Math 152 Problems, Week 1

1. (a) Evaluate the integral

$$\int \frac{x^2}{\sqrt{2+x^3}}\,dx$$

using the substitution $u = 2 + x^3$.

(b) Find an appropriate substitution to evaluate the following indefinite integrals:

$$\int e^x(1+e^x)^{27}\,dx \qquad\qquad \int \frac{x}{1+x^4}\,dx$$

(c) Suppose f is continuous and $\int_0^9 f(x)\,dx = 4$. Find $\int_0^3 xf(x^2)\,dx$. (*Hint:* Try a substitution in the integral.)

2. The function

$$C(x) = \int_0^x \cos(\pi t^2/2)\,dt$$

is used in the theory of diffraction of light waves.

(a) Find $C'(x)$ and plot its graph for $0 \le x \le 2$. On what intervals is $C(x)$ increasing?

(b) Find $C''(x)$ and plot its graph for $0 \le x \le 2$. On what intervals is $C(x)$ concave upwards?

(c) Use the graph in (a) to show that $1/2 < C(1) < 1$. Then use the `fnInt(` program on your calculator to find a more exact value for $C(1)$.

3. a) Sketch the region $0 \le y \le 1/x^2$, $1 \le x \le 4$.

b) Find the number a such that the line $x = a$ divides the region in a) into two parts of equal area.

c) Find the number b such that the line $y = b$ divides the region in a) into two parts of equal area.

Calculus 152 Problems, Week 2

1. a) Sketch the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

b) Find the area of the region in a).

2. Here are two different ways to approximate the function $\sin(x)$ on the interval $0 \le x \le \pi$.
a) Let $f(x) = ax(\pi - x)$, where a is a constant. Note that $f(0) = 0$ and $f(\pi) = 0$ for any choice of a. Determine a so that

$$\int_0^\pi f(x)\,dx = \int_0^\pi \sin(x)\,dx$$

(thus the areas under the curves $y = f(x)$ and $y = \sin(x)$ for $0 \le x \le \pi$ are the same).

b) Let $g(x) = bx(\pi - x)$, where b is a constant. Note that $g(0) = 0$ and $g(\pi) = 0$ for any choice of b. Determine b so that the the curves $y = g(x)$ and $y = \sin(x)$ have the same tangent lines at $(0, 0)$ and at $(\pi, 0)$.

c) Plot the graphs of $y = \sin(x)$, $y = f(x)$, and $y = g(x)$ for $0 \le x \le \pi$ on the same axes.

3. Let R be the parabolic region in the x-y plane bounded below by the curve $y = x^2$ and above by the line $y = 1$.
a) Sketch R. Set up and evaluate an integral that gives the area of R.

b) Suppose a solid has base R and the cross-sections of the solid perpendicular to the y-axis are squares. Sketch the solid and find its volume.

c) Suppose a solid has base R and the cross-sections of the solid perpendicular to the y-axis are equilateral triangles. Sketch the solid and find its volume.

4. The *Bernoulli polynomials* $B_n(x)$ are defined by $B_0(x) = 1$ and

$$B_n'(x) = B_{n-1}(x), \qquad \int_0^1 B_n(x)\,dx = 0$$

for $n = 1, 2, 3, \ldots$.
a) Find formulas for $B_1(x)$, $B_2(x)$, and $B_3(x)$ and graph these polynomials.

b) Use the fundamental theorem of calculus to show that $B_n(0) = B_n(1)$ for all values of $n \ge 2$.

Math 152 Problems, Week 3

1. Evaluate each indefinite integral using substitutions and integration by parts, as needed:

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx \qquad\qquad \int x^3 e^{x^2}\,dx$$

$$\int \frac{\cos\left[\ln\left(x^2\right)\right]}{x}\,dx \qquad\qquad \int x\arctan x\,dx$$

2. The following information is known about a function T and its derivatives:

x	$T(x)$	$T'(x)$	$T''(x)$
1	2	-2	2
2	3	6	5
3	7	4	-4
4	2	5	7

a) Compute $\displaystyle\int_2^3 T'(x)\,dx$.

b) Compute $\displaystyle\int_2^3 T''(x)\,dx$.

c) Compute $\displaystyle\int_2^3 x dx$.

d) Compute $\displaystyle\int_2^3 xT''(x)\,dx$.

e) Compute $\displaystyle\int_2^3 x^2 T'''(x)\,dx$.

