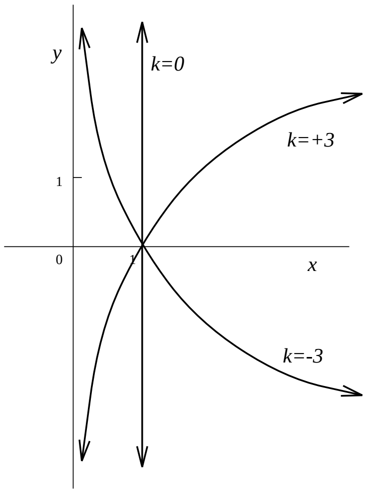
640:251:01-03 answers to the **Diagnostic Exam** **1/22/97**

1. Sketch the curves $x = e^{ky}$ on the same axes for $k = 0$, $k = +3$, and $k = -3$. Be sure to label which value of k corresponds to which curve.

Answer When $\boxed{k = 0}$ the "curve" is just $x = e^{0\cdot y} = e^0 = 1$. This is a vertical line. When $\boxed{k = +3}$ the curve is $x = e^{3y}$ which is an exponential growth curve with x as a function of y, and when $\boxed{k = -3}$ the curve is flipped to exponential decay in the same variables.



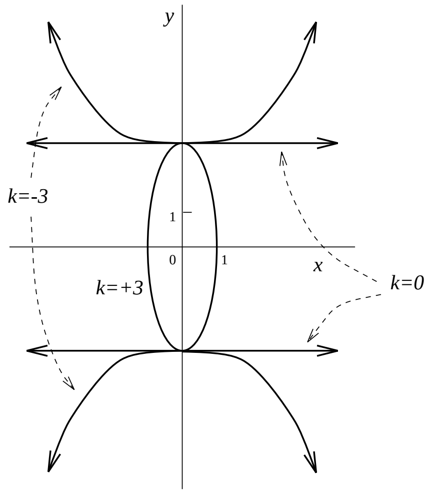
2. Sketch the curves $1 = kx^2 + y^2$ on the same axes for $k = 0$, $k = +3$, and $k = -3$. Be sure to label which value of k corresponds to which curve.

Comment These are very classical curves which should be recognized. Answering these questions by using (or, really, **ab**using [!]) a graphing calculator without any additional thought is foolish. *You've got to think!* Errors happen easily with the careless use of technology.

Answer When $\boxed{k = 0}$ the equation becomes $1 = y^2$. Solutions of this equation seem to split naturally into solutions of either the equation $1 = y$ or $-1 = y$. Therefore the "curve" is two parallel lines, as shown.

When $\boxed{k = +3}$ the equation becomes $1 = 3x^2 + y^2$. This is an ellipse as shown. Notice that the collection of points satisfying this equation is bounded in a very strong sense: neither x nor y can be too large and still satisfy the equation.

When $\boxed{k = -3}$ the equation becomes $1 = -3x^2 + y^2$. This is a hyperbola. A small change gives $1 + 3x^2 = y^2$. Therefore y must be $\geq +1$ or ≤ -1, and both y and x can be very large, unlike the situation for $k = +3$. Another difference is that the curve for $k = -3$ has two pieces.



3. Suppose $f(x) = .04(x^2 - 3)e^{-.5x}$.

a) Where is f zero?

Answer Since the exponential function is never 0, the only way this function can be 0 is if $x^2 - 3 = 0$ which happens when $x = \pm\sqrt{3} \approx \pm 1.732$.

b) Does f have any local extreme values? If yes, find them and describe what kind of extreme values they are.

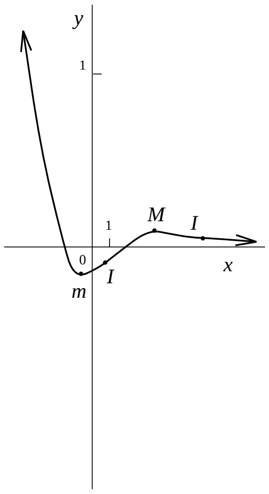
Answer We'll eventually need both f' and f'' for parts b) and c). You *must* be able to compute the derivatives of functions given by simple formulas.
So if $f(x) = .04(x^2 - 3)e^{-.5x}$
then $f'(x) = .04(2x)e^{-.5x} + .04(x^2 - 3)(-.5)e^{-.5x} = (.04)(2x + -.5x^2 + 1.5)e^{-.5x}$ and then $f''(x) = (.04)(2 - x)e^{-.5x} + (.04)(2x - .5x^2 + 1.5)(-.5)e^{-.5x} = (.04)(.25x^2 - 2x + 1.25)e^{-.5x}$. I've chosen one of many ways to compute and "simplify" these results.
Where is $f'(x) = 0$? A little work with the quadratic formula gives $x = 2 \pm \sqrt{7}$. So $x \approx 4.646$ or $x \approx -.646$. Calculators let us get these numbers and we can use calculators.
We now know that f has two critical points. What kind of critical points are they? Direct examination of f' shows that the sign of f' changes from negative to positive at $2 - \sqrt{7}$ and there's a reverse change of sign at $2 + \sqrt{7}$. Therefore f has a local minimum at $2 - \sqrt{7}$ and a local maximum at $2 + \sqrt{7}$. Or we could compute f'' at these two numbers, which seems like more work to me. You can see that $f''(4.646)$ is negative and $f''(-.646)$ is positive, so the convexity of the function at the two critical points gives the same information as the sign change of f' did.

