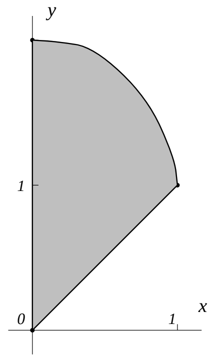
640:251:01-03 **Answers to Exam # 2** **April 15, 1997**

(14) 1. Find the local maximum and minimum values and saddle points of the function.

$$f(x,y) = x^3 - 6xy - 8y^3$$

Answer $\begin{cases} f_x = 3x^2 - 6y \\ f_y = -6x - 24y^2 \end{cases}$ and we want critical points, which are points (x,y) where both $f_x(x,y) = 0$ and $f_y(x,y) = 0$. The first function is 0 when $6y = 3x^2$ or $y = \frac{1}{2}x^2$, which can be substituted into $-6x - 24y^2 = 0$ to get $-6x - 6x^4 = 0$ so that $x + x^4 = 0$ at a critical point. Since $x + x^4 = x(1+x^3)$, critical points must occur when $x = 0$ or when $x = -1$. If $x = 0$ then $y = 0$. If $x = -1$ then $y = \frac{1}{2}$. We compute the second derivatives now so that we can "classify" the critical points: $f_{xx} = 6x$ $f_{xy} = -6$ $f_{yx} = -6$ $f_{yy} = -48y$.
I computed all four derivatives in order to have a quick check on my differentiation (since f_{xy} and f_{yx} should agree). We need $H = f_{xx}f_{yy} - (f_{xy})^2 = (6x)(-48y) - (-6)^2$. At $(0,0)$, $H = -36 < 0$, so $(0,0)$ is a saddle point. At $(-1, \frac{1}{2})$, $H = 6(48) - 36 > 0$ (I don't need the exact value, only the sign!). So $(-1, \frac{1}{2})$ is a local extremum. But $f_{xx} = 6(-1) < 0$ there, so $(-1, \frac{1}{2})$ is a local maximum. And $f(-1, \frac{1}{2}) = (-1)^3 - 6(-1)\left(\frac{1}{2}\right) - 8\left(\frac{1}{2}\right)^3 = 1$, the local maximum value.

(14) 2. Suppose R is the region shown on the right.



- The boundary from $(0,0)$ to $(1,1)$ is part of the straight line $y = x$.
- The boundary from $(0,0)$ to $(0,2)$ is part of the y-axis.
- The boundary from $(0,2)$ to $(1,1)$ is part of the parabola $y = 2 - x^2$.

Suppose $H(x,y)$ is an unknown function, and that I is the double integral defined by

$$I = \iint_R H(x,y)\,dA$$

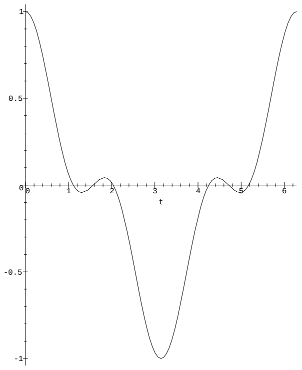
a) Write I as a sum of one or more iterated integrals in $dy\,dx$ order. **Answer** $\displaystyle\int_0^1 \int_x^{2-x^2} H(x,y)\,dy\,dx$.

b) Write I as a sum of one or more iterated integrals in $dx\,dy$ order. **Answer** $\displaystyle\int_0^1 \int_0^y H(x,y)\,dy\,dx + \int_1^2 \int_0^{\sqrt{2-y}} H(x,y)\,dy\,dx$. Of course it is necessary to "solve" for x in terms of y in the equation $y = 2 - x^2$. Since this is a calculus exam, the equation has a simple solution.