3. A water storage tank has the shape of a cylinder with diameter 10 ft. It is mounted so the axis of the cylinder is horizontal (the cylindrical cross-sections are vertical). If the depth of the water is 7 ft., what percentage of the total capacity of the tank is filled?

Draw a picture and set up the problem. Then solve it in three (3) ways:

a) By elementary geometry (areas of circular sectors)

b) Express the answer in terms of a definite integral, and then obtain an approximate numerical value for the integral using the `fnInt(` function on your calculator.

c) Evaluate the integral in b) *exactly* in terms of elementary functions using a trig substitution, and then obtain approximate numerical values for these functions using your calculator.

Of course, all three methods should give the same answer!

(Over)

4. If a freely falling body starts from rest, then its velocity is given by $v(t) = gt$ and its displacement is given by $s = \frac{1}{2}gt^2$.

a) Show that the *time average* of the velocities with respect to t over the interval $0 \le t \le T$ is

$$v_{\text{time aver}} = \frac{1}{2}v(T).$$

b) Find a formula for the velocity as a function of displacement s.

c) Show that *distance average* of the velocities with respect to s over the corresponding interval $0 \le s \le \frac{1}{2}gT^2$ is

$$v_{\text{dist aver}} = \frac{2}{3}v(T).$$

Math 152 Problems, Week 4

1. Evaluate:

a) $\displaystyle\int \sqrt{4-x^2}\,dx.$

b) $\displaystyle\int \sqrt{4+x^2}\,dx$

c) $\displaystyle\int \sqrt{4+4x+x^2}\,dx$

2. a) Sketch the region R bounded below by the x-axis, bounded on the left by the line $x=2$, bounded on the right by the line $x=3$, and bounded above by the curve

$$y = \frac{1}{x^2+3x+2}$$

b) Find the area of the region R in a).

c) The region R in a) is rotated around the x-axis. Find the volume of the solid it generates.

3. Many integrals can be done with "rationalizing substitutions" which change the integrals into integrals involving rational functions. In turn, these integrals can be computed (theoretically) using partial fractions. Here are two examples of such integrals:

a) Integrate:

$$\int \frac{1}{(\sqrt{x}+1)(\sqrt{x}+2)}\,dx$$

Try $x = t^2$.

b) Integrate:

$$\int \frac{e^x+1}{e^{2x}+1}\,dx$$

Try $t = e^x$.

Math 152 Problems, Week 5

Note: The problems on numerical integration refer to the programs `TRAP` and `SIMP` that you should have already keyed into your calculator (these programs are described on a separate handout).

1. Consider the following four integrals:

a) $\displaystyle\int_0^\infty \frac{x}{1+x^4}\,dx$ b) $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}\,dx$

c) $\displaystyle\int_0^\infty \frac{x^3}{1+x^4}\,dx$ d) $\displaystyle\int_0^\infty \frac{x^4}{1+x^4}\,dx$

Which of these integrals converge? (Hint: compare to the "correct" pure powers of x.) Compute the exact value of at least one of the convergent integrals.

2. a) Approximate $\displaystyle\int_0^4 x^3\,dx$ using the Trapezoidal Rule with $n = 4$. Write out the calculations by hand, then check using the program TRAP (remember to set $y_1 = x^3$ in your calculator before running the program).

b) Sketch the graph of $y = x^3$ for $0 \leq x \leq 4$. In your sketch show the area that the trapezoidal rule calculates exactly. Compare your approximation from a) with the exact value of the integral obtained by the Fundamental Theorem of Calculus. How could you have predicted from your graph that the approximation would be larger than the true value of the integral?

c) Approximate $\displaystyle\int_0^4 x^3\,dx$ using Simpson's Rule with $N = 4$. Write out the calculations by hand, then check using the program SIMP.

d) Compare your approximation with the exact value. Before beginning the calculation you could have predicted from the error formula for Simpson's rule (text, boxed formula on page 464) that the error would be as astoundingly small as it turned out to be. Why?

