640:152:09-11 # Problems, Week 1 **1/27/97**

We'll tell you one problem whose solution should be written up for next week's meeting. Your written solutions should be of high-quality, with complete explanations given. While I encourage you to discuss the problems with other students and with your instructors, the written work you hand in must be your own explanations.

1. Suppose you know that Q is a differentiable function whose derivative is the function q (so $Q' = q$) and that you also know the values of Q in the table below. Compute the indicated definite integrals using the information given.

a) Compute $\displaystyle\int_0^1 q(x)\,dx$.

b) Compute $\displaystyle\int_1^2 xq(x^2)\,dx$.

c) Compute $\displaystyle\int_1^2 q(6-2x)\,dx$.

d) Compute $\displaystyle\int_1^{e^3} \frac{q(\ln x)}{5x}\,dx$.

x	$Q(x)$
0	7
1	4
2	6
3	2
4	10

Hint: We'll tell you the following to celebrate the beginning of the semester:

i) You'll need to use *all* of the numbers in the table.

ii) The answers are somewhere in this list: $1,\ -1,\ 2,\ -2,\ 3,\ -3$.

2. A sort of raindrop is obtained by revolving the profile curve

$$y = \sqrt{x}(x-C)^2 \text{ for } 0 \le x \le C$$

about the $x-$axis. Here C is a positive constant.

a) Sketch the profile curve and the solid of revolution.

b) For which value of C will the raindrop have volume 1? What are the rough dimensions (length and diameter) of this raindrop?

3. a) Sketch the region $0 \le y \le 1/x^2$, $1 \le x \le 4$.

b) Find the number a such that the line $x = a$ divides the region in a) into two parts of equal area.

c) Find the number b such that the line $y = b$ divides the region in a) into two parts of equal area.

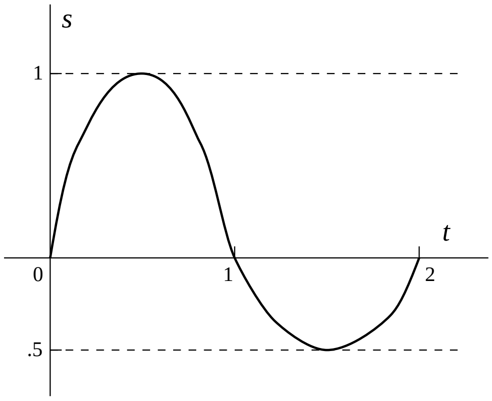
640:152:09-11 **Comments & More, Week 1** **1/27/97**

Problems 2 and 3 are meant to be direct applications of the volume and area formulas, respectively. Each has challenging "quirks" but neither is intended to be very difficult.

Problem 1 was meant to illustrate the Fundamental Theorem of Calculus and the method of substitution. Students seeking to apply mathematics should realize that functions can be defined in various ways. They can be formulas such as $\arctan(e^x)$ (with an implied domain when considering familiar functions). They can be "given" by tabular information as in problem 1, where now there's an implication of nice behavior interpolating the values in the table. Certainly such functions occur when collecting experimental data. But functions can also be "given" geometrically. Here's a rather difficult problem on the method of substitution which specifies a function geometrically, also showing the enormous difference between the effects of linear and non-linear change of variables.

Question

All that you know about the function $s = W(t)$ is given by the graph below:



Note the shape of the graph, and where the function is positive and negative.

a) Is the definite integral $\int_0^2 W(t)dt$ positive or negative?

b) Consider $\int_0^{10} W\left(\frac{t}{5}\right) dt$. Is it positive or negative? (Use substitution.) If A is a positive number, is $\int_0^{2A} W\left(\frac{t}{A}\right) dt$ positive or negative?

c) Now look at $\int_0^{256} W\left(t^{1/8}\right) dt$. You'll need to look very closely at the original graph of W to analyze whether this quantity is positive or negative. Is it? $(256 = 2^8)$

d) Is there a positive number A so that $\int_0^{2^A} W\left(t^{1/A}\right) dt$ is 0?

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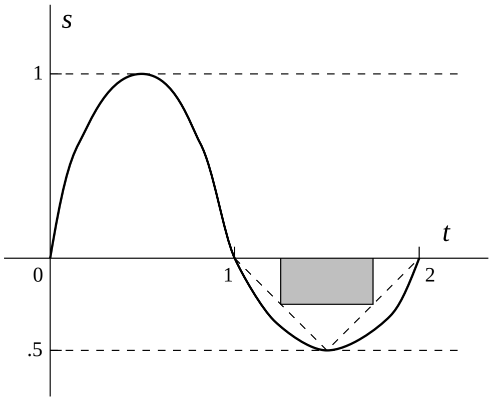
Comments and hints about the question

There might be an initial struggle: students might want to find a formula for W and not just rely upon geometry. Formulas can be "plugged in" to calculators and computers. Pictures can't (yet!).

a) Easy. Examine the picture.

b) This part of the question can be answered without much difficulty with a simple change of variables/substitution.

