a on the board in the last 5 minutes of the period was a pleasant respite from the concentration and frustration of group work. It was amusing to almost everyone, and flattering to the student, and virtually everyone in the class was able to comment and to help the presentation move along.

I have asked students to hand in as much as they can of either problem 2 or problem 3. It seems almost a shame though, to let the work done in regard to #1 go to waste. (Sigh.) I think problem 3 would have been easier to work on, too, but groups (sized either 3 or 4) worked on either problem 1 or problem 2.

Things to try next time: get them to change groups. Get them to talk. Get them to try experiments with numbers more. Ask them for suggestions. (!)

Later comments on #1

After reading the homework: they can't write. They can't write technically and they plain just can't write English. I gave them a small paragraph describing the sort of write-up I want, and have made very extensive comments on their work. I hope this will help. Here is what I handed out:

311:09 Writing workshop problems 9/14/94

I want you to write your workshop assignments so that they can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use complete English sentences.
- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Provide proper connectives between equations and between ideas.

Do not assume that the person reading the paper can read your mind. Do assume that the person reading the paper is intelligent, but also assume that the person reading the paper is busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art. Effective written exposition will be worth about 50% of the workshop grade, and, conversely, lousy exposition may be penalized as much as 50% of the grade. Please realize this!

Please proofread what you hand in. Ideally, you should read and reread and revise almost any formal communication. Neatness and clarity count, as you darn well know if you've tried to read any complicated document.

The best reference known to me discussing effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

311:09 **Workshop #2** **9/13/94**

1. (This should be a straightforward problem for those who have patience.) Suppose we have a collection of numbers which satisfy the algebraic rules. These are the rules (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4), and (D). Suppose we also have the following "two-by-two" system of linear equations:

$$\begin{cases} ax + by = c \\ dx + ey = d \end{cases}$$

Here x and y are "unknown" and a, b, c, d, e, and f are "known". Suppose in addition that

$$ae - bd \neq 0.$$

Then prove that there are unique values for x and y satisfying the system of linear equations, and find a formula for these values.

Comments: the difficulty is that I'd like you to justify *every* step by a citation of one of the nine rules above. Note, please, that there are two parts to this problem: asserting that there are solutions (and what they are!) and then asserting that these are the *only* solutions. This problem is some justification for my comment that the algebraic rules are the only ones needed to "do" linear algebra (as the subject is realized in, say, Math 250).

2. Consider the set consisting of pairs of real numbers. That is, we will look at the set of all (α, β) with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. On this set (which we will call $\mathbb{R}^2$) we will work with the following two binary operations:

$\bigoplus$ defined by $(\alpha_1, \beta_1) \bigoplus (\alpha_2, \beta_2) := (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$

$\bigotimes$ defined by $(\alpha_1, \beta_1) \bigotimes (\alpha_2, \beta_2) := (\alpha_1 \cdot \alpha_2 - \beta_1 \cdot \beta_2, \alpha_1 \cdot \beta_2 + \alpha_2 \cdot \beta_1)$

a) Prove that $\mathbb{R}^2$ together with $\bigoplus$ and $\bigotimes$ satisfies all of the algebraic rules: (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4), and (D).

Comments: this problem could actually be done in parallel (!?). That is, one person could verify (A1) while another checks (A2), etc. Some of the verifications (such as those for (A1), (A2), (M1), (M2), and (D)) should be easy. As for the others:

i) you must discover the additive and multiplicative identities for $\mathbb{R}^2$ with these operations (necessary for verification of (A3) and (M3)).

ii) you must discover the additive and multiplicative inverses for *any* element (α, β) of $\mathbb{R}^2$ with these operations (necessary for verification of (A4) and (M4)).

This shouldn't be difficult if you think back to the complex numbers,

b) Suppose you try to do the same thing with the set consisting of ordered pairs of *complex* numbers. That is, consider $\mathbb{C}^2$ and define addition and multiplication by the same formulas ($\bigoplus$ and $\bigotimes$). Then it turns out that not all of the algebraic rules (that is, (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4), and (D)) are satisfied. What goes wrong?

OVER

3. (Each part of this shouldn't be hard to do.)

a) Suppose a and b are positive numbers. "Expand" $(a-b)^2$ (what do you know about squares and order?) and prove that $a^2+b^2 \geq 2ab$.

b) Suppose a and b are positive numbers. Prove that $\frac{a}{b}+\frac{b}{a} \geq 2$.

c) Suppose a and b are positive numbers. Prove that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

d) Suppose a and b and c are positive numbers. Prove that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$.

e) Suppose that n is a positive integer, and that $a_1, a_2, \ldots, a_n$ are all positive numbers. Prove that $\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq n^2$.

Comments: I don't think that the last inequality is easy to understand or verify without some work. If you've done the earlier sections of the problem, the proof should be accessible. The inequality is almost cute.

4. We'll use the neighborhood notation that's standard in our text: if $a \in \mathbb{R}$ and $\varepsilon > 0$,

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x-a| < \varepsilon\}$$

If $V_\delta(b) \subset V_\varepsilon(a)$, prove that $a-\varepsilon < b < a+\varepsilon$ and $\delta \leq \min\{b-a+\varepsilon, a-b+\varepsilon\} \leq \varepsilon$.

Comments on #2

I am trying too hard, and forcing a bit. Students did not seem to need very much help last night, and yet I tried a bit too much to intervene. I've got to learn to lay back. I told people to do one of problems 1 or 2 (algebraic rules questions) for 40 minutes, and then one of problems 3 or 4 (order rules questions) for 40 minutes. Students found problems 1 and 2 difficult — maybe there is just too much to write.

A student gave a fairly nice presentation of problem #3 (the *whole* problem!) in the last 10 minutes of the period. The presentation wasn't sequential or well-formed enough to be ideal, but I've heard worse while auditing TA's in our department (!). I asked them to hand in two write-ups, one of problems 1 or 2 and one of problems 3 or 4.

I requested that they work with at least two new people in their groups this time. I noticed that there was a nearly complete gender separation this time. I believe I'll try cracking that next time.

Now I'll begin grading their write-ups of workshop #1. I already have prepared a paragraph on what I expect in writing style. I am fearful.

