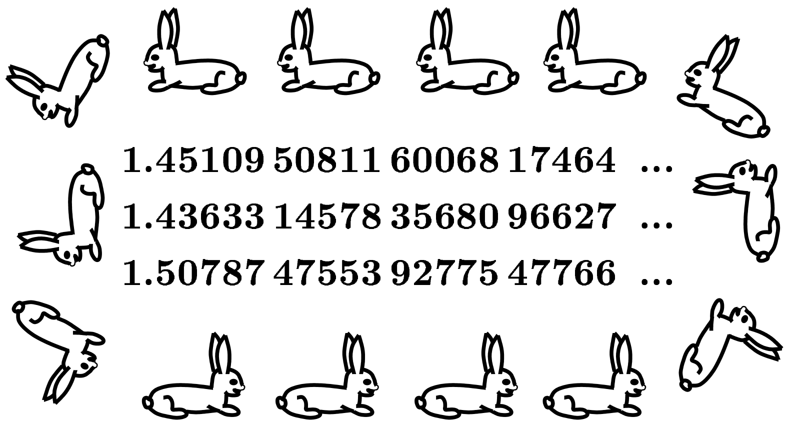


1.45109 50811 60068 17464 ...
1.43633 14578 35680 96627 ...
1.50787 47553 92775 47766 ...

A talk for the Rutgers Pi Club

Thursday, November 2, 2006

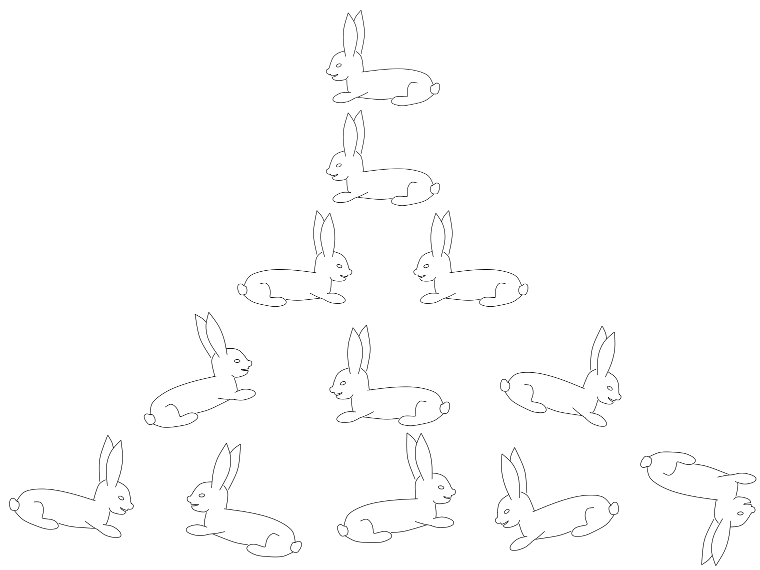
S. Greenfield, Rutgers Math Department

A most familiar sequence

Leonardo Pisano (1170–1250) published a book in 1202 which contained the following paragraph which is quoted here in translation:

> *A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?*

One of Leonardo's nicknames is *Fibonacci*. This paragraph is the first Western description of the Fibonacci sequence*: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...



The biological validity of this model is questionable, but the sequence defined by this description has amazingly many applications. Here's a standard recursive description:

$$F_{n+2} = F_{n+1} + F_n \text{ with } (F_0, F_1) = (0, 1)$$

The sequence occurs in the analysis of algorithms, and, indeed, does occur in some biological situations. There's even a scholarly journal devoted to these numbers and their generalizations. The Fibonacci numbers have many properties. For example, there is the **Cassini identity** (1680):

$$F_{n+1}F_{n-1} - (F_n)^2 = (-1)^n \text{ for } n > 0$$

The Cassini family included three generations of astronomers. The first of them discovered four moons of Saturn and saw what's now called the Cassini gap in Saturn's rings.

The asymptotic size of the Fibonacci numbers turns out to be important in practice. There is an explicit formula for F_n which easily gives useful information. **Binet's formula** states:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

* The numbers occurred a thousand years earlier in Sanskrit literature, used in the analysis of how long and short vowel sounds can be combined to form phrases!