(14) 3. The function $Q(x,y) = x^3 - xy^2$ attains a maximum value, M, when $x^2 + 4y^2 = 1$, and also attains a minimum value, m, when $x^2 + 4y^2 = 1$.
Find M and m. Explain your answer; be very careful with your analysis, please.
Answer We'll use Lagrange multipliers. Here $\begin{cases} Q_x = 3x^2 - y^2 \\ Q_y = -2xy \end{cases}$ and the constraint $C(x,y) = x^2 + 4y^2 - 1$ gives us the following collection of equations to solve: **LM** $\begin{cases} 3x^2 - y^2 = \lambda(2x) \\ -2xy = \lambda(8y) \\ x^2 + 4y^2 = 1 \end{cases}$. Extreme values will be found among the values of the solutions to this system of equations. Let's try to discover these solutions.
Consider the second equation. If $y = 0$, the second equation is always true. Then the third equation says that $x = \pm 1$. So $(\pm 1, 0)$ are two points to check for extreme values. For these points, Q is easy to compute: $Q(\pm 1, 0) = \pm 1$.
If $y \neq 0$, then we can divide the second equation by y and get $-2x = 8\lambda$ so $-\frac{1}{4}x = \lambda$. Substitute into the first equation: $3x^2 - y^2 = -\frac{1}{2}x^2$. Therefore $y^2 = \frac{7}{2}x^2$ and stuffing this into the constraint equation gives

$x^2+14x^2=1$ so that $15x^2=1$ and $x=\pm\frac{1}{\sqrt{15}}$. Notice that $Q(x,y)=x^3-xy^2=x(x^2-y^2)$ so we only need x and x^2 and y^2: but $y^2=\frac{7}{2}x^2$. Therefore when $x=\pm\frac{1}{\sqrt{15}}$, Q is $\pm\frac{1}{\sqrt{15}}\left(\left(\frac{1}{15}\right)-\left(\frac{7}{2}\right)\left(\frac{1}{15}\right)\right)=\pm\frac{1}{\sqrt{15}}\left(\frac{1}{6}\right)$. Notice that all this computation has found "solutions" which can now be discarded, since $-1<-\frac{1}{6\sqrt{15}}<\frac{1}{6\sqrt{15}}<1$! The extreme values were already detected: $m=-1$ and $M=+1$. Of course, we couldn't be sure of that until we checked every solution of the system **LM**.

Comment I was a bit startled by the answer here. Therefore I checked it using an alternative method with `Maple`'s help: I parameterized the constraint using $\begin{cases} x=\cos t \\ y=\frac{1}{2}\sin t\end{cases}$ and then substituted into Q to get a function of t alone. I had `Maple` graph this function for $0\le t\le 2\pi$, which would show Q's values on the whole ellipse. The picture shown was produced (this is, of course, with the `Unconstrained` option, so "the screen" is filled as much as possible). Note the other four bumps in the graph (besides the absolute max and min) which are exactly what the Lagrange multiplier method located.



Here are the `Maple` commands that I used, with some explanations.

`>x:=cos(t);y:=(1/2)*sint(t);` These equations establish the parameterization of the ellipse.

`>simplify(x^2+4*y^2);` The answer (which is 1) should convince you that I've described the correct ellipse.

`>z:=x^3-x*y^2;` Here we write $Q(x,y)$ over the ellipse indirectly as a function of t. The next command creates the picture displayed here.

`>plot(z,t=0..2*Pi);`

`Maple` offers some additional ways to see what's going on in three dimensions.

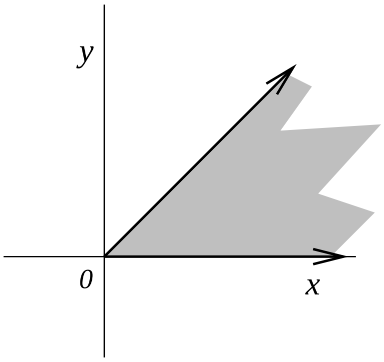
`>with(plots):spacecurve({[x,y,0],[x,y,z]},t=0..2*Pi);` This shows the ellipse and the part of the graph of Q sitting right over the ellipse – another curve. It is an interesting picture. If you still are not satisfied, you can also look at the "whole" surface.

`>plot3d(a^3-a*b^2,a=-1.1..1.1,b=-.6...6);` You'll see a piece of the graph of Q sitting over a rectangle containing the ellipse. I used `a` and `b` because the standard variables `x` and `y` are already taken. I didn't find this picture as useful as the others.