Later comments on #2

A few people are still handing in either sketchy notes of what they actually did in the workshop, or line after line of computations with very little explanation. But there is a marked improvement in style in many people. Now I can begin to go after the content more. In particular, I notice that $P \Rightarrow Q$ (that is, the implication: "if proposition P is true, then propostion Q is true") is frequently "proved" by assuming Q and deducing P from it. The grader, Mr. Dare (who is grading some of the textbook problems), suggests that the students are trying to imitate proof by contradiction. Indeed. And surely in informal analysis of problems everyone frequently tries to see what consequences the conclusion will have, to check if consistency is violated. But I must emphasize that this course needs formal presentations of proofs.

I also found that very few of the students were able to do problem 1, and I suspect that difficulty has two roots, both mathematical. The first is that I meant the mathematical content of the problem to be well-known to students. In fact, our early linear algebra course, Math 250, stresses algorithmic analysis of linear systems, and only in passing does it acknowledge a formula (Cramer's rule) for solutions. So students were severely underinformed. I found this interesting. Second, most students couldn't present existence and uniqueness of solutions adequately. They don't understand the language that is used, nor what is required. I wrote extensive notes on many papers, but I will address this in class today.

311:09 **Workshop #3** **9/20/94**

1. Suppose $S \subseteq \mathbb{R}$ and $S \neq \emptyset$. Prove that if u is an upper bound of S and $u \in S$, then $\sup S = u$.

2. Construct examples of nonempty sets S_1 and S_2 with the following properties. For each example, be sure to prove that the number which you claim is the supremum actually is the supremum.

a) S_1 is nonempty, bounded above, and $\sup S_1 \notin S_1$.

b) S_2 is nonempty, bounded above, and $\sup S_2 \in S_2$.

3. Find examples of non-empty sets T_1 and T_2, both of which are both bounded above and bounded below, so that all of the following conditions hold:

$$\sup T_1 = 1 \quad \underline{\text{and}} \quad \sup T_2 = 1 \quad \underline{\text{and}} \quad \inf T_1 = 0 \quad \underline{\text{and}} \quad \inf T_2 = 0 \quad \underline{\text{and}} \quad T_1 \cap T_2 = \emptyset.$$

Be sure to prove that your assertions about the supremums and infimums are true.

$\star$ *Special added tricky part* $\star$ Can you find examples so that *every* subinterval of the unit interval $[0,1]$ which has positive length must intersect both T_1 and T_2? Prove your assertion.

4. **stretching**

Definition: Suppose $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. The set aD is defined by the following: $v \in aD$ exactly when there is $w \in D$ so that $v = aw$.

a) Compute some examples: what is aD if $a = 0$ and $D = \{0\}$; if $a = -1$ and $D = [0,1]$; if $a = 2$ and $D = (3,5)$; and if $a = 3$ and $D = \{7\} \cup \{-2\} \cup (9,100]$?

b) Suppose that D has an upper bound. Does aD have an upper bound? Under what circumstances can you relate $\sup D$ and $\sup aD$, and what is the relationship? Prove your assertion.

5. **translating**

Definition: Suppose $D \subseteq \mathbb{R}$ and $b \in \mathbb{R}$. The set $b+D$ is defined by the following: $v \in b + D$ exactly when there is $w \in D$ so that $v = b + w$.

a) Compute some examples: what is $b+D$ if $b = 0$ and $D = \{0\}$; if $b = -1$ and $D = [0,1]$; if $b = 2$ and $D = (3,5)$; and if $b = 3$ and $D = \{7\} \cup \{-2\} \cup (9,100]$?

b) Suppose that D has an upper bound. Does $b + D$ have an upper bound? Under what circumstances can you relate $\sup D$ and $\sup b + D$, and what is the relationship? Prove your assertion.

Comments on #3

I asked that genders be split up as much as possible in the groups. I had noticed that genders were being separated in the groups. Sometimes this resulted in the women being quite quiet, as many might have predicted, but I believe it was a valuable experiment. Problems 1, 2, and 3 were done by many groups, and I asked people to move along to 4 and/or 5 about halfway through the session. Next time I will try to get people to read all the problems first, and then select groups depending upon which problem they want to work on.

1. Prove directly from the definition of limit that

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = 1.$$

Even more: Can you prove that $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+2} = 1$? Can you prove that $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\frac{1}{2}} = 1$? Can you prove that if $a > 0$ then $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+a} = 1$?

2. Prove directly from the definition of limit that, if $k \in \mathbb{N}$ is fixed, then

$$\lim_{n\to\infty} \frac{n^k}{n!} = 0.$$

Comment: You may want to begin with a special case, such as $k = 2$, and then move on to consider, say, $k = 5$, and then ...

3. Prove (if true) or give a counterexample (if false):

If $\lim_{n\to\infty} x_n = x$, and if $a < x_n < b$ for all n, then $a < x < b$.

4. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence with $x_n > 0$ for all n, and that

$$\lim_{n\to\infty} (x_n)^{1/n} = L \text{ exists.}$$

If $L < 1$ prove that

$$\lim_{n\to\infty} x_n = 0$$

Hint: Perhaps try this first with some number for L, such as $L = .9$. What happens if you take $\varepsilon = .05$? How big and how small can $(x_n)^{1/n}$ be when n is large enough (depending on ε, of course)? What then will be true about x_n itself? Once you've done this, try to generalize the reasoning to more general L's.

311:09 **Workshop #5** **11/11/94**

1. Suppose (a_n) is a sequence with the following property:

$\dagger$ For any $n \in \mathbb{N}$ there is an integer m with $m > n$ so that $a_m \cdot a_n < 0$.

a) Find a simple example of a sequence satisfying $\dagger$.

b) Suppose in addition that (a_n) converges, and its limit is A. What can you say about A? Prove your assertion.

2. Suppose (b_n) is a sequence with the following property:

$\star$ For any $n \in \mathbb{N}$ there is an integer m with $m > n$ so that $b_m > b_n$.

a) Find a simple example of a sequence satisfying $\star$.

b) Are there any examples of non-monotonic sequences satisfying $\star$? (Of course you must read $\star$ very carefully.) If there are, give one example. If there are not, prove your assertion.

c) Suppose in addition that (b_n) converges, and its limit is B. What can you say about the relation of B to any of the individual members of the sequence as a result of $\star$?