The formula is named after Binet (1786–1856) but was known to Euler (1707–1783). I think the formula is really quite remarkable: the declaration that a sequence of *integers* is equal to a simple combination of powers of irrational numbers! There are many ways of verifying this formula (mathematical induction, generating functions, linear algebra ...) Realize that $\frac{1+\sqrt{5}}{2} \approx 1.618$ and $\frac{1-\sqrt{5}}{2} \approx -.618$, so $F_n \approx .447\big((1.618)^n - (.618)^n\big)$ (because $\frac{1}{\sqrt{5}} \approx .447$). Now powers of numbers between -1 and 1 go rapidly to 0, so, in fact, $F_n \approx .447(1.618\ldots)^n$. In fact, $F_{10} = 55$ and $.447(1.618)^{10} = 54.965\ldots$ which shows that the approximation is quite good even for small n.

Any linear recurrence can be analyzed in a way that's similar to what's done for the Fibonacci numbers. The results give asymptotic information which can be quite valuable.

A slight change

Math courses are very careful about non-linearity: it's too hard! Derivatives used to tame "random" functions by locally pretending they are linear. Throw away caution and consider: $h_{n+2} = h_{n+1} + (h_n)^{\mathbf{2}}$ with $(h_0, h_1) = (0, 1)$. This is sort of a quadratic analog of the Fibonacci sequence: I'll call it the QF sequence here. Let's look at a few terms:

n	h_n
0	0
1	1
2	1
3	2
4	3
5	7
6	16
7	65
8	321
9	4546
10	1 07587
11	207 73703
12	1 15957 36272
13	43155 83320 68481
14	1 34461 53124 81085 26465
15	18624 25941 12190 84752 01821 73826

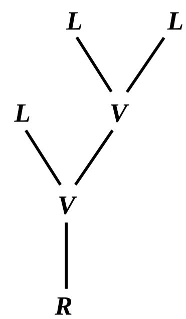
Certainly this sequence is increasing fast: very fast, much faster than the Fibonacci sequence. Why is this? We could compare the QF sequence to a sequence defined by a simpler recurrence such as $k_{n+2} = (k_n)^2$. The terms of the QF sequence are likely to be bigger than this sequence. If k_0 and k_1 were some initial values, like FROG and TOAD, then we would have determined the successive members of the sequence $\{k_n\}$. The sequence would begin like this:

$$\text{FROG},\ \text{TOAD},\ \text{FROG}^2,\ \text{TOAD}^2,\ \text{FROG}^2,\ \text{TOAD}^4,\ \text{FROG}^8,\ \text{TOAD}^8,\ \ldots$$

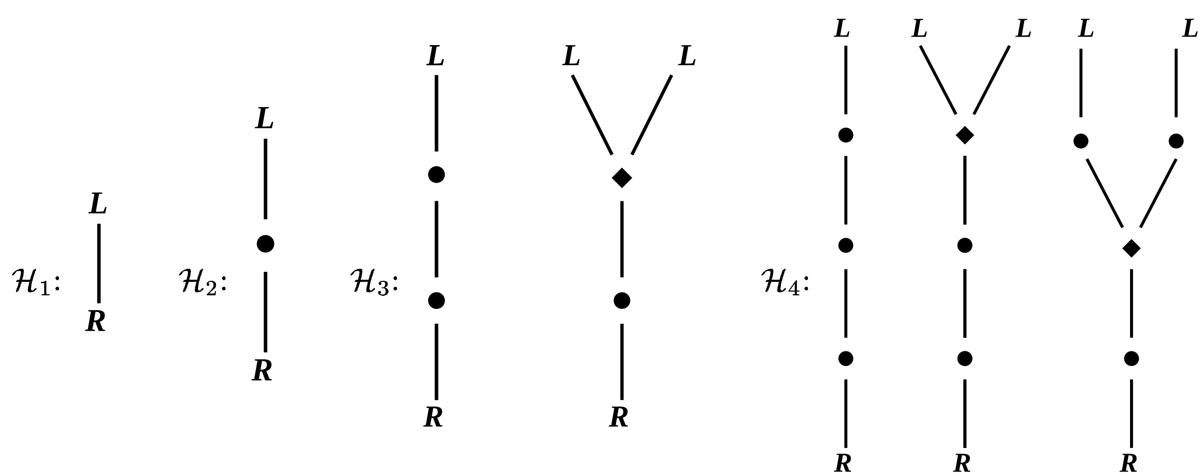
There is no interaction between the odd and even terms. Let's just look at the even terms. Then $k_{2n} = \text{FROG}^{(2^n)} = \text{FROG}^{\left((\sqrt{2})^{2n}\right)}$. Of course something similar happens to the odd terms, with a "base" of TOAD instead of FROG. This might suggest that we could expect a square root of 2 power from the QF recurrence every time n steps up by 1. You can verify that roughly: the length of the integers in the table about doubles every two steps. But, before analyzing this "abstract" sequence further, I can tell you it counts something*: certain "trees" with very restricted branching rules.