c) Here we've gotten *quite* subtle. The effect of the nonlinearity "inside" W will be felt in the whole integral, but the student needs to decompose the graph of W in some way to answer the question. For example, the following picture might help:



Then *maybe* it would be noticed that a large part of the resulting integral will be negative. A naive person might not see that easily.

d) This part of the question is quite subtle mathematically, but an engineering student might have enough of the "feel" of the subject to attempt an answer, perhaps citing continuity and the Intermediate Value Theorem. An adequate answer using "proper" theorem-proving mathematics is almost impossible for a first-year calculus student. If we assume that W is continuous, then the function $\mathcal{F}(A) = \int_0^{2^A} W\left(t^{1/A}\right)\,dt = \int_0^2 W(u)\,Au^{A-1}du$ will be "well-behaved" for $A \geq 1$. More advanced mathematics proves that $\mathcal{F}$ is continuous. Then there will be an A between 1 and 8 so that $\mathcal{F}(A) = 0$. Whew! This is too much to ask of students. Further consideration of the potential of this problem in actual use reveals that it has a large amount of "overhead", like many geometric problems. It takes some exposition before a student can really get started. Numeric or algebraic phenomena can be exhibited more briefly and then studied. Geometry seems to need more space.

640:152:09-11 # Problems, Week 2 **2/3/97**

1. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we get $v_{\text{ave}} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s we get $v_{\text{ave}} = \frac{2}{3}v_T$.

Note: this is problem 16, section 6.5, on page 410 of the textbook. Can you explain in words the seemingly paradoxical answer to this problem?

2. <u>A lesson in approximating $\sin x$</u>

a) For x near 0, $\sin x$ is well-approximated by its tangent line at $x = 0$. What is this tangent line?

b) For many purposes, approximation over an interval is preferred over approximation near a point. One way of measuring accuracy over an interval is called *mean-square error*. The mean-square error between the straight line $y = Ax$ and the function $\sin x$ over the interval $[0, 1]$ is given by the definite integral

$$\int_0^1 (\sin x - Ax)^2 \, dx$$

Find the A which <u>minimizes</u> this integral. (Hint: expand the integrand and compute everything. You'll get a simple function of A. Minimize this function.)

c) Make a rough sketch over the unit interval $[0, 1]$ of $\sin x$, the tangent line gotten in a), and the line found in b).

3. The following information is known about a function T and its derivatives:

x	$T(x)$	$T'(x)$	$T''(x)$
1	2	-2	2
2	3	6	5
3	7	4	-4
4	2	5	7

a) Compute $\int_2^3 T'(x)\, dx$.

b) Compute $\int_2^3 T''(x)\, dx$.

c) Compute $\int_2^3 x dx$.

d) Compute $\int_2^3 xT''(x)\, dx$.

Better not look at b) and c)! Integrate by parts.

e) Compute $\int_2^3 x^2 T'''(x)\, dx$.

And again and again.

640:152:09-11 **Problems, Week 3** **2/10/97**

We'll tell you one problem whose solution should be written up for next week's meeting. Your written solutions should be of high-quality, with complete explanations given. While I encourage you to discuss the problems with other students and with your instructors, the written work you hand in must be your own explanations.

1. a) Compute $\displaystyle\int_1^2 \frac{dx}{x^2}$.

b) Compute $\displaystyle\int_1^2 \frac{dx}{x(x-m)}$ if m is a small positive number.

c) What happens to your answer to b) when $m \to 0^+$?

d) Please try to support your answer to c) with relevant pictures. So you may wish to sketch the areas corresponding to the definite integrals in a) and b) (for some specific small positive m) and discuss the answer to c) with reference to these pictures.

2. Many integrals can be done with "rationalizing substitutions" which change the integrals into integrals involving rational functions. In turn, these integrals can be computed (theoretically) using partial fractions. Here are two examples of such integrals:

a) Integrate:

$$\int \frac{1}{(\sqrt{x}+1)(\sqrt{x}+2)}\,dx$$

Try $x = t^2$.

b) Integrate:

$$\int \frac{e^x+1}{e^{2x}+1}\,dx$$

Try $t = e^x$.

3. Numerically the value of

$$\int_0^4 x^2\sqrt{x^2+2}\,dx$$

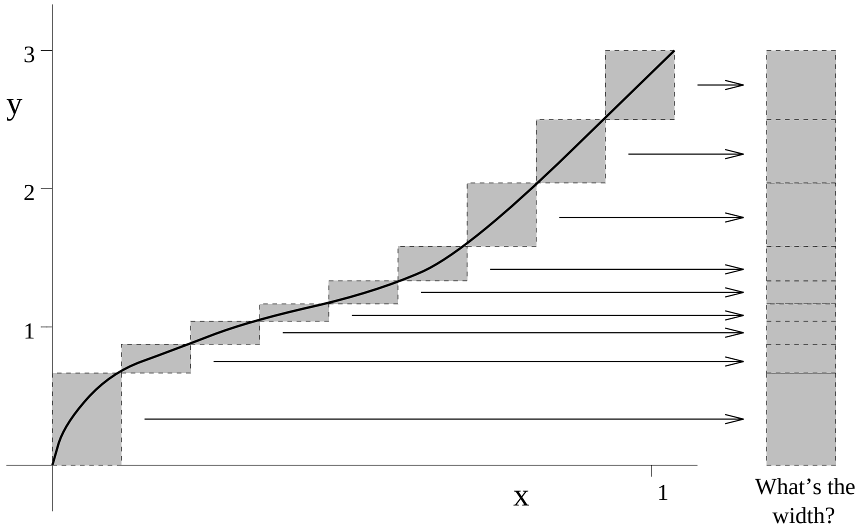
is approximately 71.243. Use the Fundamental Theorem of Calculus to compute this integral in terms of values of familiar functions and check that the reported numerical answer is correct.

