640:251:01-03 **Answers to Exam # 1** **February 28, 1997**

(15) 1. For what values of t are $\mathbf{u} = t\,\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = t\,\mathbf{i} + t\,\mathbf{j} - 2\,\mathbf{k}$ perpendicular? Are there values of t for which $\mathbf{u}$ and $\mathbf{v}$ are parallel?

Answer To test whether two vectors are perpendicular, try their dot product. So $\mathbf{u}\cdot\mathbf{v} = t^2 - t - 2 = (t-2)(t+1)$. The only values of t for which this is 0 are $t = 2$ and $t = -1$, so those are the values for which the vectors are perpendicular.

We can check if these vectors are parallel in several ways. We could take their cross product, since these vectors are never 0 and therefore their cross product will only be 0 if the vectors are parallel. Their cross product is $\langle 2-t, 3t, t^2+t\rangle$. The first component is 0 only when $t = 2$, and the second component *isn't* 0 for that value of t. Or we could argue "directly": if $\mathbf{u} = S\mathbf{v}$ for some scalar S, then (looking at the first components of that vector equation) we know $t = St$. Therefore either $t = 0$ or $S = 1$. The second component of the vector equation is $-1 = St$. Therefore t *can't* be 0 and we must choose S to be 1 to make the first equation true. But $-1 = (1)t$ gives us $t = -1$, and if we now look at the last component of the vector equation we see $1 = -2S$ which is certainly not true when $S = 1$. Wow: the logic is a bit involved, and maybe looking at the cross product is easier. Looking at this last component first may make the S approach easier: we see immediately that if an S exists, then S must be $-\frac{1}{2}$. We then still must analyze whether any t's will work. Note that checking when the *sum* of the components of $\mathbf{u}\times\mathbf{v}$ is zero doesn't tell us when the vector is 0. That was a common error.

(15) 2. Suppose Q is the point $(2,1,3)$ and P is the plane defined by the equation

$$3x - y + z = 2$$

a) Find parametric equations for the line through Q perpendicular to P.

Answer This is easy, if you remember that the coefficients of the variables in P provide the components of a vector perpendicular to P. You can just write the answer with no computation:

$$\begin{cases} x = 2 + 3t \\ y = 1 + (-1)t \\ z = 3 + 1t \end{cases}$$

b) Find the coordinates of the point where the line found in a) intersects P.

Answer Substitute the right-hand side of the equations above in the equation for P and solve for t: $3(2+3t) - (1 + (-1)t) + (3 + 1t) = 2$ becomes $6 + 9t - 1 + t + 3 + t = 2$ which becomes $11t = -6$ and finally $t = -\frac{6}{11}$.

Now insert this value of t in the equations to get $x = 2 + 3(-\frac{6}{11}) = \frac{4}{11}$ and $y = 1 + (-1)(-\frac{6}{11}) = \frac{17}{11}$ and $z = 3 + 1(-\frac{6}{11}) = \frac{27}{11}$. So $(\frac{4}{11}, \frac{17}{11}, \frac{27}{11})$ should be the answer.

You can substitute these values of x and y and z in the original equation for P and check that the point is on the plane: $3(\frac{4}{11}) - \frac{17}{11} + \frac{27}{11} = \frac{22}{11} = 2$.

(15) 3. Suppose $F(x,y,z) = \dfrac{x}{y} + \dfrac{y^2}{z+1}$.

a) What is the maximum rate of change of F at the point $(2,-2,1)$? What is a unit vector in the direction of this maximum rate of change?

Answer I hope this is an expected question on gradient. We need some partial derivatives:
$F_x = \dfrac{1}{y}$ and $F_y = -\dfrac{x}{y^2} + \dfrac{2y}{z+1}$ and $F_z = -\dfrac{y^2}{(z+1)^2}$. Therefore at the point $(2,-2,1)$, the gradient vector vector will be $\nabla F(2,-2,1) = \langle F_x(2,-2,1), F_y(2,-2,1), F_z(2,-2,1)\rangle =$
$\langle \frac{1}{(-2)}, -\frac{2}{(-2)^2} + \frac{2(-2)}{1+1}, -\frac{(-2)^2}{(1+1)^2}\rangle = \langle -\frac{1}{2}, -\frac{5}{2}, -1\rangle$.

The magnitude of $\nabla F(2,-2,1)$, which is $\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{5}{2}\right)^2 + (-1)^2}$, is the maximum rate of change. I hope that you did nothing further to this answer!