(Over)

3. a) The following improper integral converges: $\int_0^\infty xe^{-x^2}\,dx$. What is its value?

b) The value of $\int_0^\infty e^{-x^2}\,dx$ (another convergent improper integral) is $\dfrac{\sqrt{\pi}}{2}$ (this amazing fact is easily explained with some of the tools in third semester calculus). Improper integrals involving polynomials and e^{-x^2} often arise in statistics (and hence in statistical analyses of experiments). Use integration by parts to get a formula relating $\int_0^\infty x^n e^{-x^2}\,dx$ and $\int_0^\infty x^{n-2}e^{-x^2}\,dx$, where n is a positive integer bigger than 2. (The parts you take will be slightly tricky.)

c) Now find the values of

i) $\int_0^\infty x^2e^{-x^2}\,dx$ ii) $\int_0^\infty x^3e^{-x^2}\,dx$ iii) $\int_0^\infty x^4e^{-x^2}\,dx$

You will need the reduction formula and the two initial values given in a) and b).

4. Consider the definite integral $\int_0^1 \cos(x^2)\,dx$. Read all parts of the problem before doing part a), so as to save recaculating the needed numbers (you can use the `ANS` key on your calculator to do calculations on the output of the numerical integration programs.)

a) Compute the results of approximating this integral with both the Trapezoidal Rule and Simpson's Rule with $n = 8$ and $n = 16$ and $n = 32$ subdivisions. (If you have already keyed the `TRAP` and `SIMP` programs into your graphing calculator, this will take less than two minutes to get the six approximations; otherwise, it's impossibly time-consuming.) Before making the calculations, set your calculator to 9 digits of accuracy and `Sci` using the `MODE` key.

b) A twenty-digit approximation of the true value of this integral is

$$.90452423790027208145$$

Compute the differences between each of the approximations obtained in a) and the true value. (It will save time if you store this value in your calculator using the `STO` key BEFORE doing the calculations in a).) Multiply the Trapezoidal Rule errors by n^2 and the Simpson's Rule errors by n^4 in each case.

c) Make a table exhibiting all the numbers you have computed. Use the information in the table to answer the following questions:

> When n is doubled, what happens to the error? Can you predict (without calculation) what the error would be when $n = 64$, based on the cases $n = 8, 16, 32$? Check your prediction by doing the calculation.

Math 152 Problems, Week 7

Calculators and Sequences* The TI-82 has several built-in functions that may be useful when considering problems about sequences. You can read Chapter 6, *Sequence Graphing*, of the Texas Instruments *Guidebook*. To graph in sequence mode, press `MODE` and select `Seq` and `Dot`. Press `Y=` and you will see the special functions `U`*n* and `V`*n* which are used for graphing and calculating sequences. These sequences are usually written U_n and V_n, pronounced "U-sub-n" and "V-sub-n", the n^{th} terms of two infinite sequences. You'll need to use the variable *n* to write formulas for the `U`*n* and `V`*n* functions. *n* is the second function of the `9` key on the calculator. Don't confuse *n* with the alphabetic letter `N`. A key's "second function" is gotten by pressing the `2nd` key and then the other key.
For some sequences, you may need the variables `U`*n-1* (the second function of the `7` key) and `V`*n-1* (the second function of the `8` key).
After you have entered the formula for `U`*n*, go to the `WINDOW` menu to set `U`*n*`Start` to what you want for the initial element of the sequence. Also set the appropriate values for the other variables: for example, the *Guidebook* remarks "*n*`Min` should usually equal *n*`Start+1` to obtain a meaningful graph." Now press `TRACE` to plot the sequence in dot mode. Use the cursor keys to move along the dots. You can read the values of the sequence at the bottom of the screen. You can also use the `TABLE` menu to get values of `U`*n* and `V`*n*.

Some Warm-Up Exercises:

A Graph the first 10 terms of the sequence whose n^{th} term is $\dfrac{1}{n^2}$.

B Graph the first 10 terms of the sequence whose first term is 4, and whose n^{th} term is the square root of the sum of 3 and the $(n-1)^{\text{st}}$ term.

Note: Such sequences are called "recursively defined": here the second term is $\sqrt{3+\text{the first term}} = \sqrt{3+4} = \sqrt{7}$ and the third term is $\sqrt{3+\text{the second term}} = \sqrt{3+\sqrt{7}}$, etc. Recursively defined sequences occur in many applications.