3. Suppose α and β are positive numbers. Decide whether the following sequences converge. If they converge, try to find their limits. (Your answers may involve both numbers and their relationship.) I suggest you try some experiments first. (Please given reasons [such as citations from the text] for your assertions.)

i) $d_n = \sqrt[n]{\alpha^n + \beta^n}$ ii) $e_n = \sqrt[n]{\alpha^n + \beta}$

4. a) Prove: if $0 \le w \le 87$, then $0 \le \sqrt{3w + 42} \le 87$.

b) Prove: if $0 \le u \le v \le 87$, then $0 \le \sqrt{3u + 42} \le \sqrt{3v + 42} \le 87$.

c) Consider the following recursively defined sequence:

$$v_1 = 1$$
$$v_{n+1} = \sqrt{3v_n + 42}$$

and compute its first five terms. Prove that this sequence converges.

d) Suppose V is the limit of the sequence defined in c). What is V? (Square the recursive equation and take limits using some limit theorems.)

311:09 **Workshop #6** **10/18/94**

The last one entirely about sequences!

1. Find all starting values x_1 such that the sequence defined by $x_{n+1} = x_n^2$ converges to a limit L and prove that this convergence takes place. Prove that you have found all such choices of x_1.

2. a) Either find an example (and briefly verify the desired properties of the example) or explain why one can't exist (citing theorems to prove your assertions).

i) Find an example of a monotone sequence which is not Cauchy.

ii) Find an example of a monotone sequence which is not bounded.

iii) Find an example of a Cauchy sequence which is not monotone.

iv) Find an example of a Cauchy sequence which is not bounded.

v) Find an example of a bounded sequence which is not monotone.

vi) Find an example of a bounded sequence which is not Cauchy.

b) Which subsets of the three attributes $\{$bounded, Cauchy, monotone$\}$ imply that a sequence *must* converge? (There are a total of $2^3 = 8$ such subsets. For each of them, supply an example or a citation to verify your assertion.)

3. While dancing with my darling, we may take steps forward or backward, left or right. As we dance, we get tired, and each step is at most half the length of the previous step. Does the sequence of our steps converge to a point in the plane? (Explain your conclusion in detail, substantiating your reasoning in the language of this course.)

4. (In the following problem C is some unspecified positive number.)

a) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{n \cdot m}\ .$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

b) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{n + m}\ .$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

c) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{n} + \frac{C}{m}\ .$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

OVER

d) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{\max(n,m)}\ .$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

e) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{\min(n,m)}\ .$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

Comment

Replace dancing with my darling with a tired frog problem: that is, have a tired frog dancing on the $x-$axis, so that each step is at most one-half the length of the preceding step. Does the position of the frog converge? The problem above is a two-dimensional version of this, and seems susceptible to misunderstanding as stated. I had originally hoped to pose it so that readers would understand that the dancers would be able to move *either* in the x direction *or* in the y direction. Some students interestingly interpreted the problem as allowing both motions at once *or* as allowing motion "at an angle" with the max distance halving at each step: a more interesting problem to analyze. I think that for most students a "tired frog" would be difficult enough.

1. In all of the functions listed in this problem, the *domain* of the function is all of $\mathbb{R}$ and the *range* of the function is the set $\{0, 1\}$. For each function given, answer the following questions:

   i) For which $a \in \mathbb{R}$ does $\lim\limits_{x \to a} f$ exist?

   ii) If this limit exists, what is it?

Prove your assertions!

a) $f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

b) $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

c) $f(x) = \begin{cases} 0, & \text{if } 0 < x < 1 \\ 1, & \text{all other } x \end{cases}$

d) $f(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{all other } x \end{cases}$

e) $f(x) = \begin{cases} 0, & \text{if } x \neq \frac{1}{n} \text{ for all } n \in \mathbb{N} \\ 1, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \end{cases}$

2. a) The *domain* of g is all of $\mathbb{R}$. The following inequality is true for all $x \in \mathbb{R}$:

$$|g(x) - 17| \leq 56\sqrt{|x - 5|}$$

Prove that $\lim\limits_{x \to 5} g$ exists and that the value of the limit is 17.

b) The *domain* of g is all of $\mathbb{R}$. There is a fixed real number L so that the following inequality is true for all $x \in \mathbb{R}$:

$$|g(x) - L| \leq 56\sqrt{|x - 5|}$$

Prove that $\lim\limits_{x \to 5} g$ exists and that the value of the limit is L.

c) The *domain* of g is all of $\mathbb{R}$. There is a fixed real number L and a fixed real number a so that the following inequality is true for all $x \in \mathbb{R}$:

$$|g(x) - L| \leq 56\sqrt{|x - a|}$$

Prove that $\lim\limits_{x \to a} g$ exists and that the value of the limit is L.

OVER

d) The *domain* of g is all of $\mathbb{R}$. There is a fixed real number L and a fixed real number a and a fixed positive number K so that the following inequality is true for all $x \in \mathbb{R}$:

$$|g(x) - L| \leq K\sqrt{|x-a|}$$

Prove that $\lim_{x \to a} g$ exists and that the value of the limit is L.

e) The *domain* of g is all of $\mathbb{R}$. There is a fixed real number L and a fixed real number a and a fixed positive number K and a fixed positive real number p so that the following inequality is true for all $x \in \mathbb{R}$:

$$|g(x) - L| \leq K|x-a|^p$$

Prove that $\lim_{x \to a} g$ exists and that the value of the limit is L.

One purpose of this problem is to make a bit of a joke about the increasing amount of abstraction. Which part of the problem is "harder", and which part is "hardest"?

3. In this problem suppose that F has domain all of $\mathbb{R}$ and that $\lim_{x \to 0} F = L$.

a) If $G(x) = F(4x)$, prove that $\lim_{x \to 0} G(x) = L$.

b) If a is a non-zero constant, and if $H(x) = F(ax)$, prove that $\lim_{x \to 0} H(x) = L$.

c) If $K(x) = F(x^2)$, prove that $\lim_{x \to 0} K(x) = L$.

311:09 **Workshop #8** **11/1/94**

1. Consider the following properties of functions. In addition to what is listed, each of the functions considered in this problem should have domain all the real numbers except 0, that is, $\mathbb{R}\backslash\{0\}$.