Weird trees

In theoretical computer science, a graph is a set of vertices together with a set of edges where the edges connect pairs of distinct vertices. A vertex connected by an edge is called *incident* with that edge. A *tree* here will be a connected graph without closed paths of edges. The *degree* of the vertex is the number of edges it touches. A *rooted tree* has one distinguished vertex with degree 1. The root vertex will be labeled ***R***. Any other vertices of degree 1 in a rooted tree are called *leaves* and will be labeled ***L***. Trees will be drawn here with their roots at the bottom of their pictures. The *level* of a vertex is its distance to the root where the *distance* between two vertices is the number of edges required to travel from one to the other. A tree with three leaves is displayed. One leaf has level 2 and the two others have level 3. This tree also has two vertices designated ***V*** which are neither leaves nor the root. They both have degree 3. One of these has level 1 and the other has level 2.



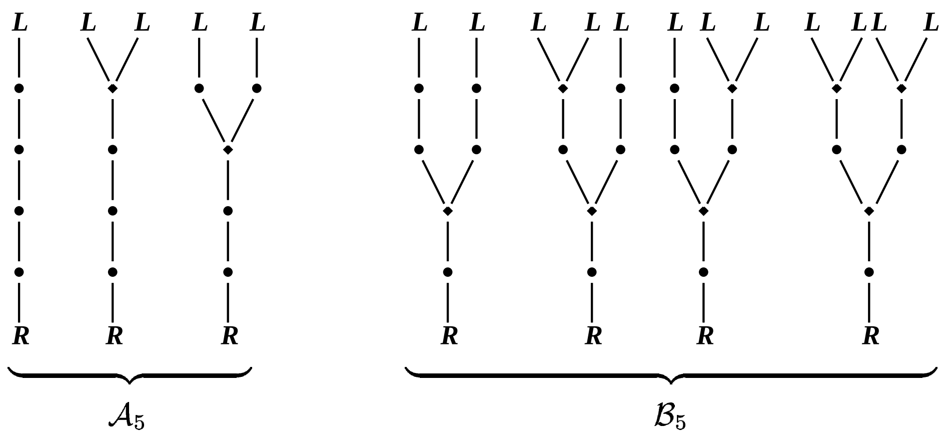
$\mathcal{H}_n$ will be a set of all rooted trees of a certain type for each integer $n \geq 1$. Every leaf of each tree in $\mathcal{H}_n$ will have level n. Any vertices of degree greater than 1 in trees in $\mathcal{H}_n$ will be one of two types: the diamond (◆) and the circle (●). The diamond will always have degree 3 so the tree must "branch" at a diamond. The distance between two diamonds must always be at least 2, and the level of a diamond must be at least 2. All other vertices of degree greater than 1 will be circles and each circle will have degree 2. There will be no branching at a circle – just a "trunk". Here is a display of some small forests of this species, a rather peculiar sort of binary tree.



The trees in $\mathcal{H}_5$ can be grouped suggestively. The illustration following shows $\mathcal{H}_5$ divided into two groups, $\mathcal{A}_5$ and $\mathcal{B}_5$. $\mathcal{A}_5$ contains the trees in $\mathcal{H}_5$ whose level 2 vertex is a circle.

* It is frequently useful to find *something* that is counted by a sequence of integers!

$\mathcal{B}_5$, the trees in $\mathcal{H}_5$ whose level 2 vertex is a diamond. The trees are "oriented": the left and right branches are distinct. If you look carefully, the trees in $\mathcal{A}_5$ are the trees in $\mathcal{H}_4$ pushed one level "up", while the trees in $\mathcal{B}_5$ have branching beginning at the second level, and the left and right subtrees shown are independently chosen from $\mathcal{H}_3$. Since the right and left branches are each from $\mathcal{H}_3$, the number of choices is the square of the number of trees in $\mathcal{H}_3$. Therefore the number of trees in $\mathcal{H}_5$ are exactly equal to the number in $\mathcal{A}_5$ (that's h_4) plus the number in $\mathcal{B}_5$ (and that's $(h_3)^2$). So the number of trees in $\mathcal{H}_n$ satisfies the QF recurrence.