Factorials will be important in many of the applications we'll study. "Six factorial" is written 6! and means $6\cdot5\cdot4\cdot3\cdot2\cdot1 = 720$. Try computing 10! by entering the number 10; then press the `MATH` key, select `PRB`, and then go down to `!` and press `ENTER`. If you need $n! = n(n-1)\cdots2\cdot1$ in a formula, first enter `n` (using `2nd` followed by `9`), then `MATH` and `PRB` followed by `!`. Of course the sequence of factorials is itself recursively defined.

C Graph and inspect the first 10 terms of the sequence $\dfrac{n^3}{n!}$.

D These **Fibonacci numbers** are very famous: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. Create this sequence on your calculator. You'll need to be a bit ingenious, and use *both* of the sequence variables. Each term is the sum of the two preceding it, with obvious initial conditions.

(**Hint:** Try setting $V_n = U_{n-1}$ and then express U_n in terms of U_{n-1} and V_{n-1}.)

OVER

* What follows is for the TI-82. Other calculators should have equivalent capabilities. Check the manuals.

After you've practiced with A, B, C and D, you're ready for some problems:

1. Consider the following sequences:

$$a_n = \left(1 + \frac{1}{n}\right)^n \qquad b_n = \left(1 + \frac{1}{n^2}\right)^n \qquad c_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n$$

a) Use your calculator to plot the first ten terms of each of these sequences. Try to guess $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ and $\lim_{n\to\infty} c_n$ from your calculator information.

b) Treat n as a continuous variable which gets *really* large and use calculus (l'Hospital's rule) combined with some algebraic transformations to prove your guesses in a).

Note This is a *very* important example. Your intuition may be weak initially (we will train it!): all of these sequences "sort of" look like $(1+0)^\infty$ in a very loose way (when $n \approx \infty$), but their asymptotic behavior is remarkably different!

2. An interesting recursive sequence: define $a_1 = 1$ and $a_n = \sqrt{3a_{n-1} + 42}$ for $n > 1$.

a) Graph this sequence. Compute a_n for $n = 2, 3, \ldots, 10$. Does it appear that $\lim_{n\to\infty} a_n$ exists?

b) Verify that the sequence of a_n's is increasing. Do this by showing two things:

i) $0 \le a_1 \le a_2$. ii) If $0 \le A \le B$ then $0 \le \sqrt{3A+42} \le \sqrt{3B+42}$.

Explain briefly why verifying these **two** facts gives you some confidence to state that the following **infinite** number of facts is true:

$$a_1 \le a_2 \le a_3 \le a_4 \le \ldots \le a_n \le a_{n+1} \le a_{n+2} \le \ldots$$

Note This isn't silly. For example, is $a_{345,678} \le a_{345,679}$? You certainly don't want to compute all these numbers exactly!

c) Verify that every number in this sequence is less than 10. Do this by showing two things:

i) $0 \le a_1 < 10$. ii) If $0 \le A < 10$, then $0 \le \sqrt{3A+42} < 10$.

Note Again, the conclusion of this section of the problem is not at all obvious. I don't want to compute $a_{123,456,789}$ and, yet, it *is* less than 10 for sure! Why? Explain, please.

Important Theorem: If $\{a_n\}$ is an *increasing* and *bounded* sequence, then $\lim_{n\to\infty} a_n$ exists.

d) Let $L = \lim_{n\to\infty} a_n$. Since $a_n = \sqrt{3a_{n-1} + 42}$, what happens if we take the limit of this equation as $n \to \infty$ and do a little algebra? What information do you get about L and how does the information compare to the numbers gotten in a)?

Math 152 Problems, Week 8

Calculators and Series:* The TI-82 has several built-in functions that are useful for these problems. You should read Chapter 11, *Lists*, of the Texas Instruments *Guidebook* for the details.

To use these functions to calculate a partial sum such as $\sum_{N=4}^{10} 1/N^3$, press the 2nd key LIST (the alternate to the STAT key) and move the cursor to **Seq**. When you press ENTER you will see `seq(` . You must fill in the arguments in the `seq` function. For this example you would enter

$$\texttt{seq}(1/N^3, N, 4, 10, 1)$$

Here N is the ordinary *alphabetic variable*, not the special n on the shift-9 key. Now press STO L_1 to store this sequence as the list named L_1 (the special List variables L_1, L_2 ... are the 2nd keys of 1, 2, ...). You can view the list by pressing L_1. You can calculate the sum of all the elements in the list L_1 (which is the desired partial sum) by pressing 2nd LIST, moving the cursor to MATH, selecting `sum` and then pressing L_1. You can generate and store up to 6 lists this way, each with up to 99 items. You can also do arithmetic on lists. For example, if you have two series with terms a_n and b_n stored as lists L_1 and L_2, then the expression $3L_1 + 4L_2^5$ will give the list with terms $3a_n + 4b_n^5$.