1 f has the property that $|f(x)| < \frac{5}{|x|}$ for all $x \neq 0$.

2 f has the property that $|f(x)| < 19$ for all $x \neq 0$.

3 f has the property that $\lim_{x \to 0} f(x)$ exists and that $|\lim_{x \to 0} f(x)| < 7$.

4 f has the property that $\lim_{x \to 0} f(x)$ exists and equals 0.

a) Find examples of functions where

1 is true but # **2** is not.

2 is true but # **3** is not.

3 is true but # **4** is not.

b) Prove each of the following statements or give an example showing that it is false.

If g satisfies # **1** and if h satisfies # **2**, then $\lim_{x \to 0} g(x) \cdot h(x)$ exists.

If h satisfies # **2** and if k satisfies # **4** then $\lim_{x \to 0} h(x) \cdot k(x)$ exists.

If h satisfies # **2** and if j satisfies # **3** then $\lim_{x \to 0} h(x) \cdot j(x)$ exists.

If g satisfies # **1** and if k satisfies # **4** then $\lim_{x \to 0} g(x) \cdot (k(x))^2$ exists.

2. Suppose f and g both have domains all of $\mathbb{R}$. Suppose also that $\lim_{x \to a} f$ and $\lim_{x \to a} g$ exist, and that both of the limits are L. Now define a function h whose domain is all real numbers except a by

$$h(x) = \begin{cases} f(x), & \text{if } x < a \\ g(x), & \text{if } x > a \end{cases}$$

Prove that $\lim_{x \to a} h$ exists and is equal to L.

3. Suppose that f and g are functions whose domains are all of $\mathbb{R}$. For each of the following statements, prove if always true or give a counterexample if sometimes false.

a) If $\lim_{x \to 0} f(x)$ does not exist and $\lim_{x \to 0} g(x)$ does not exist, then $\lim_{x \to 0} f(x) \cdot g(x)$ does not exist.

b) If $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} f(x) + g(x)$ both exist, then $\lim_{x \to 0} g(x)$ exists.

c) If $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} f(x) \cdot g(x)$ both exist, then $\lim_{x \to 0} g(x)$ exists.

311:09 **Workshop #9** **11/8/94**

Introduction

Contemporary literary criticism urges readers to consider classical literature very carefully. Here is an example of a textual fragment together with some questions which might be asked about it:

$$\overbrace{\text{Then}}^{\alpha}\ \overbrace{\text{the wolf crept}}^{\beta}\ \overbrace{\text{closer and closer}}^{\gamma}\ \text{and little Red Riding Hood said,}$$

$$\text{“My, what great, big, enormous, yellow}\ \overbrace{\text{teeth}}^{\delta}\ \text{you have, Grandma.”}$$

α At what time did this occur? Before breakfast?

β How can a wolf creep? How is this different from a wolf slinking? What does this fragment reveal about the author's prejudice against wolves?

γ Does this involve some sort of limiting relationship between lRRH and the wolf? Who suggests the ε and who must come up with the δ?

δ Teeth: what does this tale have to contribute to the current debate over government-financed and planned health care? Do you think that the author is in favor of involving dental care in such discussions?

***

Below you are asked to analyze some results of our course (indeed, some of the true gems of the course!) in a similar spirit:

Continuous functions are bounded.

$$\text{If } f \text{ is a } \overbrace{\text{continuous}}^{\alpha} \text{ function whose domain is a}$$

$$\overbrace{\text{closed}}^{\beta} \text{ and } \overbrace{\text{bounded}}^{\gamma} \text{ interval, then } f \text{ is bounded.}$$

α If this word is omitted, the result is false. Give an example of a function whose domain is a closed and bounded interval, which is continuous at all but one point of its domain, and which is neither bounded above nor bounded below.

β If this word is omitted, the result is false. Give an example of a continuous function defined on an interval which isn't closed (but which is bounded) which is neither bounded above nor bounded below.

γ If this word is omitted, the result is false. Give an example of a continuous function which is neither bounded above nor bounded below whose domain is a closed interval.

Continuous functions attain highest and lowest values.

$$\text{If } f \text{ is a } \overbrace{\text{continuous}}^{\alpha} \text{ function whose domain is a } \overbrace{\text{closed}}^{\beta} \text{ and } \overbrace{\text{bounded}}^{\gamma} \text{ inter-}$$

$$\text{val, then } f \text{ attains } \overbrace{\text{a maximum and a minimum value}}^{\delta} \text{ in that interval.}$$

α If this word is omitted the result is false. Give an example of a bounded function whose domain is a closed and bounded interval which is continuous at all points of its domain but one and which does not attain either the supremum or the infimum of its values.

β If this word is omitted, the result is false. Give an example of a continuous function whose domain is a non-closed but bounded interval, which is a bounded function, and which does not attain either the supremum or the infimum of its values.

γ If this word is omitted, the result is false. Give an example of a function which is both continuous and bounded whose domain is a closed interval and which does not attain either the supremum or the infimum of its values.

δ Must f have only one maximum or minimum point? (Recall that f has a maximum at x^* in an interval I if $f(x^*) \geq f(x)$ for all $x \in I$. A minimum is defined similarly by reversing the inequality.) Give an example of a continuous function defined on a closed and bounded interval which has exactly 56,000 distinct maxima and exactly 2 minima. Give another example of a continuous function defined on a closed and bounded interval which has exactly 3 maxima and an infinite number of distinct minima.

Continuous functions attain intermediate values.

> If f is a $\overbrace{\text{continuous}}^{\alpha}$ function whose domain is a closed and bounded $\overbrace{\text{interval}}^{\beta}$, and $\overbrace{\text{if}}^{\delta}$ f is positive at one endpoint of the interval and negative at the other, $\overbrace{\text{then}}^{\delta}$ f must be zero $\overbrace{\text{at some point}}^{\gamma}$ in the interval.

α If this word is omitted the result is false. Give an example of a function whose domain is a closed and bounded interval, and which is positive at one endpoint of the interval and negative at the other, which is not continuous at exactly one point, and which is never zero in the interval.

β If this word is omitted the result is false. Give an example of a function whose domain is not a closed and bounded interval, which is continuous at every point in its domain, which has both positive and negative values on its domain, and which is never zero.

