Mathematics 251:01-03 # Curvature: Lab 1 **Some background**

Years ago, computations of curvature and its companion, torsion, were left to small communities of pure mathematicians and theoretical physicists. The computations were difficult because the formulas for these quantities are very intricate, and the uses of curvature were not easy to describe. In the last decade, `maple` and comparable programs have made curvature computations easier. The concept of curvature and related ideas have found increasing application in fields as diverse as computer graphics, material science, and biochemistry (describing how and why long molecules bend). `maple` is used in this lab to explore the the curvature of several curves.

You'll need both your textbook and a `maple` window, and you will also need some time to explore using `maple` before you actually start the problems below.

Preparation This material is supplied to help you "warm up" for the real computations which follow. Please *don't* hand in the preparation material, but (after you've practiced) do the problems. Remember that `maple` commands must terminate with one of `:` and `;` or else `maple` won't listen to them! For simplicity I've left out these punctuation marks in the following discussion. Also, `help(command_name)` will give you information about the command whose name is `command_name` (I generally look at the examples in the `help` screens first!).

The text has several formulas for curvature (see, for example, pages 716 and 717 and problem 28 on page 720). The formulas involve computations of several derivatives and algebraic manipulations of the derivatives. Since space curves are given by three functions, the computations can be very tedious.

Learn how `maple` computes derivatives. For example, compute the first three derivatives of $x^3 \sin(e^x)$. Call p and q and r the first three derivatives of this function. Compute the quantity $\dfrac{p^2+q}{r}$ and evaluate it when $x = \pi$. Something near .47869 is what I get.

Summary You'll need to use the commands `diff` and and `subs` and `evalf` and to look at `constants` (to see that `maple` believes that `E` is e and to check what `maple` calls π).

We'll also need to do some vector computations. `maple` is a large and complicated program with many ways to do vector computations. You can create a vector with a simple assignment statement such as `v:= [3,2,4]`. `maple` has routines to handle vector manipulation. To save active storage space in computers, these routines are in a package called `linalg`. If you want to use these routines in any `maple` session, you'll need to type the command `with(linalg)` before you use them. Do this. Then you can compute dot products with `dotprod` and cross products with `crossprod`. Take the square root of the dot product of a vector with itself to compute its length (use `sqrt`). The curvature formulas need derivatives of vector functions. `maple` will take derivatives of vector expressions "in parallel" (some computer languages will not allow such "overloading"). Try the following, please. Write `a:= [2*t,t^2,cos(t)]` and follow this with the command `diff(a,t)`. See if you can find the *speed* and *acceleration* of this "particle" when $t = 10$. My answers were approximately 20.107 and $(0, 2, .83907)$. The first quantity is a scalar and the second is a vector. I also computed the cross product of the velocity and the acceleration when $t = 10$: my answer was approximately $(15.693, -1.678, 4)$ (use `subs` before you use `evalf`).

Summary You'll need to use the commands mentioned previously, and these new commands: the command `sqrt` which takes square roots of (scalar) quantities, the package `linalg` (loaded by the command `with(linalg)`) and (in that package) `dotprod` and `crossprod`. Vectors can be added with the `add` command in the `linalg` package. Scalar multiplication can be done with the `scalarmul` command, also in the package.

Everyone makes typing errors and other mistakes while doing more complex calculations. You can always type `restart` to clear `maple` 's memory of past assignments of variables and any other garbage that may have accumulated. You would then need to reload packages like `linalg`, though. Let's try a more intricate example, a "real" curvature of a "real" curve in space. Consider the simple curve $\mathbf{r}(t) = \langle t^4, t^2, 2t+1\rangle$. First, tell `maple` what `r` is. Call the function `r` and *don't* use parentheses. Look at a picture of `r` for $-3 \le t \le 2$. To do this, you'll need to load another package called `plots`. Give `maple` the command `with(plots)`. You now have access to a large number of intricate plotting commands. Type the command `spacecurve(r,t=-3..3)` and look at the result. "Click" on `Projection` and look *very* closely at the difference between `Constrained` and `Unconstrained`. Click on `Axes` (that will help you see what's happening in `Constrained` compared to `Unconstrained`) and try `Boxed` and learn how the mouse buttons and motions can make your point of view change. After you have "played" enough with the picture (that is, gotten comfortable with it and its variations), go on and read about how to compute the curvature of this curve.