640:152:09-11 **Problems, Week 4** **2/17/97**

1. Suppose $f(x) = \sqrt{3x + 6x^4}$.

a) Prove that f is increasing on the interval $[0, 1]$.

b) Write a finite sum which will be within 10^{-10} of the true value of $\int_0^1 f(x)\, dx$. You are *not* asked to actually compute the sum, just describe it in any convenient fashion and explain why your sum is within the desired tolerance.
Hint: your reasoning and your explanation may be guided by the picture below:



The axes have different scales. The picture "traps" the area between right & left sums, allowing an error estimate.

2. The computer program `Maple` is *not* able to find an antiderivative of $\left(1 + x^3\right)^{5/2}$. It does report the numerical approximation

$$\int_0^1 \left(1 + x^3\right)^{5/2}\, dx \approx 1.92$$

Here you will use the Trapezoid Rule to get this approximation.

a) First, find n (the number of subdivisions) so that you will be sure that the Trapezoidal Rule estimate with n subdivisions will be within 10^{-2} of the true value of the definite integral. (Use the appropriate error bound for the Trapezoid Rule.)

b) Write the sum resulting from applying the Trapezoid Rule which you assert is within $\pm.01$ of the integral's value. Then use your calculator to actually compute this Trapezoid Rule approximation to **5** decimal places and report this result.

3. a) Suppose $\mathcal{A}$ is the region in the first quadrant bounded by the curves $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ for x between 0 and 1. Sketch this region as well as you can.

b) Compute the area of $\mathcal{A}$ if it is finite. If it is not, explain carefully why your computation shows that it is not.

c) Suppose $\mathcal{B}$ is the region in the first quadrant bounded by the $y-$axis and the two curves $y = \tan x$ and $y = \sec x$ for x between 0 and $\frac{\pi}{2}$. Sketch this region as well as you can.

d) Compute the area of $\mathcal{B}$ if it is finite. If it is not, explain carefully why your computation shows that it is not.

640:152:09-11 **Problems, Week 6** **3/3/97**

We begin the core of the course this week: sequences, series, and approximation of functions. You may be interested in using your calculator to do some numerical exploration of these topics, although you should always remember that

Our goal is to teach calculus, not how to use a calculator!

Calculators and Sequences* The TI-82 has several built-in functions that may be useful when considering problems about sequences. You can read Chapter 6, *Sequence Graphing*, of the Texas Instruments *Guidebook*. To graph in sequence mode, press `MODE` and select `Seq` and `Dot`. Press `Y=` and you will see the special functions `U`*n* and `V`*n* which are used for graphing and calculating sequences. These sequences are usually written U_n and V_n, pronounced "U-sub-n" and "V-sub-n", the n^{th} terms of two infinite sequences. You'll need to use the variable *n* to write formulas for the `U`*n* and `V`*n* functions. *n* is the second function of the `9` key on the calculator. Don't confuse *n* with the alphabetic letter `N`. A key's "second function" is gotten by pressing the `2nd` key and then the other key.
For some sequences, you may need the variables `U`*n-1* (the second function of the `7` key) and `V`*n-1* (the second function of the `8` key).
After you have entered the formula for `U`*n*, go to the `WINDOW` menu to set `U`*n*`Start` to what you want for the initial element of the sequence. Also set the appropriate values for the other variables: for example, the *Guidebook* remarks "*n*`Min` should usually equal *n*`Start+1` to obtain a meaningful graph." Now press `TRACE` to plot the sequence in dot mode. Use the cursor keys to move along the dots. You can read the values of the sequence at the bottom of the screen. You can also use the `TABLE` menu to get values of `U`*n* and `V`*n*.

Stop just reading! Do A and B for practice:

A Try an example now: graph the first 10 terms of the sequence whose n^{th} term is $\frac{1}{n^2}$.

B Try another example: graph the first 10 terms of the sequence whose first term is 4, and whose n^{th} term is the square root of the sum of 3 and the $(n-1)^{\text{st}}$ term.

Note Such sequences are called "recursively defined": here the second term is $\sqrt{3 + \text{the first term}} = \sqrt{3+4} = \sqrt{7}$ and the third term is $\sqrt{3 + \text{the second term}} = \sqrt{3+\sqrt{7}}$, etc. Recursively defined sequences occur in many applications.