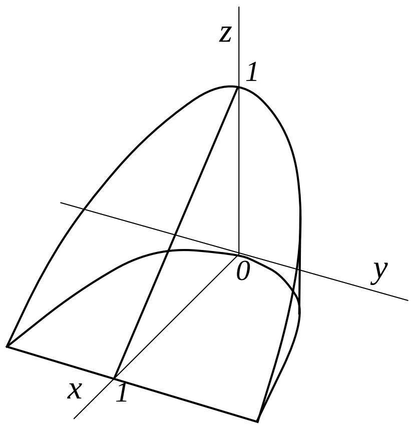
`>implicitplot3d({a^2+4*b^2=1,c=a^3-a*b^2},a=-1.5..1.5,b=-1..1,c=-2..2);` This picture shows the cylinder over the ellipse and the graph of Q. I liked it. I recommend the choices `Constrained` and `Framed` and `Patch w/o Grid` and `XY` color.

(12) 4. Suppose W is the region in the upper half-plane (where $y\ge 0$) between the lines $y=x$ and the x-axis. Compute $\displaystyle\iint_W \frac{1}{(1+x^2+y^2)^2}\,dA$.

Answer The region and the integrand can both be easily described in polar coordinates. First, the region: W has $0\le r<\infty$ and $0\le\theta\le\frac{\pi}{4}$. (A simple sketch may help you see the bounds on r and θ for the region.) We recognize that $x^2+y^2=r^2$ so the integrand becomes $\dfrac{1}{(1+r^2)^2}$. Therefore we need to compute $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\infty}\frac{1}{(1+r^2)^2}\,r\,dr\,d\theta$. The inner integral is improper so it should be computed as a limit. First, suppose A is large and positive: $\displaystyle\int_0^A\frac{1}{(1+r^2)^2}\,r\,dr=-\frac{1}{2}\cdot\frac{1}{1+r^2}\Big]_0^A=-\frac{1}{2}\cdot\frac{1}{1+A^2}+\frac{1}{2}$. As $A\to\infty$, the value of this integral $\to\frac{1}{2}$. So we only need to compute $\displaystyle\int_0^{\frac{\pi}{4}}\frac{1}{2}\,d\theta$ which is $\dfrac{\pi}{8}$, and that's the answer desired.



(15) 5. Use a triple integral to find the volume of the solid bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.



Answer Here's my sketch of the volume described by the given equations. $z = 0$ is the "floor", and $x + z = 1$ is a tilted "top", so that the height is given by $z = 1 - x$. The solid is sitting over a region in the xy-plane. The region is bounded by the line $x = 1$ and an arc of the parabola $x = y^2$. I'll do the integral in $dz\,dy\,dx$ which seems easiest to me. But actually I think any of the six rectangular orders wouldn't be too difficult. To get the volume, the integrand should just be 1. The triple integral is $\displaystyle\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} 1\,dz\,dy\,dx$.

Let's compute it. The inside integral is easy, and gives us $1 - x$. Then we've got $\displaystyle\int_{-\sqrt{x}}^{\sqrt{x}} 1-x\,dy = (1-x)y\Big]_{-\sqrt{x}}^{\sqrt{x}} = (1-x)\sqrt{x} - (1-x)(-\sqrt{x}) = 2(1-x)\sqrt{x}$. There's only one more integral left (stay calm!):

$$\int_0^1 2(1-x)\sqrt{x}\,dx = 2\int_0^1 \sqrt{x} - x^{3/2}\,dx = 2\left(\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right)\Bigg]_0^1 = 2\left(\frac{2}{3} - \frac{2}{5}\right) = \frac{8}{15}.$$

(14) 6. Evaluate the line integral, where C is the given curve.

$$\int_C xy\,dx + (x-y)\,dy,\ C \text{ consists of the line segments from } (0,0) \text{ to } (2,0) \text{ and from } (2,0) \text{ to } (3,2).$$

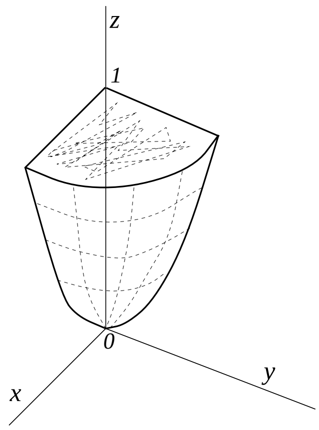
Answer Parameterize the two line segments separately. That is, describe C as $C_1 + C_2$, where C_1 is the line segment from $(0,0)$ to $(2,0)$ and C_2 is the line segment from $(2,0)$ to $(3,2)$.