γ Does f have to be zero at exactly one point? Give an example of a function satisfying the hypotheses of this result which is zero in the interval at exactly 200 points. Give another example of a function satisfying the hypotheses of this result which is zero in the interval at infinitely many points.

δ Suppose that f is continuous in some closed and bounded interval, I. Is the converse of this result true? That is, if f is zero somewhere in I, must f have different signs at the endpoints of I? Give an example to show that this converse is false.

311:09 **Workshop #10** **11/15/94**

1. a) Suppose that $f : I \to \mathbb{R}$, and that there are positive constants M and p so that

$$|f(x) - f(y)| \le M|x-y|^p$$

for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on I.

b) Show that

$$|x^{1/3} - y^{1/3}| \le |x-y|^{1/3}$$

for all $x, y \in [0,\infty)$. (Thus $x^{1/3}$ is uniformly continuous on $[0,\infty)$.)

c) Show that there is <u>no</u> $M > 0$ such that

$$|x^{1/3} - y^{1/3}| \le M|x-y|$$

for <u>all</u> $x, y \in [0,\infty)$. (So a change in the value of p appearing in the inequality of part a) may affect the validity of the inequality.)

2. a) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous, "one-to-one" (that is, if $f(a) = f(b)$ then $a = b$), and $f(0) < f(1)$. Prove that $f(0)$ is the minimum value of f on $[0,1]$ and that $f(1)$ is the maximum value of f on $[0,1]$.

b) Prove that f is strictly increasing on $[0,1]$. (Hint: derive a contradiction if $0 \le a < b \le 1$ but $f(b) < f(a)$ by finding a point c with $f(a) = f(c)$.)

3. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$$

a) Prove that f is continuous only at $x = \dfrac{1}{2}$.

b) Prove that f is one-to-one.

c) Prove that f is "very" non-monotone: specifically, in any subinterval of positive length there are choices of a and b with $a < b$ so that $f(a) < f(b)$ can occur and $f(a) > f(b)$ can occur. (You may contrast this with the result of problem 3 which says, essentially, that functions which are continuous and one-to-one on an interval must be monotone.)

4. Prove if true in general, or give a counterexample otherwise:

> If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$ and is also strictly increasing, then the inverse function f^{-1} is uniformly continuous on $\mathbb{R}$.

311:09 **Workshop #11** **11/29/94**

1. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ so that f has all of the following properties:
   - f is continuous on $\mathbb{R}$ except for $\{1, 2, 4, 8\}$.
   - f is *not* continuous at 1, 2, 4, and 8.
   - f is differentiable on $\mathbb{R}$ except for $\{0, 1, 2, 3, 4, 8\}$.
   - f is *not* differentiable at 0, 1, 2, 3, 4, and 8.

Either prove that your example has the properties you assert, or prove that no such function exists.

2. (*Pasting together differentiable functions.*) Suppose that f and g are functions with domain $[a, b]$ where $a < b$. Suppose also that c is a fixed number in (a, b).

a) If f and g satisfy the following two additional hypotheses:

i) f and g are continuous at c and $f(c) = g(c)$.

ii) f and g are differentiable at c and $f'(c) = g'(c)$.

then prove that the function F with domain $[a, b]$ defined by

$$F(x) = \begin{cases} f(x), & \text{if } a \le x \le c \\ g(x), & \text{if } c < x \le b \end{cases}$$

is differentiable at c. (What does $F'(c)$ turn out to be?)

b) Give an example to show that if requirement ii) above is not true (while all the others are) then the conclusion is no longer valid.

3. Suppose

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ x^3, & \text{if } x \text{ is not rational} \end{cases}$$

Where is f continuous? Where is f differentiable? Prove your assertions.

4. Consider the following functions all of whose domains are all of $\mathbb{R}$:

$$f_1(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \qquad f_2(x) = \begin{cases} \frac{(-1)^n}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

$$f_3(x) = \begin{cases} n, & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \qquad f_4(x) = \begin{cases} \frac{1}{n^2}, & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Which of these is continuous at 0? Which of these is differentiable at 0? Prove your assertions.

YA-HOOOO!!!!! THE LAST ONE!!!!!

not really a celebration, just a memorial: quiet and distinguished . . . YA-HOOOO !!!!!

THE BIG FROG/TOAD/LEMUR/SLOTH RACE !!!

Every team has 30 minutes to prepare for the competition: a five minute presentation of its function and the answers to the questions asked.

THE FROGS Suppose $f(x) := \begin{cases} 1, & \text{if } x = \dfrac{1}{n} \text{ where } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$.

The domain of f is $[0,1]$. Sketch a graph of f on its domain. What are the upper and lower sums $L(f,P)$ and $U(f,P)$ for the partition

$$\left(0, a, \frac{1}{3} - b, \frac{1}{3} + b, \frac{1}{2} - b, \frac{1}{2} + b, 1 - b, 1\right)$$

of $[0,1]$? Here a is a number *slightly* larger than $\frac{1}{4}$ and b is a small positive number. Be very careful; this is certainly *not* a random partition but is one nicely adapted to this function.

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

THE TOADS Suppose $f(x) := \begin{cases} 1, & \text{if } x \text{ has a "3" in its decimal expansion} \\ 0, & \text{otherwise} \end{cases}$.

Thus $f(0.400) = 0$ and $f(0.4\mathbf{3}7) = 1$ and $f(0.427\mathbf{3}) = 1$ and $f(0.42729) = 1$. If a number has *two* decimal expansions (for example, $\frac{1}{10} = .1000\ldots$ and also $\frac{1}{10} = .0999\ldots$ while $\frac{3}{100} = .0300\ldots$ and also $\frac{3}{100} = .0299\ldots$) consider only the expansion without the concluding digit string, "999...".

The domain of f in this problem is $[0,1]$. Sketch a graph of f on its domain as well as you can. What are the upper and lower sums $L(f,P)$ and $U(f,P)$ for the partition

$$(0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1)$$

of $[0,1]$?

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

OVER

THE LEMURS Suppose $f(x) := \begin{cases} 1, & \text{if } 0 \le x \le \dfrac{1}{4} \\ 0, & \text{if } \dfrac{1}{4} < x \le \dfrac{1}{2} \\ 3, & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}$.

Sketch a graph of f on its domain. What are the upper and lower sums $L(f;P)$ and $U(f;P)$ for every possible partition P of $[0,1]$?