Factorials will be important in many of the applications we'll study. "Six factorial" is written 6! and means $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$. I computed this using `!` which is available under the `MATH` key: select `MATH`, then go to `PRB`, and then down to `!`. If you need $n! = n(n-1)\cdots 2 \cdot 1$ in a formula, first enter `n` (using `2nd` followed by `9`), then `MATH` and `PRB` followed by `!`. Of course the sequence of factorials is itself recursively defined.

More practice: do C and D now, please:

C Try to graph and inspect the first 10 terms of the sequence $\frac{n^3}{n!}$.

D These **Fibonacci numbers** are very famous: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

OVER

* What follows is for the TI-82. Other calculators should have equivalent capabilities. Check the manuals.

Create this sequence on your calculator. You'll need to be a bit ingenious, and use *both* of the sequence variables. Each term is the sum of the two preceding it, with obvious initial conditions.

Hint Try to describe U_n in terms of the one-step back elements of both sequences, and use V_n to pull things forward. (That's sufficiently mysterious!)

After you've practiced with A and B and C and D, you're ready for some problems:

1. Consider the following sequences:

$$a_n = \left(1 + \frac{1}{n}\right)^n \qquad b_n = \left(1 + \frac{1}{n^2}\right)^n \qquad c_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n$$

a) Use your calculator to plot the first ten terms of each of these sequences. Try to guess $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ and $\lim_{n\to\infty} c_n$ from your calculator information.

b) Treat n as a continuous variable which gets *really* large and use calculus (l'Hospital's rule) combined with some algebraic transformations to prove your guesses in a).

Note This is a *very* important example. Your intuition may be weak initially (we will train it!): all of these sequences "sort of" look like $(1 + 0)^\infty$ in a very loose way (when $n \approx \infty$), but their asymptotic behavior is remarkably different!

2. An interesting recursive sequence: define $a_1 = 1$ and $a_n = \sqrt{3a_{n-1} + 42}$ for $n > 1$.

a) Graph this sequence. Compute a_n for $n = 2, 3, \ldots, 10$.

b) Verify that the sequence of a_n's is increasing. Do this by showing two things:

i) $0 \le a_1 \le a_2$. ii) If $0 \le A \le B$ then $0 \le \sqrt{3A + 42} \le \sqrt{3B + 42}$.

Explain briefly why verifying these **two** facts gives you some confidence to state that the following **infinite** number of facts is true:

$$a_1 \le a_2 \le a_3 \le a_4 \le \ldots \le a_n \le a_{n+1} \le a_{n+2} \le \ldots$$

Note This isn't silly. For example, is $a_{345,678} \le a_{345,679}$? You certainly don't want to compute all these numbers exactly!

c) Verify that every number in this sequence is less than 10. Do this by showing two things:

i) $0 \le a_1 < 10$. ii) If $0 \le A < 10$, then $0 \le \sqrt{3A + 42} < 10$.

Note Again, the conclusion of this section of the problem is not at all obvious. I don't want to compute $a_{123,456,789}$ and, yet, it *is* less than 10 for sure! Why? Explain, please.

d) Every sequence which is increasing and has an upper bound must converge. So $\lim_{n\to\infty} a_n$ must exist – call the limit L. Since $a_n = \sqrt{3a_{n-1} + 42}$, what happens if we take the limit of this equation as $n \to \infty$ and do a little algebra? What information do you get about L and how does the information compare to the numbers gotten in a)?

640:152:09-11 **Problems, Week 8 ±1** **3/10/97**

1. a) Two students are sharing a loaf of bread. Student Alpha eats half of the loaf, then student Beta eats half of what's left, then Alpha eats half of what's left, and so on. How much of the loaf will each student eat?

b) Two students are sharing a loaf of bread. Student Alpha, now hungrier and more ferocious, eats two-thirds of the loaf, then student Beta eats eats half of what's left, then Alpha eats two-thirds of what's left, then Beta eats half of what's left, and so on. How much of the loaf will each student eat?

c) Now let's start with three students: Alpha, Beta, and Gamma. They decide to share a loaf of bread. Alpha eats half of the loaf, passes what's left on to Beta who eats half, and then on to Gamma who eats half, and then back to Alpha who eats half, and so on. How much of the loaf will each student eat?

2. a) If you enter 5 in your 10 digit calculator, and then hit the square root button 20 times, you'll probably get:

$$1.00000\,15348$$

and if you then subtract 1 and multiply by 1,048,576 you'll get:

$$1.60943\,91475$$

but on the other hand, the same calculator will tell you that $\ln 5$ is:

$$1.60943\,79124$$

and I wonder, is this just a coincidence? **Well, is it?** Hint: 1,048,576 is 2^{20}.

b) Given a positive number, x, outline a strategy for computing $\ln x$ only with the "primitive" arithmetic operations ($+$, $\times$, $-$, $/$) and square root ($\sqrt{\ }$). Your strategy should involve asserting (and verifying) that a certain sequence which can be easily computed with the listed operations always converges to $\ln x$.

3. We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

a) Why?

b) Find a specific partial sum of the series which is within .00001 of the sum of the whole series. Verify your assertion, and compute this partial sum.