The C_1 integral: here use $\begin{cases} x = t \\ y = 0 \end{cases}$ with $0 \le t \le 2$ so that $\begin{cases} dx = dt \\ dy = 0\,dt \end{cases}$ and for this portion of C we get $\displaystyle\int_0^2 t(0)\,dt + (t-0)\,0\,dt = 0$.

The C_2 integral: again I'll use $x = t$ with $2 \le t \le 3$ but then to stay "on" the line I need the slope which is $\dfrac{2-0}{3-2} = 2$ so that $y = 2t +$ something and that "something" should give me $y = 0$ when $t = x = 2$. So the something is -4. Therefore I will parameterize this portion of the line integral by $\begin{cases} x = t \\ y = 2t - 4 \end{cases}$ with $2 \le t \le 3$. Then $\begin{cases} dx = dt \\ dy = 2\,dt \end{cases}$ so that $xy\,dx + (x-y)\,dy = t(2t-4)\,dt + (t-(2t-4))\,2\,dt = 2t^2 - 6t + 8\,dt$. And we need the integral: $\displaystyle\int_2^3 2t^2 - 6t + 8\,dt = \frac{2}{3}t^3 - 3t^2 + 8t\Big]_2^3 = \frac{17}{3}$. (O.k.: I'll admit it: I had `Maple` do that last step. When I worked the exam out by hand I left the answer as $\left(\frac{2}{3}(3)^3 - 3(3)^2 + 8(3)\right) - \left(\frac{2}{3}(2)^3 - 3(2)^2 + 8(2)\right)$ which is what I hope you did!)

(17) 7. Suppose $J = \displaystyle\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{z}} \rho(p)\,r dr\,d\theta\,dz$ (in cylindrical coordinates) where $\rho(p)$ is some unknown density function of the point p in the solid.

a) Sketch the solid region in $\mathbb{R}^3$ over which the integral J is computed, and accompany your sketch with equations describing the boundary surfaces of the region using the standard rectangular coordinates x, y, and z.



Answer Here is an approximate picture of the region. The limits of the integrals provide me with some surfaces: z goes between 0 and 1 and θ varies between 0 and $\frac{\pi}{2}$. Therefore the region is in the first octant: $x \geq 0$, $y \geq 0$, and $z \geq 0$. Probably the most puzzling boundary is given by the limits on r: points with $r = 0$ are on the z-axis, but what is the surface $r = \sqrt{z}$? Of course, $z = r^2$ there, so we have $z = x^2 + y^2$, our old friend the paraboloid, a parabolic cup opening upwards. So the solid region has three flat boundaries whose equations are $x = 0$, $y = 0$, and $z = 1$, and has a curved boundary surface with rectangular equation $z = x^2 + y^2$.

b) Suppose the density $\rho(p)$ at p is directly proportional to the distance between p and the z-axis with constant of proportionality equal to K. Compute J.

Answer The distance from p to the z-axis (in cylindrical coordinates) is just r. So $\rho(p) = Kr$, and we need to compute $\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{z}} Kr\, rdr\; d\theta\; dz = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{z}} Kr^2 dr\; d\theta\; dz = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{z}} \frac{K}{3} r^3 \Big]_0^{\sqrt{z}} d\theta\; dz =$ $\int_0^1 \int_0^{\frac{\pi}{2}} \frac{K}{3} z^{3/2} d\theta\; dz = \frac{K\pi}{6} \int_0^1 z^{3/2}\; dz = \left(\frac{K\pi}{6}\right) \frac{2}{5} z^{5/2}\Big]_0^1 = \frac{K\pi}{15}$.