Some Warm-Up Exercises:

A Create and store a list L_1 whose entries are $\{1, 2, 3, \ldots, 20\}$. Now use operations on this list to calculate $\sum_{n=1}^{20} n$

B Operate on the list L_1 to create a list L_2 whose entries are $\{1^2, 2^2, 3^2, \ldots, 20^2\}$. Now use operations on this list to calculate $\sum_{n=1}^{20} n^2$.

C Now use the formulas on p. 324 of the text to check your answers to **A** and **B**.

D Create and store a list L_3 whose entries are $\{1, 1/2, 1/3, \ldots, 1/20\}$. Now use operations on this list to calculate $\sum_{n=1}^{20} n^{-1}$

E Use operations on the lists L_1, L_2 and L_3 to find $\sum_{n=1}^{20}(23n - 13n^2 + 5n^{-1})$

After you've practiced with A, B, C and D, you're ready for some problems on infinite series.

OVER

* What follows is for the TI-82. Other calculators should have equivalent capabilities. Check the manuals.

1. Let $a_n = 5^{2n}/7^{3n+1}$ for $n = 0, 1, 2, \ldots$ and let $s_k = \sum_{n=0}^{k} a_n$ be the kth partial sum of the series with terms a_n.

a) Write out the expressions for s_1, s_2, and s_3 as sums of rational numbers *without* using your calculator. You do not have to carry out the arithmetic.

b) Create a list L_4 whose entries are $\{a_0, a_1, a_2, \ldots, a_{20}\}$.

c) Find the numerical value of $\sum_{n=0}^{20} a_n$.

d) Use the formula for the sum of a geometric series to find $\sum_{n=0}^{\infty} a_n$. Compare with the numerical value of the partial sum in c).

2. If $f(x)$ is a positive, decreasing function defined for $N \le x < \infty$, then

$$(*) \qquad \int_{N+1}^{\infty} f(x)\,dx \le \sum_{n=N+1}^{\infty} f(n) \le \int_{N}^{\infty} f(x)\,dx.$$

(see Figures 1 and 2 on p. 599 of the text). These inequalities prove the **Integral Test**:

The series $\sum_{n=N}^{\infty} f(n)$ converges if and only if the improper integral $\int_N^{\infty} f(x)\,dx$ converges.

Use the integral test to determine which of the following series converge and which diverge. For the series that converge, calculate the sum of the terms up to $n = 10$ (see calculator instructions above) and obtain upper and lower bounds for the remainder R_{10} by using $(*)$ with $N = 10$.

a) $\sum_{n=1}^{\infty} n\, e^{-n^2}$ b) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ c) $\sum_{k=1}^{\infty} \frac{k}{(1+k^2)^3}$

Math 152 Problems, Week 9

Remarks on Calculators and Series: Graphing calculators such as the TI-82 have several built-in functions that make it easy to generate a list of terms in a series and to evaluate partial sums of series. The workshop sheet for week 8 described these functions briefly and had some practice exercises. Refer to that sheet and the instruction manual for your calculator to refresh your memory if necessary.

In this workshop you will analyze two aspects of an infinite series $\sum_{n=0}^{\infty} a_n$:

(1) Determine if the series converges.

(2) If the series converges, find a good numerical approximation to $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$

For answering question (1) first check whether $\lim_{n\to\infty} a_n = 0$. If the answer is 'NO', then the series diverges. If the answer is 'YES', you will need to use one of the 'convergence tests', such as the integral test, the comparison test, or the alternating series test. After you know a series converges, you can use your calculator and the information from the convergence tests to answer question (2)

1. Use the comparison test or limit comparison test to determine which of the following series converge and which diverge. In the series d) the parameter $p > 0$ and you should determine for what values of p (if any) the series converges.

a) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin(n)}{1+n^2}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4+1}}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n^4+1)^{1/2}}{(n^2+1)^p}$

2. For each of the following series, determine whether the three conditions of the alternating series test are satisfied. If the answer is 'YES', calculate the sum of the terms up to $n = 20$ and estimate the 'tail' of the series $\sum_{n=21}^{\infty} a_n$.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^3+4)^2}$

b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n+\ln n}$

c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$

3. Use the remainder estimate for an alternating series to answer the following:

a) We will show soon that $\ln 2 = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. How many terms in this series do you need to add in order to determine the first and second digits in the decimal expansion of $\ln 2$?

b) We will show soon that $\dfrac{1}{e} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. If you add up the same number of terms of this series as you did in part a), how close will you be to the true value?
(*Note:* Your calculator can't do this problem!)