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

THE SLOTHS Suppose $f(x) := \begin{cases} 0, & \text{if } \dfrac{1}{2^n} < x < \dfrac{1}{2^{n-1}} \text{ and } n \in \mathbb{N} \text{ with } n \text{ even} \\ 1, & \text{if } \dfrac{1}{2^n} \le x \le \dfrac{1}{2^{n-1}} \text{ and } n \in \mathbb{N} \text{ with } n \text{ odd} \\ 0, & \text{if } x = 0. \end{cases}$.

Sketch a graph of f on its domain. What are the upper and lower sums of f for the partition

$$\left(0, a, \frac{1}{4} - b, \frac{1}{4} + b, \frac{1}{2} - b, \frac{1}{2} + b, 1 - b, 1\right)$$

of $[0,1]$? Here a is a number *slightly* larger than $\dfrac{1}{8}$ and b is a small positive number. Be very careful; this is certainly *not* a random partition but is one nicely adapted to this function.

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

THE BIG FROG/TOAD/LEMUR/SLOTH RACE !!!

Every team has 30 minutes to prepare for the competition: a five minute presentation of its function and the answers to the questions asked.

THE FROGS Suppose $f(x) := \begin{cases} 1, & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$.

The domain of f is $[0,1]$. Sketch a graph of f on its domain. What are the upper and lower sums $L(f,P)$ and $U(f,P)$ for the partition

$$\left(0, a, \frac{1}{3} - b, \frac{1}{3} + b, \frac{1}{2} - b, \frac{1}{2} + b, 1 - b, 1\right)$$

of $[0,1]$? Here a is a number *slightly* larger than $\frac{1}{4}$ and b is a small positive number. Be very careful; this is certainly *not* a random partition but is one nicely adapted to this function.

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

THE BIG FROG/TOAD/LEMUR/SLOTH RACE !!!

Every team has 30 minutes to prepare for the competition: a five minute presentation of its function and the answers to the questions asked.

THE TOADS Suppose $f(x) := \begin{cases} 1, & \text{if } x \text{ has a "3" in its decimal expansion} \\ 0, & \text{otherwise} \end{cases}$.

Thus $f(0.400) = 0$ and $f(0.4\mathbf{3}7) = 1$ and $f(0.427\mathbf{3}) = 1$ and $f(0.42729) = 1$. If a number has *two* decimal expansions (for example, $\frac{1}{10} = .1000\ldots$ and also $\frac{1}{10} = .0999\ldots$ while $\frac{3}{100} = .0300\ldots$ and also $\frac{3}{100} = .0299\ldots$) consider only the expansion without the concluding digit string, "999...".

The domain of f in this problem is $[0,1]$. Sketch a graph of f on its domain as well as you can. What are the upper and lower sums $L(f,P)$ and $U(f,P)$ for the partition

$$(0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1)$$

of $[0,1]$?

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

THE BIG FROG/TOAD/LEMUR/SLOTH RACE !!!

Every team has 30 minutes to prepare for the competition: a five minute presentation of its function and the answers to the questions asked.

THE LEMURS Suppose $f(x) := \begin{cases} 1, & \text{if } 0 \leq x \leq \dfrac{1}{4} \\ 0, & \text{if } \dfrac{1}{4} < x \leq \dfrac{1}{2} \\ 3, & \text{if } \dfrac{1}{2} < x \leq 1 \end{cases}$.

Sketch a graph of f on its domain. What are the upper and lower sums $L(f;P)$ and $U(f;P)$ for every possible partition P of $[0,1]$?

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

THE BIG FROG/TOAD/LEMUR/SLOTH RACE !!!

Every team has 30 minutes to prepare for the competition: a five minute presentation of its function and the answers to the questions asked.

THE SLOTHS Suppose $f(x) := \begin{cases} 0, & \text{if } \dfrac{1}{2^n} < x < \dfrac{1}{2^{n-1}} \text{ and } n \in \mathbb{N} \text{ with } n \text{ even} \\ 1, & \text{if } \dfrac{1}{2^n} \leq x \leq \dfrac{1}{2^{n-1}} \text{ and } n \in \mathbb{N} \text{ with } n \text{ odd} \\ 0, & \text{if } x = 0. \end{cases}$.

Sketch a graph of f on its domain. What are the upper and lower sums of f for the partition

$$\left(0, a, \frac{1}{4} - b, \frac{1}{4} + b, \frac{1}{2} - b, \frac{1}{2} + b, 1 - b, 1\right)$$

of $[0,1]$? Here a is a number *slightly* larger than $\dfrac{1}{8}$ and b is a small positive number. Be very careful; this is certainly *not* a random partition but is one nicely adapted to this function.

Is f integrable on $[0,1]$? If it is, find the integral and prove that your answer is correct. If it is not, explain why.

311:09 ~~**Workshop #5**~~ **10/4/94**

A PRACTICE EXAM, GIVEN LAST YEAR, INSTEAD!

1. Find all numbers $n \in \mathbb{N}$ for which $2^n \leq n!$. Prove that your answer is correct.

2. Fix $a, b \in \mathbb{R}$ with $a < b$. Let $I = (a, b) = \{x \in \mathbb{R} : a < x < b\}$. Prove that $\sup I = b$.

3. Let $S = \{x \in \mathbb{Q} : 0 < x < 1\}$. <u>(Note: S consists only of rational numbers.)</u> Find $\sup S$ and prove that your answer is correct.

4. Prove that there is no rational number c with $c^2 = 7$.

5. Suppose $a > 0$, $b > 0$, and $n \in \mathbb{N}$. Prove that $a < b$ implies $a^n < b^n$.

6. Let "$-$" be the usual binary operation given on $\mathbb{R}$ by $a - b$. Prove or disprove each of the following properties.

NA1) $a - b = b - a$ for all $a, b \in \mathbb{R}$.
NA2) $(a - b) - c = a - (b - c)$ for all $a, b, c \in \mathbb{R}$.
NA3) There exists an element $z \in \mathbb{R}$ such that $z - a = a$ and $a - z = a$ for all $a \in \mathbb{R}$.
NA4) For each element $a \in \mathbb{R}$ there exists an element $b \in \mathbb{R}$ such that $a - b = 0$ and $b - a = 0$.