c) Does f have any inflection points? If yes, find them.

Answer Where is $f''(x) = 0$? Another use of the quadratic formula tells me that this occurs when $x = 4 \pm \sqrt{11}$, which happens when $x \approx 7.317$ or $x \approx .683$. Since f'' changes sign at the roots of f'', these values of x are actually inflection points. Roots of the second derivative of a function don't have to be inflection points. For example, look at the function x^6.

d) Draw a rough sketch of the graph of $y = f(x)$. Label any interesting features you have discovered. What can you say about any asymptotic behavior of the graph?

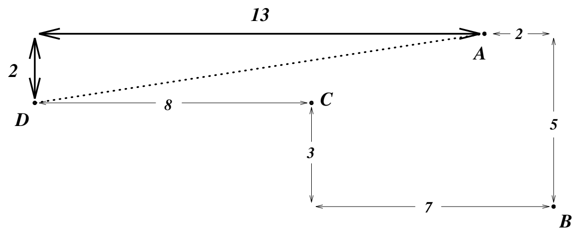
Answer You've got a graphing calculator, but more importantly you must know calculus. The asymptotics of f are easy, since the exponential function is much more "powerful" than any polynomial. Therefore $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = +\infty$. Here's a rough graph, with labels (M for local maximum, m for local minimum, and I for inflection point):



The numbers .04 and the $-.5$ were used so that the "interesting features" on the graph of the function would fit neatly into a little box. The mismatched vertical and horizontal scales in the graph show that I didn't succeed in scaling the picture as well as I wanted!

4. Point A is five miles north and two miles west of point B. Point B is three miles south and seven miles east of point C. Point D is eight miles west of point C. Assuming all of these points are on a flat plane, what is the distance from D to A?

Answer I thought this problem would be difficult. Most people drew some variant of the picture below and then correctly concluded that the distance from D to A is $\sqrt{13^2+2^2} = \sqrt{173} \approx 13.153$.

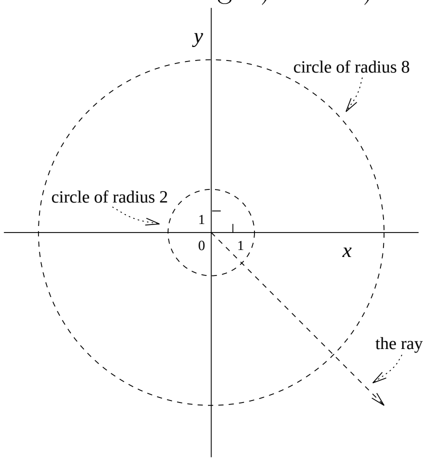


5. In this problem and in the whole course, the two-dimensional vector $<1,0>$ will be called $\mathbf{i}$ and the two-dimensional vector $<0,1>$ will be called $\mathbf{j}$.

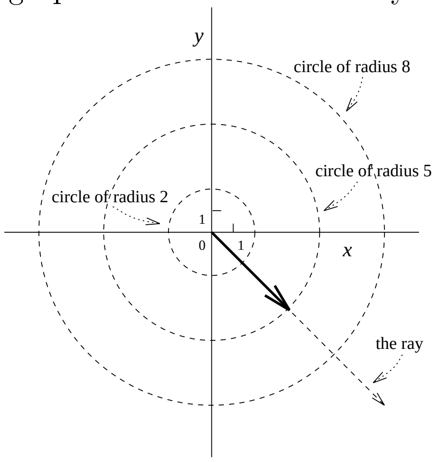
> We say that two vectors have *the same direction* if each of them is a **positive** scalar multiple of the other. For example, the two vectors $\mathbf{i}+\mathbf{j}$ and $3\mathbf{i}+3\mathbf{j}$ <u>do</u> have the same direction (the positive scalar is either 3 or $\frac{1}{3}$ depending on your point of view!), while the two vectors $\mathbf{i}+\mathbf{j}$ and $3\mathbf{i}+\mathbf{j}$ do <u>not</u> have the same direction, and the two vectors $\mathbf{i}+\mathbf{j}$ and $-2\mathbf{i}-2\mathbf{j}$ also do <u>not</u> have the same direction.

a) Suppose $\mathbf{a}$ and $\mathbf{b}$ are two-dimensional vectors, and you know that $|\mathbf{a}| = 8$ and $|\mathbf{b}| = 2$. Draw a picture of suitable vectors so that $|\mathbf{a}+\mathbf{b}| = 5$ **and** the vector $\mathbf{a}+\mathbf{b}$ has the same direction as $\mathbf{i}-\mathbf{j}$.

