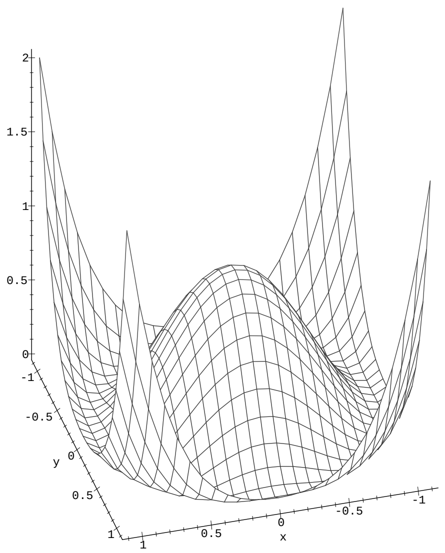
640:251:01-03 answers to **Another Quiz** **4/2/97**

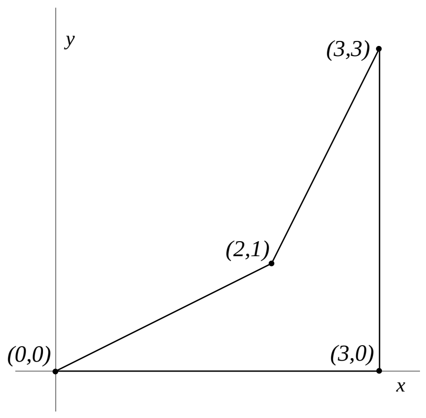
1. Find all critical points of $P(x,y) = \left(x^2+y^2-1\right)^2$.

Answer Since $\begin{cases} P_x = 2(x^2+y^2-1)^1(2x) \\ P_y = 2(x^2+y^2-1)^1(2y) \end{cases}$, any point with $x^2+y^2=1$ will be a critical point: these are the points on the standard unit circle whose radius is 1 and whose center is $(0,0)$. If $x^2+y^2-1 \neq 0$, then the only way both P_x and P_y could both be 0 is for x and y to both be 0 also. So $(0,0)$ is another critical point.



Comment You weren't asked to "classify" the critical points of P. The Second Derivative Test could be used to show that $(0,0)$ is a local maximum. But that Test would *not* apply at the other critical points belonging to the unit circle because $P_{xx}P_{yy}-(P_{xy})^2 = 0$ at those points if you compute it. But $P(x,y) = (\text{something})^2$, so its values must be ≥ 0. Therefore, since $P(x,y)=0$ on the unit circle, $P(x,y)$ has local (and absolute!) minima at *all* of the critical points located on the unit circle. At the left is a picture generated by `Maple` of part of this function's graph, an `Unconstrained` view with `x=-1.1..1.1` and `y=-1.1..1.1`, with `Theta=74`, `Phi=59`, and `R=1.00`, with `Axes=Framed`, and with `Shading=Z`.

2. Suppose $\mathcal{R}$ is the region shown (bounded by four straight line segments) and $H(x,y)$ is an unknown function. Write $\iint_{\mathcal{R}} H(x,y)\,dA$ as a sum of one or more iterated integrals.



Answer The side from $(0,0)$ to $(2,1)$ is described by $y=\frac{1}{2}x$ or by $x=2y$. The side from $(2,1)$ to $(3,3)$ is described by $y=2x-3$ or by $x=\frac{1}{2}y+\frac{3}{2}$. The other two sides are described by $y=0$ and $x=3$.

We can replace dA by either $dx\,dy$ or by $dy\,dx$. Either answer is acceptable, but only one needs to be given. Look at both answers, please.

For $dx\,dy$: $\displaystyle\int_0^1 \int_{2y}^3 H(x,y)\,dx\,dy + \int_1^3 \int_{\frac{1}{2}y+\frac{3}{2}}^3 H(x,y)\,dx\,dy$

and for $dy\,dx$: $\displaystyle\int_0^2 \int_0^{\frac{1}{2}x} H(x,y)\,dy\,dx + \int_2^3 \int_0^{2x-3} H(x,y)\,dy\,dx.$

3. Compute: $\displaystyle\int_0^\infty \int_0^\infty \frac{1}{\left(1+x^2+y^2\right)^2}\, dx\, dy.$

Answer That the region over which we're integrating is the first quadrant and the presence of x^2+y^2 in the integrand should signal "polar coordinates". In fact, even `Maple` is *not* able to find symbolically a repeated antiderivative of the integrand in rectangular coordinates (necessary to use the Fundamental Theorem of Calculus directly twice)! In polar coordinates, the exchange of $dx\, dy$ for $r\, dr\, d\theta$ has just the right factor to make finding an antiderivative very easy. Of course this isn't a coincidence; the question was designed for the technique presented here. The first quadrant in polar coordinates is described by $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r < \infty$. The integral requested becomes:

$$\int_0^{\frac{\pi}{2}} \int_0^\infty \frac{1}{(1+r^2)^2}\, r\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \lim_{A\to+\infty} \int_0^A \frac{1}{(1+r^2)^2}\, r\, dr\, d\theta =^*$$

$$\int_0^{\frac{\pi}{2}} \lim_{A\to+\infty} -\frac{1}{2}\left(\frac{1}{1+r^2}\right)\Bigg]_{r=0}^{r=A} d\theta = \int_0^{\frac{\pi}{2}} \lim_{A\to+\infty} -\frac{1}{2}\left(\left(\frac{1}{1+A^2}\right)-1\right) d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\, d\theta = \frac{\pi}{4}$$

4. Compute: $\displaystyle\int_0^1 \int_0^x \int_0^{xy} z\, dz\, dy\, dx.$

Answer This is repeated polynomial integration.

$$\int_0^1 \int_0^x \int_0^{xy} z\, dz\, dy\, dx = \int_0^1 \int_0^x \frac{z^2}{2}\Bigg]_{z=0}^{z=xy} dy\, dx = \int_0^1 \int_0^x \frac{(xy)^2}{2}\, dy\, dx =$$

$$\int_0^1 \int_0^x \left(\frac{x^2}{2}\right) y^2\, dy\, dx = \int_0^1 \left(\frac{x^2}{2}\right)\frac{y^3}{3}\Bigg]_{y=0}^{y=x} dx = \int_0^1 \left(\frac{x^2}{2}\right)\left(\frac{x^3}{3}\right) dx = \int_0^1 \frac{x^5}{6}\, dx =$$

$$\frac{x^6}{36}\Bigg]_{x=0}^{x=1} = \frac{1}{36}.$$

* with $u = 1+r^2$ and $du = 2r\, dr$ so $\frac{1}{2}\, du = r\, dr$, etc.