Mathematics 251: Lab 3 MAXIMUM AND MINIMUM VALUES

In this lab, we use `Maple` to help visualize and compute the maximum and minimum values of a function of two variables.

Please turn in only the printout of your `Maple` worksheet. Include explicit answers to all questions asked, using the **`text`** feature of `Maple` to insert them in the worksheet (don't write them by hand). Insert graphs directly into your worksheet; instructions were given in the initial handout. Remove from the worksheet any extraneous material such as the output of commands like **`with(plots)`** and any errors you have made.

1 In this problem we wish to find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = \frac{y^3}{9} + 3x^2y + 9x^2 + y^2 + xy + 9$$

a) First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \le x \le 1.5$, $-7 \le y \le 1$. Use the `axes=BOXED` option so you can see the values of f.

b) Use `Maple`'s `diff` and `solve` commands to find all the critical points of f. This can be done by first assigning names such as `fx` and `fy` to the expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ obtained by using the `diff` command (`fx:=diff(f,x)` and `fy:=diff(f,y)` will do that). Then the `solve` command can be used to obtain the values of x and y for which the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, are satisfied: `solve({fx=0,fy=0},{x,y})`.

c) Again using `Maple`'s `diff` command, evaluate f_{xx} and $D = f_{xx}f_{yy} - (f_{xy})^2$ at each critical point to determine whether it is a local maximum, local minimum, or saddle point. Clearly label in your `Maple` worksheet the type of each critical point.

Note To evaluate an expression such as `fx` at the point $(1, 2)$, the `subs` command can be used: `subs({x=1,y=2},fx)`. Evaluating both `fx` and `fy` at the point $(1, 2)$, can be done by `subs({x=1,y=2},{fx,fy})`. Also, you can avoid retyping solutions gotten from `solve`. Do this by assigning a name to the result: `soln:= solve({fx=0,fy=0},{x,y})`. Also, if there is more than one solution, use the syntax `soln[3]` to refer to the third one. Hence, to evaluate both `fx` and `fy` at `soln[3]`, you can type `subs(soln[3],{fx,fy})`.

d) For each critical point, obtain a plot of f in a small region around the critical point, which clearly shows whether it is a local maximum, a local minimum, or a saddle point. Make sure it is clear which point each plot is for.

2. Now consider the function $f(x,y) = y(1 - 10xy)e^{-x^2-y^2}$. Here we want to find the absolute minimum and absolute maximum of f. (You may want to `restart` because many of the variables you might want to use may already be defined.)

a) First obtain a rough idea of what this function looks like by plotting it over the region $-5 \le x \le 5$, $-5 \le y \le 5$ and then over the region $-2 \le x \le 2$, $-2 \le y \le 2$.

b) What happens if you use `Maple`'s `diff` and `solve` commands to try to find the critical points of f?

c) Now use the `fsolve` command to find the critical points which lie in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$. There are exactly four of them, as indicated by the plot in part a). Also find the values of the function f at all of these critical points; be sure that your worksheet makes it clear which values are obtained at which points. It is **not** necessary to use a second derivative test here or in part d) below.

The `fsolve` command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. If `fx` and `fy` are the names assigned to the expressions obtained for the partial derivatives f_x and f_y, respectively what happens if you use the command in the form `fsolve({fx=0,fy=0},{x,y})`? To improve the chance of convergence, add a region in which you expect the root to lie. You can obtain such a region by using the second of the plots obtained in part a). For example, one of the critical points can be found by typing `p1:= fsolve({fx=0,fy=0},{x,y}, x=-1..0, y=0..1)`. As in Problem 1, if you call `p1` the solution of the `fsolve` command, you can then evaluate $f(x, y)$ at the values of x and y given by `Maple` in response to this command by typing `evalf(subs(p1,f))` assuming that `f` is the name assigned to the expression for the function f. `evalf` will convert the result of the `subs` command to a numerical value.

d) From your answer in c) and the plot in a), determine the absolute minimum and absolute maximum values of the function f.

BONUS (for up to **4** extra points) Try to find the (approximate) maximum and minimum of the function, f, in problem 2 on the boundary of the tilted ellipse, $x^2 + xy + y^2 = 1$.

Here are some suggestions using methods discussed in the lecture given March 4. Before beginning any of them, you probably should check (with `implicitplot` for example) that the solutions to the equation do indeed look like an ellipse.

Method α: `Maple` can solve the equation for y in terms of x (with several solutions), then substitute each of the solutions in $f(x, y)$, getting a function of x alone. Then you can determine the max and min approximately. I had more success graphing the resulting one variable function(s) and looking at the pictures to get numbers than by using `fsolve`.

Method β: There are numbers A and B so that the ellipse is $\begin{cases} x = A\cos t + B\sin t \\ y = A\cos t - B\sin t \end{cases}$ for t in $[0, 2\pi]$. Have `Maple` help you find the correct values of A and B, then substitute these functions into f and extremize the resulting one variable function of t either graphically or numerically.

Method γ: Here's the "real" method for higher dimensions. Just write the Lagrange multiplier equations, solve them easily, and then evaluate the function at these solutions: `Maple` can do all these things. You'll need some skill and persistence with `Maple` to get it to do all these things and yield the answers.

You should work on this extra part of the lab only if you have enough time (and are up to date with the other aspects of the course) and if you want to get more comfortable with `Maple`. I'll be happy to help you with any of the approaches sketched above.