Comment If **a** and **b** have the indicated magnitudes and if their **tails** of **a** and **b** at the origin, then their heads must lie on the circles indicated. Their sum (with its tail also at the origin) must be a *positive* multiple of $\mathbf{i} - \mathbf{j}$, so its head must lie on the ray indicated. All this information prepares us for answering a) and b) and c).

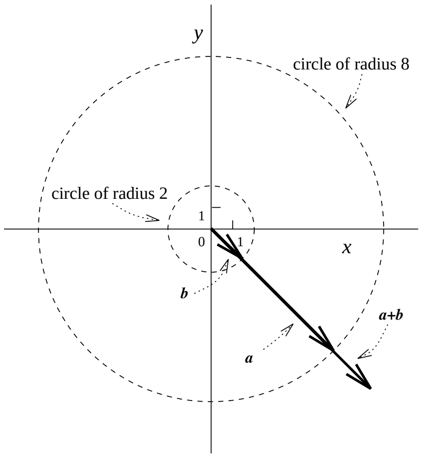


Answer How can we get an answer to a)? If we position the vector we want to be $\mathbf{a}+\mathbf{b}$ so that its tail is at the origin, then its head must be on a circle of radius 5 centered at $(0, 0)$. If the direction is what I've indicated, there's exactly *one* vector that could be $\mathbf{a}+\mathbf{b}$. To create this $\mathbf{a}+\mathbf{b}$ we'll need to start at $(0, 0)$, travel out to the circle of radius 8 (this is the **a** part) and then travel somewhere a distance of 2 units (this would be the effect of adding on **b**). But that will $\boxed{\textbf{never}}$ get us to the circle of radius 5! (I think because of the rather important fact that $5 < 8 - 2$) Darn it, this problem doesn't have a solution: I'm embarrassed. If I had specified (for example) $|\mathbf{a}| = 8$ and $|\mathbf{b}| = 6$ and $|\mathbf{a}+\mathbf{b}| = 3$, there would have been a nice graphical solution. I'm sorry.



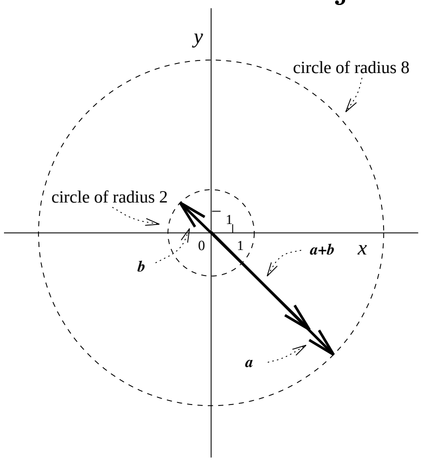
b) Suppose **a** and **b** are two-dimensional vectors, and you know that $|\mathbf{a}| = 8$ and $|\mathbf{b}| = 2$. Draw a picture of suitable vectors so that $|\mathbf{a}+\mathbf{b}|$ is as *large* as possible **and** the vector $\mathbf{a}+\mathbf{b}$ has the same direction as $\mathbf{i} - \mathbf{j}$.

Answer Here to get the magnitude of $\mathbf{a}+\mathbf{b}$ as large as possible, their directions must line up. And to get their sum or resultant to point in the direction desired, they each must be a positive multiple of $i + j$.



c) Suppose **a** and **b** are two-dimensional vectors, and you know that $|\mathbf{a}| = 8$ and $|\mathbf{b}| = 2$. Draw a picture of suitable vectors so that $|\mathbf{a} + \mathbf{b}|$ is as *small* as possible **and** the vector $\mathbf{a} + \mathbf{b}$ has the same direction as $\mathbf{i} - \mathbf{j}$.

Answer To get the result as small in magnitude as possible but still pointing in the direction we want, place **a** and **b** so that they point opposite to each other, making sure that their resultant still points in the direction of $\mathbf{i} - \mathbf{j}$.



General Comment Every skill tested by this Diagnostic Exam is used in Math 251. Be sure you know how to do these problems.