Note: you may care to read page 11-8 discussing `seq(` and page 11-10 discussing `sum` in the TI-82 guidebook in order to conclude that the arithmetic you are asked to do is very easy in the late 20th century. Or compute what's needed by hand.

c) Find a specific partial sum of the series which is within 10^{-100} of the sum of the whole series. Verify your assertion.

Please note: you are not* asked to compute this partial sum, just to select one and verify that it satisfies the condition of this part of the problem. Also, you are not asked to find "least" partial sum fitting the approximation criterion given. Don't make this problem harder than it is!

* Thank goodness!

Problems, Week 9 ?

1. We know that $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

a) Why?

b) Find a specific partial sum of the series which is within .00001 of the sum of the whole series. Verify your assertion, and compute this partial sum.
Note: you may care to read page 11-8 discussing `seq(` and page 11-10 discussing `sum` in the TI-82 guidebook in order to conclude that the arithmetic you are asked to do is very easy in the late 20^{th} century. Or compute what's needed by hand.

c) Find a specific partial sum of the series which is within 10^{-100} of the sum of the whole series. Verify your assertion.

Please note: you are not* asked to compute this partial sum, just to select one and verify that it satisfies the condition of this part of the problem. Also, you are not asked to find "least" partial sum fitting the approximation criterion given. Don't make this problem harder than it is!

2. We know that $\sum_{n=1}^{\infty} \frac{1}{n^3+\sqrt{n}}$ converges.

a) Why?

b) Find N so that $\sum_{n=N}^{\infty} \frac{1}{n^3+\sqrt{n}} < .0001$.

c) Find $\sum_{n=1}^{\infty} \frac{1}{n^3+\sqrt{n}}$ to within an accuracy of $\pm .0001$.

3. a) Suppose that $a_n \geq 0$ for all n, and that $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} a_n^2$ must also converge.
Hint: look at the a_n's: they must get "small". Then *compare* the size of a small [positive] number and its square.)

b) Give an example to show that $\sum_{n=1}^{\infty} a_n^2$ can converge even though $\sum_{n=1}^{\infty} a_n$ does not converge.

4. Suppose $z_1 = \frac{1}{2}$ and that $z_{n+1} = (z_n)^2$ when $n \geq 1$. Prove either that $\sum_{1}^{\infty} z_n$ diverges, or (if it converges) find its sum to 10 decimal place accuracy.

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* Thank goodness!

Department of Amazing Coincidences

`Maple` reports that $\displaystyle\sum_{n=1}^{20} \frac{1}{n^n}$ is (approximately)

$$1.29128599706266354040728$$

while $\displaystyle\int_0^1 x^{-x}\, dx$ is said to have an approximate value of

$$1.29128599706266354040728$$

Weird, huh?

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Weird, huh?

Problems, Week 10.37

1. In the following series x is a real number. In each case use the ratio test to determine all values of x for which the series converges absolutely.

$$\sum_{n=1}^{\infty} n x^{2n} \qquad \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \sum_{n=1}^{\infty} \frac{x^n}{(n!)^2} \qquad \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

2. Consider the function F defined by the series

$$F(x) = \sum_{n=1}^{\infty} \frac{(\cos nx)^n}{3^n}$$

a) Verify that this series converges for all x.

b) Calculate $F(1.23)$ to an accuracy of $\pm .0001$

c) Sketch a reasonable graph of F and explain why F is periodic. Use the graph you have sketched to determine the range of F approximately.

3. Use the remainder estimate for an alternating series to answer the following:

a) We will soon see that $\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. How many terms in this series do you need to add in order to determine the first and second digits in the decimal expansion of $\ln 2$?

b) We will soon see that $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. If you add up the same number of terms of this series as you did in part a), how close will you be to the true value?

4. Manipulate the formula

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \cdots \qquad \text{(valid when } |r| < 1\text{)}$$

for the sum of a geometric series to express each of the following functions as a power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

Determine the coefficients a_n and x values for which the series converges in each case.

a) $f(x) = \dfrac{x}{1-x}$ b) $g(x) = \dfrac{2}{3x+4}$ c) $h(x) = \dfrac{1}{x^4+16}$

Problems, Week 11

1. Use the ratio test or root test to find the radius of convergence of each of the following power series. Also test the endpoints to find the interval of convergence for each series.

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}(3x)^n \qquad \sum_{n=1}^{\infty} \frac{n^3}{(2n+1)!}(x-1)^n \qquad \sum_{n=1}^{\infty} \sqrt{n}(3x+2)^n$$

2. Define the *Rutgers function* $R(x) = 1766 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = 1766 + \sum_{n=1}^{\infty} \dfrac{x^n}{n!}$

a) Find the radius of convergence of this series.

b) Differentiate the series for $R(x)$ term-by-term to show that $R'(x) = R(x) - 1765$.

c) Solve the differential equation $y' = y - 1765$ with the initial condition $y(0) = 1766$. Use this to express $R(x)$ in terms of e^x.