The desired unit vector is $\frac{\nabla F(2,-2,1)}{|\nabla F(2,-2,1)|} = \frac{\langle -\frac{1}{2}, -\frac{5}{2}, -1\rangle}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{5}{2}\right)^2 + (-1)^2}}$. Again, I hope that you did nothing further to this answer!

b) Find the equation of a plane tangent to the surface $\frac{x}{y} + \frac{y^2}{z+1} = 1$ at the point $(2,-2,1)$.

Answer The gradient vector is perpendicular to level surfaces. So $\nabla F(2,-2,1)$ is normal to the plane mentioned, which also goes through the point $(2,-2,1)$. So we now have enough information to just write an equation for the plane: $-\frac{1}{2}(x-2) + -\frac{5}{2}(y-(-2)) + -1(z-1) = 0$.
You need *not* "simplify" this equation any further!

(18) 4. Suppose $Q(x,y)$ is defined by the equation

$$Q(x,y) = e^{xf(y)}$$

where f is a differentiable function of one variable with

$$f(0) = A \qquad f'(0) = B \qquad f''(0) = C$$

Use this information to compute the following six quantities:

- $Q(0,0)$ **Answer** $Q(0,0) = e^{0f(0)} = e^0 = 1$.
- $\frac{\partial Q}{\partial x}(0,0)$ **Answer** $Q_x = e^{xf(y)}f(y)$ so $Q_x(0,0) = e^0 f(0) = A$.
- $\frac{\partial Q}{\partial y}(0,0)$ **Answer** $Q_y = e^{xf(y)}xf'(y)$ so $Q_y(0,0) = e^0 0 = 0$.
- $\frac{\partial^2 Q}{\partial x^2}(0,0)$ **Answer** $Q_{xx} = e^{xf(y)}(f(y))^2$ so $Q_{xx}(0,0) = e^0(f(0))^2 = A^2$.
- $\frac{\partial^2 Q}{\partial x \partial y}(0,0)$ **Answer** $Q_{xy} = e^{xf(y)}xf'(y)f(y) + e^{xf(y)}f'(y)$ so $Q_{xy}(0,0) = e^0 0 + e^0 B = B$.
- $\frac{\partial^2 Q}{\partial y^2}(0,0)$ **Answer** $Q_{yy} = e^{xf(y)}(xf'(y))^2 + e^{xf(y)}xf''(y)$ so $Q_{yy}(0,0) = e^0 0 + e^0 0 = 0$.

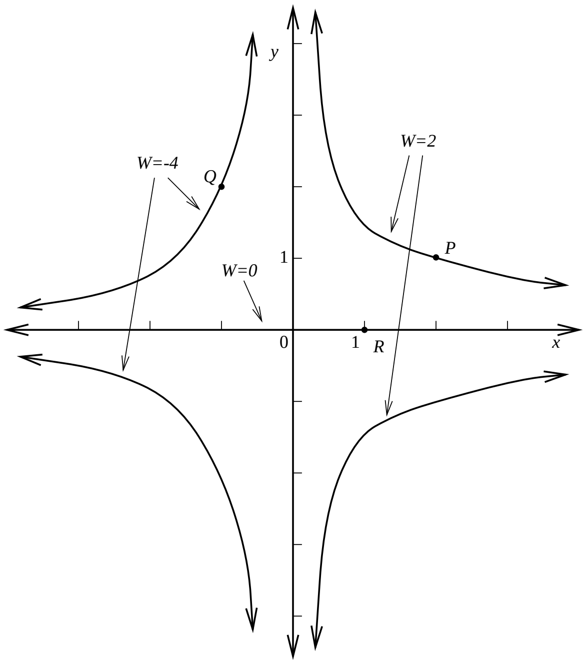
Note: in these computations you must use the chain rule repeatedly, and in a few of them, the product rule is necessary.

(15) 5. Sketch the three level curves of the function

$$W(x,y) = xy^2$$

which pass through the points $P = (2,1)$ and $Q = (-1,2)$ and $R = (1,0)$. Be sure to label each curve with the appropriate function value and be sure that your drawing is clear and unambiguous.

Answer Since $W(2,1) = 2$ we need to sketch the level curve $xy^2 = 2$. If you want y as a function of x then $y = \pm\frac{\sqrt{2}}{\sqrt{x}}$. The "$\pm$" is needed: the level curve has two pieces.
Similarly, $W(-1,2) = -4$, so the curve is $xy^2 = -4$ and $y = \pm\frac{4}{\sqrt{-x}}$. Again the level curve has two pieces.
For R we know $W(1,0) = 0$ and the level curve is $xy^2 = 0$. Any point with either x or y equal to 0 is on this curve: it is the union of the two axes. This curve has one piece, but it has a crossing point.