Math 152 Problems, Week 10

1. In the following series x is a real number. In each case use the ratio test to determine all values of x for which the series converges absolutely.

$$\sum_{n=1}^{\infty} nx^{2n} \qquad \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \sum_{n=1}^{\infty} \frac{x^n}{(n!)^2} \qquad \sum_{n=1}^{\infty} n!x^n$$

2. Manipulate the formula

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \cdots \qquad \text{(valid for } |r| < 1)$$

for the sum of a geometric series to express each of the following functions as a power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

Determine the coefficients a_n and the range of x values for which the series converges in each case.

a) $f(x) = \dfrac{x}{1-x}$

b) $g(x) = \dfrac{2}{3x+4}$

c) $h(x) = \dfrac{1}{x^4+16}$

3. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!(n+1)!} = 1 - \frac{1}{1!2!}x^2 + \frac{1}{2!3!}x^4 - \frac{1}{3!4!}x^6 + \cdots$$

(remember that $0! = 1$).

a) Use the ratio test to show that the power series converges for all values of x.

b) If N is an integer, we write $s_N(x)$ for the partial sum of the terms of this series up to and including x^N. For example, $s_0(x) = 1$, $s_1(x) = 1$ (there isn't any term with x) and $s_2(x) = 1 - x^2/2$. Plot $s_2(x)$, $s_4(x)$ and $s_6(x)$ on the same axes in the window $-3 \le x \le 3$ and $-.5 \le y \le 1.5$. (*Calculator Hint:* Key in $s_4(x)$ by adding one term to $s_2(x)$ and likewise key in $s_6(x)$ by adding one term to $s_4(x)$.)

c) Notice in the plots in b) that the three curves are quite different when $|x| > 1$, but are quite close when $|x| \le 1$. Change the window to $-1 \le x \le 1$ and $.5 \le y \le 1.5$ to verify this property.

d) Suppose you use the partial sum $s_6(x)$ to approximate the infinite series over the range $-.5 \le x \le .5$. Use the alternating series error formula to obtain an upper bound for the error in your approximation. Your answer should be a single number that applies to all x values in the range $-.5 \le x \le .5$.

Math 152 Problems, Week 11

1. Use the ratio test or root test to find the radius of convergence of each of the following power series. Also test the endpoints to find the interval of convergence for each series.

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}(3x)^n \qquad \sum_{n=1}^{\infty} \frac{n^3}{(2n+1)!}(x-1)^n \qquad \sum_{n=1}^{\infty} \sqrt{n}(3x+2)^n$$

2. Define the *Rutgers function* $R(x) = 1766 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = 1766 + \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

a) Find the radius of convergence of this series.

b) Differentiate the series for $R(x)$ term-by-term to show that $R'(x) = R(x) - 1765$.

c) Solve the differential equation $y' = y - 1765$ with the initial condition $y(0) = 1766$. Use this to express $R(x)$ in terms of e^x.

3. In this problem, $f(x) = \dfrac{x^2}{16 + x^4}$

a) Use the formula for the sum of a geometric series to express $f(x)$ as a power series in x. What is the radius of convergence of this series?

b) Integrate the power series in a) term-by-term to find a power series for the indefinite integral $\displaystyle\int f(x)\,dx$

c) Use the series in b) to express $\displaystyle\int_0^1 \frac{x^2\,dx}{16 + x^4}$ as a convergent infinite series.

d) Use the alternating series remainder estimate to determine how many terms of the series in c) are needed to approximate the definite integral to 5 decimal places (this means that the error must be less than 5×10^{-6}). Then calculate the partial sum of the series with this number of terms, and compare with the approximate value from the `fnint(` program.