3. In this problem, $f(x) = \dfrac{x^2}{16 + x^4}$

a) Use the formula for the sum of a geometric series to express $f(x)$ as a power series in x. What is the radius of convergence of this series?

b) Integrate the power series in a) term-by-term to find a power series for the indefinite integral $\displaystyle\int f(x)\,dx$

c) Use the series in b) to express $\displaystyle\int_0^1 \frac{x^2\,dx}{16+x^4}$ as a convergent infinite series.

d) Use the alternating series remainder estimate to determine how many terms of the series in c) are needed to approximate the definite integral to 5 decimal places (this means that the error must be less than 5×10^{-6}). Then calculate the partial sum of the series with this number of terms, and compare with the approximate value from the `fnint(` program.

4. Define a function $f(x)$ by the power series

$$\sum_{n=0}^{\infty} (n+2)(n+1)x^n = 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots$$

a) Find the radius of convergence of this series.

b) Define $g(x) = \displaystyle\int_0^x f(t)\,dt$, where $f(x)$ is the function defined above. Use term-by-term integration to find the power series for $g(x)$.

c) Let $h(x) = \displaystyle\int_0^x g(t)\,dt$, where $g(x)$ is the function defined in b). Use term-by-term integration to find the power series for $h(x)$.

d) Use the formula for the sum of a geometric series to show that $h(x) = 1/(1-x)$. Now differentiate $h(x)$ twice to find an expression for $f(x)$ as a rational function.

Problems, Week 12.99432

In combinatorics, probability, statistics and theoretical computer science, the idea of a *GENERATING FUNCTION* is very important. Here's the definition:

A function $F(x)$ is the *GENERATING FUNCTION* for a sequence $\{a_n\}_{n=0}^{\infty}$ if

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

We loooked at the generating function of the Fibonacci numbers during one of the lectures.

1. Find the generating functions for the following sequences.

a) $\{1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{n!}, \ldots\}$

b) $\{1, 1, 1, \ldots$ (all 1's)$\}$

c) $\{1, 3, 3, 1, 0, \ldots$ (all the rest 0's) $\}$

2. Suppose we know the generating function $F(x)$ for some sequence $\{a_n\}_{n=0}^{\infty}$.

a) Show that $xF'(x)$ is the generating function for $\{na_n\}_{n=1}^{\infty}$. (*Hint:* Differentiate the series for $F(x)$ term-by-term and then multiply by x.)

b) Suppose $a_0 = 0$. Show that $\int_0^x \frac{F(t)}{t}\,dt$ is the generating function for $\{\frac{1}{n}a_n\}_{n=1}^{\infty}$. (*Hint:* Integrate the series for $F(t)/t$ term-by-term.)

3. Below is a list of sequences and a list of functions. Match the sequences with their generating functions. Use the Taylor series for the functions that you know and the results of problems #1 and #2.

Sequences :	Functions :
a) $\{0, 1, 2, 3, \ldots, n, \ldots\}$	α) $\dfrac{x + x^2}{(1-x)^3}$
b) $\{1, 2, 4, 8, \ldots, 2^n, \ldots\}$	β) $\ln\left(\dfrac{1}{1-x}\right)$
c) $\{0, 1, 4, 9, \ldots, n^2, \ldots\}$	γ) $\dfrac{1}{1-2x}$
d) $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$	δ) $\displaystyle\int_0^x \ln\left(\frac{1}{1-t}\right)\frac{dt}{t}$
e) $\{0, 1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\}$	ϵ) $\dfrac{x}{(1-x)^2}$

4. Consider the power series $\sum_{n=1}^{\infty}(n^2 + 3n + 7)x^n$.

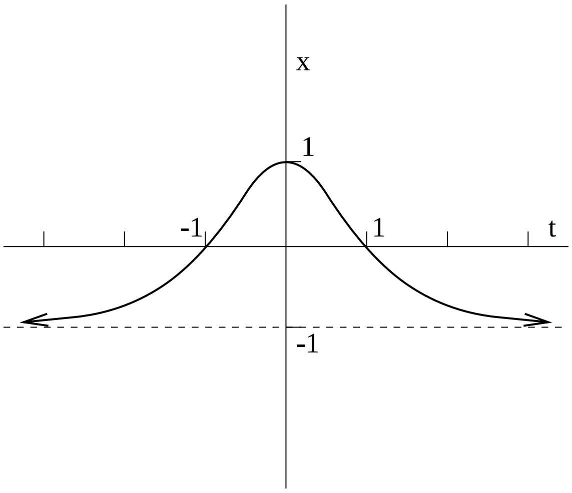
a) What is the radius of convergence of this series?

b) Inside its radius of convergence, the sum of this series is a rational function (a quotient of polynomials). What is this function?