I used the formula on page 716 of the text to compute the curvature of **r**. I built up the pieces slowly – step by step. I named the pieces to correspond with their logical places (so I called a cross product of two vectors `cp`). You could call the first derivative of `r` something like `rprime` or `vel` or `vel_of_r` or whatever is comfortable or useful or meaningful to *you* (!). Names can be many letters long. I find that longer names are useful in computations (they help me to avoid getting lost!) but that longer names are more liable to be mistyped. Please compute the curvature and compare your answer to mine:

$$\kappa = 4\frac{\sqrt{1+36t^4+16t^6}}{(16t^6+4t^2+4)^{3/2}}$$

I hope you got something similar. What happens to κ as $t \to \pm\infty$ (just look at the highest degree terms in the top and bottom and make a rough estimate)? I hope you can "see" this flattening occurring in the picture of the curve.

I definitely don't want to compute the curvature of even such a low-degree polynomial curve "by hand".

Summary You'll need to use the commands mentioned previously, and these new commands: the package `plots` (loaded by the command `with(plots)`) and the command `spacecurve` in that package.

Mathematics 251:01-03 **Curvature: Lab 1** **The problems**

Please turn in only the printout of your `maple` worksheet. Include explicit answers to all questions asked, using the **`text`** feature of `maple` to insert them in the worksheet (don't write them by hand). Insert all graphs directly into your worksheet; instructions were given in the initial handout. Remove from the worksheet any extraneous material such as the output of commands like `with(plots);` and any errors you have made.

1. **The curvature of an unknown curve**

Consider the curve which is given parametrically by

$$\begin{cases} x = & \dfrac{1-t^2}{1+t^2} \\ y = & \dfrac{2t}{1+t^2} \end{cases}$$

a) Compute the curvature of this curve. Simplify your final answer. (You may find `maple`'s `simplify` command useful in this computation and in other computations.)

b) You should have gotten a suspiciously simple answer to a). Use `maple` to do enough algebra to verify that the curve is what you think it is.

c) Plot this curve. (You can use either `plot` with the proper syntax or you can use `spacecurve`.)

d) Explain in a sentence or two why all of a) and b) and c) are consistent.

2. **The curvature of an ellipse**

If c and d are positive constants, then the parametric curve $\mathbf{r}(t) = (c\cos t, d\sin t, 0)$ (a curve in the xy-plane) is an ellipse.

a) Compute the curvature, κ, of this ellipse.

b) Use `maple` to show that when $c = d$, your expression for κ can be simplified a great deal. (Use `subs` followed by `simplify`.)

c) For what values of t in $[0, 2\pi)$ is the curvature greatest when $c > d$? What about when $c < d$? (You can find the extreme values of a function of one variable using calculus and `maple` can do the computations for you. Simply `diff`erentiate and then see where the resulting continuous function is 0 by trying to `solve` an appropriate equation. `maple` will do all of this easily except the final step when you will need to compare values. Please include some text describing and explaining your final results.)

d) Suppose n = the number of letters in your first name and m = the number of letters in your last name. Verify that your answers to c) are correct in a special case by plotting the curvature function, κ, on the interval $[0, 2\pi]$ for the choice $c = n$ and $d = m$*. What is the largest value and the smallest value of κ for the choices you have made? Include this graph in your worksheet. Also include a graph of the specific ellipse whose curvature you are examining and indicate by hand on the ellipse where the extreme values of the curvature are attained and the nature of these extreme values (max or min). (Use the `subs` command to specialize your formula for curvature, and then use `plot`.)

3. **The curvature of a space curve**

The two surfaces $x = y^2$ and $z = 2x^2 + y^4$ intersect in a curve. The point $P = (1, 1, 2)$ is on this curve.

a) Plot the two surfaces together near P, showing their intersection. Include this graph in your worksheet. (You'll need to use the command `implicitplot3d` which is in the package `plots`. There are *many* ways to look at the resulting picture. I've gotten something nice by trying $-2 \le x \le 2$ and $-2 \le y \le 2$ and $0 \le z \le 5$ or 6, with the axes included and with contour lines and with greyscale `Z` – but I've also had to play a bit with the orientation of the viewing window to "see" the surfaces and the resulting curve.)

b) Decide how to parameterize the curve. Plot a portion of the curve which contains the point P. Include this graph in your worksheet. (Use `spacecurve`.)

c) Compute the curvature of the curve at P.

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* If $n = m$ (for example, if your name is "Fred Fred") then just take the square root of one of the numbers!