7. Suppose $S \subset \mathbb{R}$ and $a \in S$, $\varepsilon > 0$ are such that $V_\varepsilon \subset S$. (Recall that V_ε is the ε-neighborhood of a.) Let $T = \{x \in \mathbb{R} : 3x \in S\}$.

a) Is $a/3 \in T$?
b) Is there a number $\delta > 0$ such that $V_{\delta/3} \subset T$? If there is, produce δ explicitly; if not, give an example.

8. Suppose that $S \subset \mathbb{R}$ is nonempty and bounded above.

a) If u is an upper bound for S, is it automatically true that $u \in S$? If so, prove it; if not, give a counterexample.
b) If u is an upper bound for S <u>and</u> $u \in S$, prove that $u = \sup S$.

9. Prove or give a counterexample:
If $H_n = [a_n, b_n)$, $n \in \mathbb{N}$, is any nested sequence of half-open-on-the-right bounded intervals, then

$$\bigcap_{n=1}^{\infty} H_n \neq \emptyset.$$

10. Let $S = \{x = m/(10)^n : n \in \mathbb{N}, m \in \mathbb{N}, m < (10)^n\}$. Label <u>all four</u> of the following statements as true or false, and prove <u>one</u> of your answers.

S is a closed interval	True	False
S is a open interval	True	False
S is bounded above	True	False
$\sup S$ is in S	True	False

Comments on last year's exam:

This exam was given last fall in one section of 311. The first two questions were given "closed book" — no books or notes were permitted. In this section of the exam, emphasis was placed on relatively simple consequences of the most important definitions, which needed to be known totally perfectly.

Use of any texts or notes were permitted in the second section of the exam. Students were asked to "do **any 6 of the 8** problems" in the second section.

This exam was given slightly earlier in its semester than our exam will be. Our first test will cover material up to and including section 3.1 of the text. Certainly the most important topics in the course so far include: mathematical induction, completeness (including the precise definition and verification of the supremum of a set), and the limit of a sequence. The knowledgeable use of these ideas of course involves skillful manipulation of inequalities.

My exam will be similar, so be sure to **bring your text and notes!** I will give you two problems to work on for 20 minutes without any assistance, and then six problems (no choice from me, sorry!) to work on with assistance from texts and notes. Please note that you may *not* share written material during the exam.

Comments on last year's exam:

This exam was given last fall in one section of 311. The first two questions were given "closed book" — no books or notes were permitted. In this section of the exam, emphasis was placed on relatively simple consequences of the most important definitions, which needed to be known totally perfectly.

Use of any texts or notes were permitted in the second section of the exam. Students were asked to "do **any 6 of the 8** problems" in the second section.

This exam was given slightly earlier in its semester than our exam will be. Our first test will cover material up to and including section 3.1 of the text. Certainly the most important topics in the course so far include: mathematical induction, completeness (including the precise definition and verification of the supremum of a set), and the limit of a sequence. The knowledgeable use of these ideas of course involves skillful manipulation of inequalities.

My exam will be similar, so be sure to **bring your text and notes!** I will give you two problems to work on for 20 minutes without any assistance, and then six problems (no choice from me, sorry!) to work on with assistance from texts and notes. Please note that you may *not* share written material during the exam.

311:09 **11/17/94**

An exam given last year in Math 311 at about this time in the semester.

1. Show directly from the definition of limit that

$$\lim_{n\to\infty} \frac{n^2}{3n^2+1} = \frac{1}{3}.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \frac{x^2}{3x^2+1}.$$

Show directly from the definition of continuity that f is continuous at $x = 1$.

3. For each of the following statements, prove (if true) or give a counterexample (if false).

a) If (x_n) is any convergent sequence with limit x,and $a < x_n < b$ for all n, then $a < x < b$.

b) If (x_n) is any convergent sequence with limit x, and $a < x < b$, then $a < x_n < b$ for all n.

c) If (x_n) is any convergent sequence with limit x, and $a < x < b$, then $a < x_n < b$ for some n.

4. Let (x_n) be the sequence given by

$$x_1 = \frac{1}{1}, x_2 = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}, x_3 = \frac{1}{1\cdot 3}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 3}, \ldots, x_n = \frac{1}{1\cdot n}+\frac{1}{2\cdot n}+\frac{1}{3\cdot n}+\ldots+\frac{1}{n\cdot n}.$$

Prove that (x_n) converges.

5. Let the sequence (x_n) be defined as follows: $0 < x_1 < 1$ is some fixed number, and

$$x_{n+1} = \sqrt{x_n} \quad \text{for } n \geq 1.$$

Prove that (x_n) converges, and find the limit.
What happens if $x_1 = 0$ instead?

6. Suppose $f : \mathbb{R} \to \mathbb{R}$, and

$$\lim_{x\to 0} f(x) = L.$$

Let $g(x) = f(2x)$. Show that

$$\lim_{x\to 0} g(x) = L.$$

7. Let $I = [a, b]$ be a closed, bounded interval, and let $f : I \to \mathbb{R}$. Label <u>each</u> of the following statements as true or false, and prove <u>one</u> of your answers.

a) If f is increasing on I, then f is automatically continuous on I.

b) If f is increasing on I, then f is automatically bounded on I.

c) If f is increasing on I, then $|f|$ is automatically continuous on I.

d) If $|f|$ is increasing on I, then f is automatically increasing on I.

8. Let $I = [a, b]$ be a closed bounded interval, and suppose $f : I \to \mathbb{R}$ has the following properties:

a) f is continuous,

b) $f(x) \geq 0$ for all $x \in I$, and

c) for every $x \in I$ there is a point $c_x \in I$ with $f(c_x) \leq \left(\frac{f(x)}{2}\right)$.

Prove that $f(x) = 0$ for some $x \in I$.

9. Show that the function

$$f(x) = \sqrt{x}$$

is uniformly continuous on the interval $[1, \infty)$.

10. Suppose that $f : A \to \mathbb{R}$ is uniformly continuous on A and that $|f(x)| \geq 4$ for all $x \in A$. Show that $\frac{1}{f}$ is uniformly continuous on A.

Rules for our second exam:

You may bring **one** standard-sized ($8\frac{1}{2}''$ by $11''$) sheet of paper with notes to use during the exam. The exam will principally cover the material we have done in chapters 3, 4, and 5, but of course the earlier material cannot be neglected since the course is cumulative.