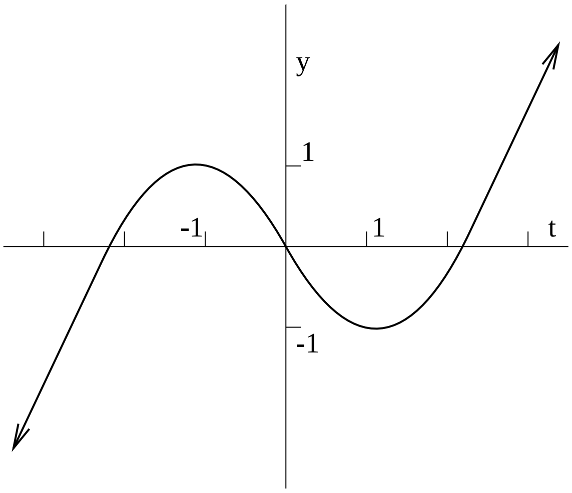
640:152:09-11 — **Problems, Week 12.4?** — **4/30/97**

From a final exam given in the spring, 1996, semester

1. A curve is described by parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ where the graphs of x as a function of t and y as a function of t are given below.



x=f(t)



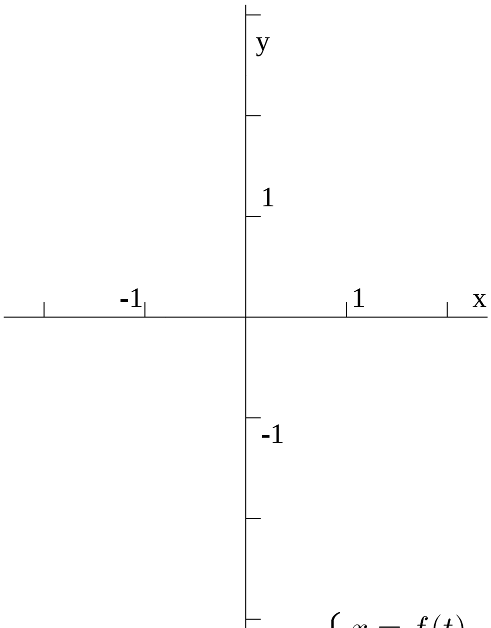
y=g(t)

a) What is the position of the point on the curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ when $t = 0$? ______

b) For which value(s) of t is the tangent line to the curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ horizontal? ______

c) For which value(s) of t is the tangent line to the curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ vertical? ______

d) Sketch the curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ using all the information given.



e) As $t \to +\infty$, which line (if any) does the curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ approach? ______

From a final exam given in the fall, 1996, semester

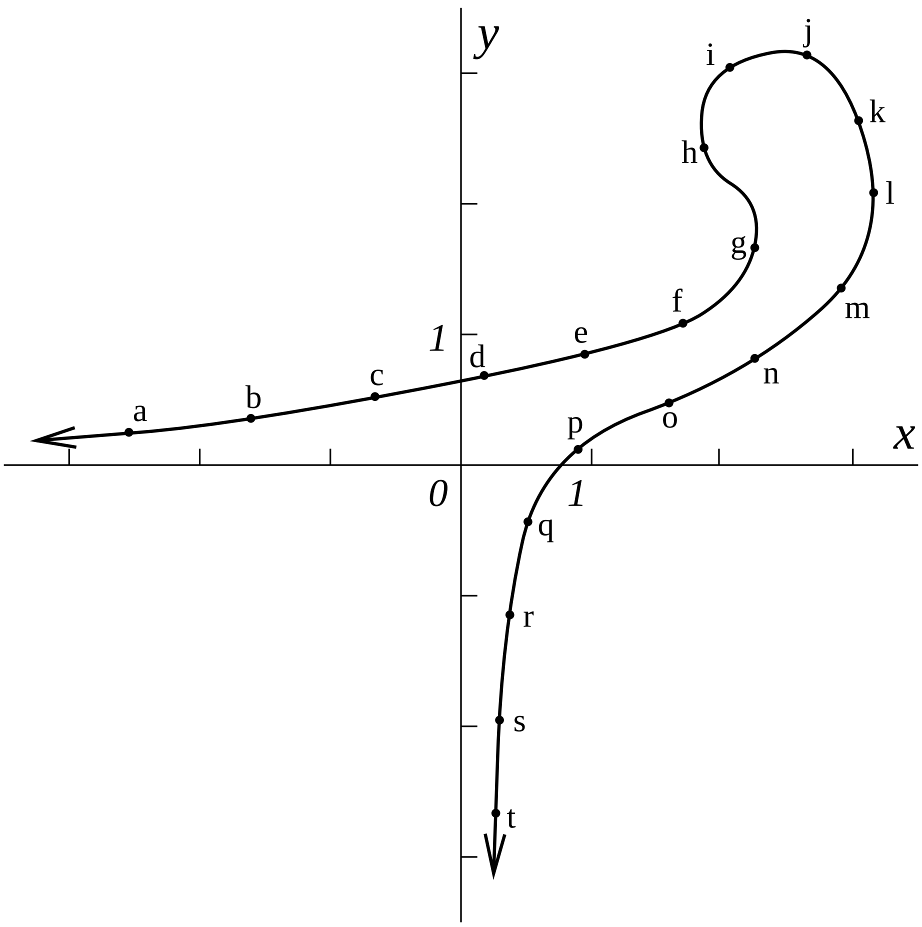
2. Below is the graph of a parametric curve. The alphabetic letters close to it are, in order, the increasing values of the parameter, t, which describes the curve using two functions, $x = \mathcal{F}(t)$ and $y = \mathcal{G}(t)$.