Trees with restricted branching rules are useful tools in some aspects of physics and chemistry and in many areas of computer science. Studying and counting trees can be useful.

Asymptotics

As mentioned, we might expect "superexponential" behavior like $(\text{something})^{\left((\sqrt{2})^n\right)}$. Let us try to find the value of the base of this superexponential growth. I should mention that almost *everything* discussed here had its first appearance as a result of experimentation, assisted by `Maple`. The numbers appearing are so large (and, as you will see, so small!) that machine-based help is essential for exploration. What's more, conventional calculators and computer programs which have fixed sizes for numbers would rapidly be insensitive and useless for much of this exploration.

The QF sequence increases, but some control is possible:

Lemma For $n > 0$, $0 < h_n \le h_{n+1} \le 2(h_n)^2$.

Proof This is true for $n = 1$. Since the sequence is increasing, $h_n \le h_{n+1}$ always, and $h_{n+2} = h_{n+1} + (h_n)^2 \le (h_{n+1})^2 + (h_{n+1})^2 = 2(h_{n+1})^2$. ■

Now define a_n by $h_{n+2} = (h_n)^2 \overbrace{\left(1 + \frac{h_{n+1}}{(h_n)^2}\right)}^{a_n}$. The lemma implies that $1 \le a_n \le 3$.

I'll consider a specific example, say h_8, of the QF sequence, and use the definition of the a_n's repeatedly. I'll do some algebraic juggling in order to help you see a general pattern.

$$h_8 = (h_6)^2\left(1+\frac{h_5}{(h_6)^2}\right) = a_6(h_6)^2 = a_6\left((h_4)^2\left(1+\frac{h_3}{(h_4)^2}\right)\right)^2 =$$

$$a_6(a_4)^2\left((h_4)^2\right)^2 = a_6(a_4)^2(h_4)^4 = a_6(a_4)^2\left((h_2)^2\left(1+\frac{h_1}{(h_2)^2}\right)\right)^4 =$$

$$a_6(a_4)^2(a_2)^4(h_2)^8 = a_6(a_4)^2(a_2)^4(a_0)^8 = \left((a_6)^{1/8}(a_4)^{1/4}(a_2)^{1/2}(a_0)^{1/1}\right)^{(\sqrt{2})^8}$$

Here I made $a_0 = 1$ since $h_2 = h_0 = 1$. But what I am really trying to do is to somehow guess a formula which imitates what we've seen for the much simpler $\{k_n\}$ sequence. For example, I know the $k_8 = \text{FROG}^{((\sqrt{2})^8)}$. Since the QF sequence is more complicated, I don't think we should expect something so explicit. The h_8 formula seems to suggest that the h_{2n} might approximately be a $(\sqrt{2})^{2n}$ power of some "base" and that base would be an infinite product of the a_{2j}'s to some powers.

This can be done in general, because we could write $h_{n+2} = (h_{n-2})^4(a_{n-2})^2a_n = (h_{n-4})^8(a_{n-4})^4(a_{n-2})^2a_n = \ldots$ and so on down to 1, where "1" is either h_1 or h_2. There are powers of 2 in the exponent. I will rewrite things using logs, because then the products become sums and infinite sums are likely to be more familiar than infinite products. There are two cases depending on n's parity (evenness or oddness).

Even QF terms

$$h_{2n} = \left(\exp\left(\sum_{j=0}^{n-1}\frac{1}{2^{j+1}}\log a_{2j}\right)\right)^{(\sqrt{2})^{2n}} \quad \text{which is} \approx A^{(\sqrt{2})^{2n}} \text{ if } A = \exp\left(\sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\log a_{2j}\right).$$

Odd QF terms

Here h_{2n+1} is close to $B^{(\sqrt{2})^{2n+1}}$ if $\log B = \dfrac{1}{\sqrt{2}}\displaystyle\sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\log a_{2j+1}$.

