Mathematics 251: Lab 4 TRIPLE INTEGRALS

In this lab, we use Maple to help visualize and compute the value of a triple integral. In particular, we see that choosing the order of integration carefully can simplify the process.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet (don't write them by hand). Insert graphs directly into your worksheet; instructions were given in the initial handout. Remove from the worksheet any extraneous material such as the output of commands like `with(plots);` and any errors you have made.

The basic command for integration in Maple is `int`. To evaluate $\int_0^1 (x+1)\,dx$, we type `int(x+1, x=0..1);`. To evaluate the double integral $\int_0^1 \int_0^{1-x} (xy+x+1)\,dy\,dx$, we type `int(int(x*y + x +1, y=0..1-x),x=0..1);`. To evaluate the triple integral $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (xyz+x+1)\,dz\,dy\,dx$, we type
`int(int(int(x*y*z + x + 1, z=0..1-x-y),y=0..1-x),x=0..1);`

1. If a solid has mass density $\delta = 1$, then its moment of inertia about the z-axis is given by

$$I_z = \int\int\int (x^2+y^2)\,dV.$$

We wish to calculate I_z for a solid S defined to be the region between

$$y = 8 - z^2 - 2x^2 \qquad (1) \qquad \text{a paraboloid}$$

and

$$y = z^2 \qquad (2) \qquad \text{a parabolic cylinder}$$

1a. To see what this region looks like, plot the two surfaces given above on the same set of axes for $-2 \le x \le 2$, $-\sqrt{8} \le z \le \sqrt{8}$.

1b. Since the boundaries of S are given in terms of functions of x and z, it is easiest to calculate I_z by doing the y integration first. What are the upper and lower limits of the y integral?

1c. The next step in finding I_z is to determine the x and z limits of integration. To get these, it may be helpful to view the projection of the intersection of the paraboloid and the parabolic cylinder onto the x-z plane. To get this projection, use the `implicitplot` command, i.e., type `with plots:` followed by

```
implicitplot(8-z^2-2*x^2=z^2,x=-2..2, z=-sqrt(8)..sqrt(8),
scaling=CONSTRAINED);
```

What type of curve is the intersection of the two surfaces?

1d. Now formulate and evaluate the triple integral using the Maple command `int`.

1e. Suppose now we want to compute the same integral by doing the z integration first. It is then useful to vizualize the solid with x and y as independent variables. To do this, solve (by hand computation) equations (1) and (2) for z, assigning the names `z1` and `z2` to the Maple expressions for the positive and negative solutions, respectively, of equation (1) and the names `z3` and `z4` to the Maple expressions for the positive and negative solutions, respectively, of equation (2). These four equations each specify a portion of the boundary of S. Plot these four surfaces on a single set of axes over the region $-2 \leq x \leq 2$, $0 \leq y \leq 8$.

It may be useful to view the plot of part (1e) from several viewpoints to do the remaining parts of this assignment.

1f. The difficult part is now to determine the limits of integration. Note that from your plot, it is clear that for some values of x and y the z values inside S range from the bottom of the paraboloid to the top of the paraboloid, while for other values of x and y, the z values inside S range from the bottom of the parabolic cylinder to the top of the parabolic cylinder. Thus, to evaluate I_z we need to write it as the sum of two integrals I_1 and I_2. Determine (and state) the z limits of integration for each of these two integrals.

1g. We now determine the x and y limits of integration for each of the two integrals in part (1f). First note that if we project the graph of the solid S onto the xy plane, we will obtain a region bounded by the line $y = 0$ and the curve $y = f(x)$. Determine (by hand) $f(x)$ and assign the name `f` to the Maple expression for $f(x)$.

1h. To determine the x and y limits of integration of I_1 and I_2, we next observe that the curve which divides the xy region of integration found in part (1g) into the region used for I_1 and the region used for I_2 is the projection on the xy plane of the intersection of the surfaces $z1$ and $z3$. Determine an equation for this curve in the form $y = g(x)$ and assign the name `g` to the Maple expression for $g(x)$.

1i. To see what the two regions look like, plot the curves $y = 0$, $y = f(x)$, and $y = g(x)$ for $-2 \leq x \leq 2$ on the same set of axes using the `plot` command with the `axes=BOXED` option.

1j. Using these results set up and use Maple to evaluate the sum of the integrals I_1 and I_2. It is easier if you first integrate in z, then in y, and finally in x. As a check, verify that you get the same answer as in part (1d).

BONUS: For (up to) three bonus points, use Maple to compute this integral doing the x integration first (inside). Include all necessary steps (see for example 1e through 1j above) and explain briefly what you are doing at each step.