On the next page, sketch graphs of the functions $\mathcal{F}$ and $\mathcal{G}$ as well as you can. Note any intercepts of the coordinate axes of these graphs, and any local maxima and minima, and any asymptotic properties of these curves. You should write brief explanatory statements next to any interesting features of the graphs, such as maxima or minima or asymptotic behavior.

. . . .

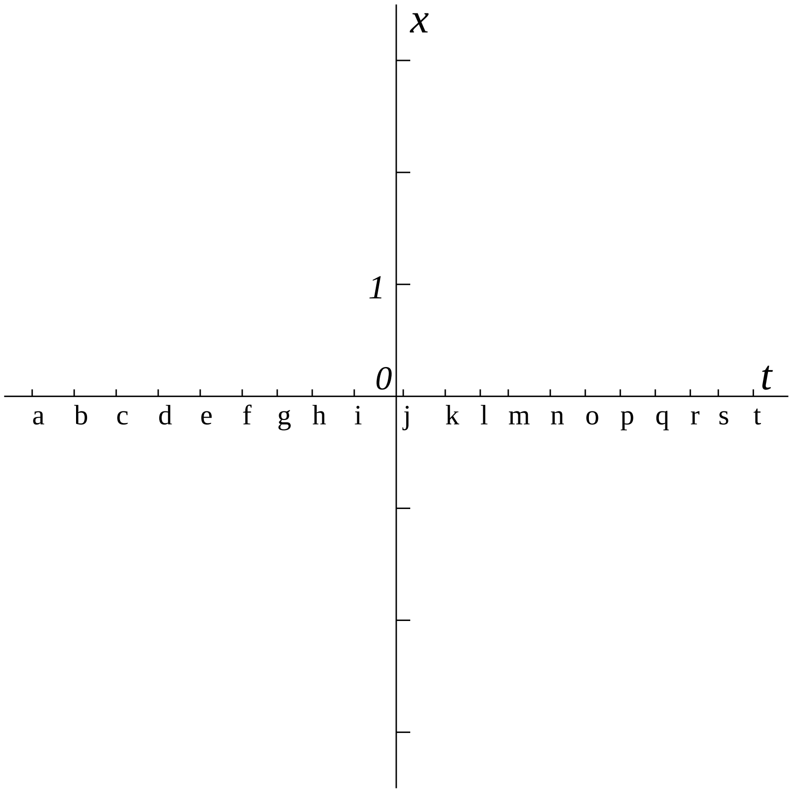
You may wish to detach this page here to make it easier to complete the graphs on the next page.

. . . .

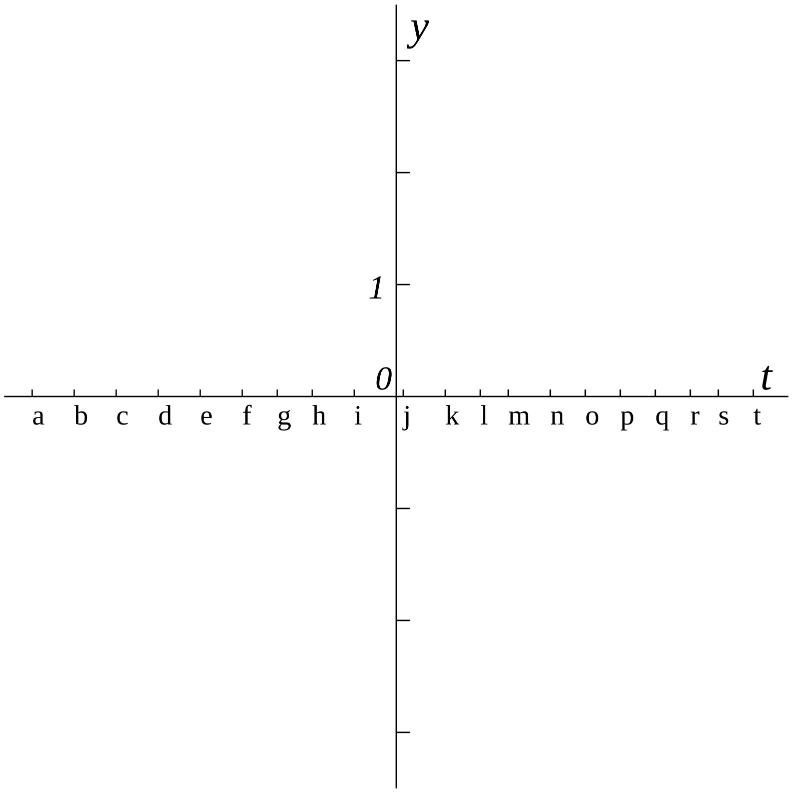


Remember to indicate as well as you can on each graph:

- Any intercepts of coordinate axes
- Any local maxima or minima
- Any asymptotic behavior



Your graph of $x = \mathcal{F}(t)$



Your graph of $y = \mathcal{G}(t)$