4. Define a function $f(x)$ by the power series

$$\sum_{n=0}^{\infty} (n+2)(n+1)x^n = 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots$$

a) Find the radius of convergence of this series.

b) Define $g(x) = \displaystyle\int_0^x f(t)\,dt$, where $f(x)$ is the function defined above. Use term-by-term integration to find the power series for $g(x)$.

c) Let $h(x) = \displaystyle\int_0^x g(t)\,dt$, where $g(x)$ is the function defined in b). Use term-by-term integration to find the power series for $h(x)$.

d) Use the formula for the sum of a geometric series to show that $h(x) = 1/(1-x)$. Now differentiate $h(x)$ twice to find an expression for $f(x)$ as a rational function.

Math 152 Problems, Week 13

In combinatorics, probability, statistics and theoretical computer science, the idea of a *GENERATING FUNCTION* is very important. Here's the definition:

A function $F(x)$ is the *GENERATING FUNCTION* for a sequence $\{a_n\}_{n=0}^{\infty}$ if

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

1. Find the generating functions for the following sequences.

a) $\{1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{n!}, \ldots\}$

b) $\{1, 1, 1, \ldots$ (all 1's)$\}$

c) $\{1, 3, 3, 1, 0, \ldots$ (all the rest 0's) $\}$

2. Suppose we know the generating function $F(x)$ for some sequence $\{a_n\}_{n=0}^{\infty}$.

a) Show that $xF'(x)$ is the generating function for $\{na_n\}_{n=1}^{\infty}$. (*Hint:* Differentiate the series for $F(x)$ term-by-term and then multiply by x.)

b) Suppose $a_0 = 0$. Show that $\int_0^x \frac{F(t)}{t}\, dt$ is the generating function for $\{\frac{1}{n} a_n\}_{n=1}^{\infty}$. (*Hint:* Integrate the series for $F(t)/t$ term-by-term.)

3. Below is a list of sequences and a list of functions. Match the sequences with their generating functions. Use the Taylor series for the functions that you know and the results of problems #1 and #2.

Sequences :	Functions :
a) $\{0, 1, 2, 3, \ldots, n, \ldots\}$	α) $\frac{x + x^2}{(1-x)^3}$
b) $\{1, 2, 4, 8, \ldots, 2^n, \ldots\}$	β) $\ln\left(\frac{1}{1-x}\right)$
c) $\{0, 1, 4, 9, \ldots, n^2, \ldots\}$	γ) $\frac{1}{1-2x}$
d) $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$	δ) $\int_0^x \ln\left(\frac{1}{1-t}\right) \frac{dt}{t}$
e) $\{0, 1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\}$	ϵ) $\frac{x}{(1-x)^2}$

(OVER)

Remarks on Graphing Curves in Parametric Form: To plot the curve with parametric equations $x = f(t)$ and $y = g(t)$, press MODE key, move cursor down to fourth line, and over to `Par`. Now press $Y =$ key and insert the parametric formulas $X_{1T} = f(T)$ and $Y_{1T} = g(T)$. Then press WINDOW key and set the range for the parameter T and the T_{step} size. (Typically the step size can be .1, but you should experiment to see what happens when you change the step size.) You must also set the $X - Y$ window size as usual. Then press GRAPH. The TRACE command is very useful here, since it displays the T values as well as the $X - Y$ values.

4. Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$, where $c > 0$ is a constant.

(a) Graph the curves for $c = 1/4$, $c = 1$ and $c = 4$. What features do all the curves have in common? (You may need to adjust the graphing window as you change c.) How does the shape change as c increases?

(b) How many tangent lines does the curve have at the point $(c, 0)$? What are their slopes? Check your answer (for $c = 1/4$, $c = 1$ and $c = 4$) by drawing the tangent lines on your graphing calculator. Press `TRACE`, move the cursor to the desired spot on the curve, press the `DRAW` key, select `Tangent(` and then press `ENTER`. You can also use the `CALC` key and select dy/dx (press `ENTER` twice to get the numerical value for the slope dispayed on the screen.

5. A point P travels on the parabola $y = x^2$.

(a) Give parametric formulas for the location of P. Take the parameter to be the angle θ that the ray from the origin to P makes with the positive x-axis. The parametric formulas will have to be given "piecewise".

(b) Give parametric formulas for the location of P. Take the parameter to be the distance r from the origin to P. The parametric formulas again will have to be given "piecewise".