WITH A FEW FURTHER COMMENTS ON OUR EXAM AT THE END.

1. Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

2. Let S be defined as follows:

$$S := \left\{ \frac{n}{2n+1} : n \in \mathbb{N} \right\} .$$

Find $\sup S$ and $\inf S$. Prove directly from the definitions of supremum and infimum that your answers are correct.

3. Let the sequence (a_n) be given by

$$a_n = \frac{n}{2n+1}, \quad n \in \mathbb{N} .$$

Find the limit of this sequence and prove directly from the definition of limit that your answer is correct.

4. Prove directly from the definition of uniform continuity that the function $f(x) = x^2$, $0 \le x \le 1$, is uniformly continuous on that interval.

5. Let f be the function defined on the interval $0 \le x \le 1$ by

$$f(x) := \begin{cases} \frac{1}{n!}, & \frac{1}{n+1} < x \le \frac{1}{n}, n \in \mathbb{N} \\ 0, & x = 0 \end{cases} .$$

(Draw the graph of f for $\frac{1}{4} \le x \le 1$ to get a feel for this function.)
Show that f is differentiable at $x = 0$.

6. Suppose that f is continuous on the interval $[0, 4]$, differentiable on the interval $(0, 4)$, $f(0) = 1$, and $5 < f'(x) < 7$ for $0 < x < 4$. Find optimal numbers A and B such that

$$A \le f(x) \le B \text{ for } 0 \le x \le 4 ,$$

and prove that f satisfies these inequalities.

7. Let f be defined on the interval $0 < x \le 1$ as follows:

$$f(x) := \begin{cases} x, & \frac{1}{n+1}, n \in \mathbb{N}, n \text{odd} \\ -x, & \frac{1}{n+1}, n \in \mathbb{N}, n \text{even} \end{cases} .$$

(Draw the graph of f for $\frac{1}{5} \le x \le 1$ to get a feel for this function.)
Prove that $\lim_{x \to 0} f(x) = 0$.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere, and satisfies $f(z) = 1$ for every irrational number z. Is it necessarily true that $f(x) = 1$ for every x? Prove if true or give a counterexample if false.

9. Let the sequence (x_n) be defined as follows:

$$x_1 = 5;\ x_{n+1} = 2\sqrt{x_n - 1}\ , n \geq 1\ .$$

Prove that the sequence (x_n) converges. What is its limit?

10. In the following, the function f is defined on A and the points x_n and x are points in A. Label each of the following statements as true or false, and prove one of your answers.

a) If f is monotone on A, and $x_n \to x$, then $f(x_n) \to f(x)$.

b) If f is continuous on A, and $x_n \to x$, then $f(x_n) \to f(x)$.

THIS PROBLEM HAS BEEN SHORTENED, SINCE IT CONTAINED SOME MATERIAL WE HAVE NOT COVERED.

11. Let (x_n) be a sequence satisfying

$$|x_{n+1} - x_n| \leq \frac{1}{3^n} \text{ for all } n \in \mathbb{N}\ .$$

Show that (x_n) is a Cauchy sequence.

12. Prove if true or give a counterexample if false for each of the following statements.

a) If f is differentiable for $-1 < x < 1$ and $f'(0) = 0$, then f has a relative maximum or minimum at $x = 0$.

b) If f is differentiable for $-1 < x < 1$ and $f'(0) \neq 0$, then f does not have a relative maximum or minimum at $x = 0$.

13. Show that the function

$$f(x) := \frac{x}{1 + x^2}$$

is uniformly continuous on $\mathbb{R}$.

14. Suppose $f : A \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y|^{1/3} \text{ for all } x, y \in A\ .$$

Prove that f is uniformly continuous on A.

15. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, prove that $1/f$ is integrable in $[a, b]$.

16. A PROBLEM ON MATERIAL WE DID NOT COVER.

17. As proved in class, if $f:[a,b]\to\mathbb{R}$ is integrable and $c\in[a,b]$ then

$$\int_a^b f=\int_a^c f+\int_c^b f\,.$$

(You do not have to reprove this property.)
Let $a<c_1<c_2<\ldots<c_n<b$. Prove that

$$\int_a^b f=\int_a^{c_1} f+\int_{c_1}^{c_2} f+\ldots+\int_{c_n}^b f\,.$$

I MAY PRESENT THE INITIAL EQUATION IN THIS PROBLEM IN CLASS — I CAN'T TELL, SINCE I AM PREPARING THIS MATERIAL SEVERAL DAYS IN ADVANCE OF WHEN I WILL GIVE IT OUT. YOU CAN, HOWEVER, DO THIS PROBLEM ANYWAY IF YOU LOOK CAREFULLY AT WHAT YOU ARE ALLOWED TO ASSUME AND WHAT YOU ARE THEN ASKED TO PROVE.

18. Suppose $f:[0,1]\to\mathbb{R}$ satisfies

$$|f(x)-f(y)|\le|x-y|^{1/3}\text{ for all }x,y\in[0,1].$$

Let $\mathcal{P}_n$ denote the usual partition of $[0,1]$ into n equal-length subintervals. Given $\varepsilon>0$, how large must n be to guarantee $U(f,\mathcal{P}_n)-L(f,\mathcal{P}_n)<\varepsilon$?

Please be reminded that the time and place for this section's **Final Exam** is as follows:

Thursday, December 15, from 8:00 to 11:00 PM, in HH-A4

and I will have office hours on Tuesday, Wednesday, and Thursday (December 13, 14, and 15) from 1 PM to 4 PM in Hill 542 on Tuesday and Thursday and from 11 AM to 2 PM on Wednesday. I will be accessible by e-mail (`greenfie@math.rutgers.edu`) as usual.

You may bring **one** standard-sized ($8\frac{1}{2}''$ by $11''$) sheet of paper with notes to use during the exam.

The final exam will cover everything we've studied, but the emphasis will probably be on the later part of the course. I repeat what I wrote before the second exam: the earlier material cannot be neglected since the course is cumulative. You must know definitions, principal examples and techniques, and the most important theorems. I recommend that you review previous exams and their answers because if I thought it was important before, I will probably continue to think it is important.