(12) 6. A particle's acceleration (acceleration is the second derivative of position) is given by

$$\mathbf{a}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2\,\mathbf{k}$$

a) Suppose that its position at time 0 is $(1, 0, 1)$ and its velocity (the first derivative of position) at time 0 is $\langle 0, 2, 0\rangle$. What is its position at time t?

Answer How to do it: integrate, check initial conditions, and then repeat. Here we go!

The first integration gives $\mathbf{r}'(t) = \langle \sin t + C_1, -\cos t + C_2, 2t + C_3\rangle$. When t=0, the given $\langle 0, 2, 0\rangle$ should agree with $\mathbf{r}'(0) = \langle C_1, -1 + C_2, C_3\rangle$ so $C_1 = 0$ and $C_2 = 3$ and $C_3 = 0$. Therefore $\mathbf{r}'(t) = \langle \sin t, -\cos t + 3, 2t\rangle$.

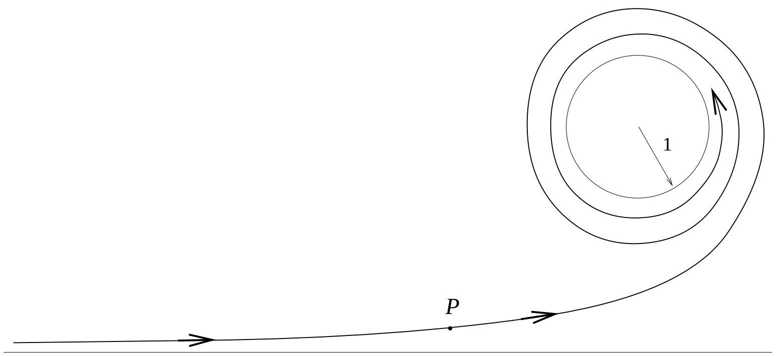
Integrating again we find $\mathbf{r}(t) = \langle -\cos t + C_4, -\sin t + 3t + C_5, t^2 + C_6\rangle$. Matching two versions of $\mathbf{r}(0)$: $(1, 0, 1)$ should agree with $(-1 + C_4, C_5, C_6)$. So $C_4 = 2$ and $C_5 = 0$ and $C_6 = 1$.

Therefore the position of the particle at time t is $(-\cos t + 2, -\sin t + 3t, t^2 + 1)$.

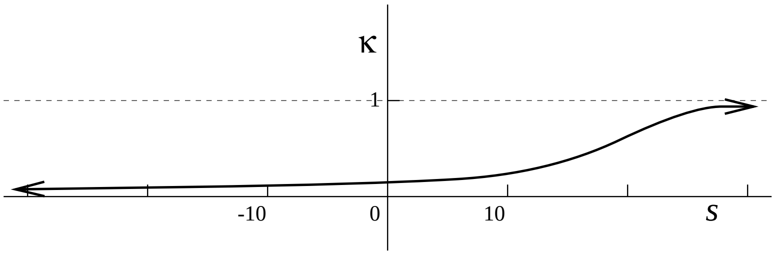
b) Write an integral for the distance the particle travels between $t = 0$ and $t = 2\pi$.
DO NOT EVALUATE THE INTEGRAL.

Answer The distance the particle travels is the integral of the length of the velocity vector over the indicated interval (distance = speed $\cdot$ time). Therefore the answer is
$\displaystyle\int_0^{2\pi} \sqrt{(\sin t)^2 + (-\cos t + 3)^2 + (2t)^2}\,dt.$

(10) 7. A point is moving along the curve below in the direction indicated. Its motion is parameterized by arc length, s, so that it is moving at unit speed. Arc length is measured from the point P (both backward and forward). The curve is intended to continue indefinitely both forward and backward in s, with its forward motion curling more and more tightly about the indicated circle, and, backward, closer to the indicated line.



Sketch as well as you can on the axes below a graph of the curvature, κ, as a function of the arc length, s. What are $\lim_{s\to+\infty} \kappa(s)$ and $\lim_{s\to-\infty} \kappa(s)$? Use complete English sentences to briefly explain the numbers you give.



Note the differing vertical & horizontal scales!

Answer $\lim_{s\to+\infty} \kappa(s) = 1$ and $\lim_{s\to-\infty} \kappa(s) = 0$. The curve gets flatter and flatter as the point travels left ($s \to -\infty$), because the curve is getting closer to the straight line, which has curvature 0. As s increases, the curvature seems to be steadily increasing. The path curls more and more tightly around the circle which has radius 1 and therefore curvature 1 (the curvature of a circle is the reciprocal of its radius).