Why "should" these infinite series converge? The QF sequence increases very rapidly. For example, the twentieth term has 370 decimal digits! The a_n's are quite close to 1 because $\frac{h_{n+1}}{(h_n)^2} \approx (h_n)^{\sqrt{2}-2}$, which is certainly a *negative* exponent! Since $\log(1+x) \approx x$ for small x, convergence of the series of logs follows. By being a bit careful of the error (no more tools than basic calculus are needed) the following result can be verified:

Theorem $\displaystyle\lim_{n\to\infty}\frac{h_{2n}}{A^{(\sqrt{2})^{2n}}} = 1$ and $\displaystyle\lim_{n\to\infty}\frac{h_{2n+1}}{B^{(\sqrt{2})^{(2n+1)}}} = 1$.

We can compute A and B. The accuracy of the calculation needs attention since there are both very large and very small numbers involved.

$$A \approx 1.45109\,50811\,60068\,17464\ldots \text{ and } B \approx 1.43633\,14578\,35680\,96627\ldots$$

So these growth constants are two numbers which govern the sequence. The odd elements of the sequence and the even elements of the sequence seem to be asymptotically decoupled.

But the bases of the superexponential growth actually must "pay attention" to each other since together they describe a single strictly increasing sequence of integers. To me this is a very strange situation.

Dependence on initial conditions

Students who study differential equations get used to thinking about the solution as a result both of the differential equation and of the initial conditions. One of the standard beginning examples is throwing a ball and getting the path as a function of time. The differential equation describes how the position of the ball "evolves" through time, but certainly the ball's position at later time will also depend on its initial position and initial velocity. Recurrences frequently have similar dependence.

The Binet formula for the Fibonacci numbers can be generalized to any sequence satisfying the recurrence $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = p$ and $F_1 = q$, for *any* real numbers p and q. Then the sequence looks like

$$p, q, p + q, p + 2q, 2p + 3q, 3p + 5q, \ldots$$

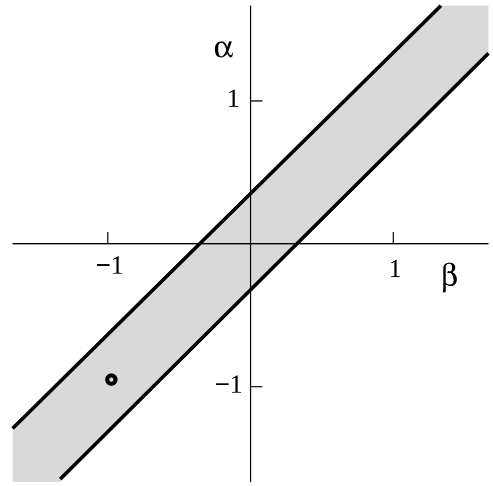
The *linearity* of the recurrence makes guessing a general formula this not too difficult. If $F_n(p, q)$ is the n^{th} term of the sequence with (p, q) initial conditions, then $F_n(p, q) = (F_{n-1})\,p + (F_n)\,q$. You can check this formula just by looking at the solutions with initial conditions $(0, 1)$ and $(1, 0)$ because of linearity. Similar results are true for all linear recurrences. But QF is not linear, and the resulting mixing seems quite complicated. Here is one result:

Theorem If p and q are any non-negative real numbers with $p + q > 0$, and if $h_n(p, q)$ is the n^{th} term of the sequence satisfying $h_{n+2} = h_{n+1} + (h_n)^2$ with $(h_0, h_1) = (p, q)$ then there are positive numbers $A(p, q)$ and $B(p, q)$ so that $\lim_{n\to\infty} \dfrac{h_{2n}}{A(p, q)^{(\sqrt{2})^{2n}}} = 1$ and $\lim_{n\to\infty} \dfrac{h_{2n+1}}{B(p, q)^{(\sqrt{2})^{(2n+1)}}} = 1$.

The initial condition $(0, 0)$ isn't covered by this result, but then the sequence produced is all 0's. The initial conditions, (p, q), here defined in the first quadrant of the plane, get changed into a pair of superexponential growth constants, $(A(p, q), B(p, q))$, which describe the sequence asymptotically.

The intertwined nature of the two superexponential subsequences is quite strange. We know, under the hypotheses of the previous theorem, that for n large enough, $h_{2n} < h_{2n+1} < h_{2n+2}$. If we assume that the asymptotic approximations satisfy the same conditions, then we know that $A(p, q)^{(\sqrt{2})^{2n}} < B(p, q)^{(\sqrt{2})^{(2n+1)}} < A(p, q)^{(\sqrt{2})^{2n+2}}$. We might make a bit of sense out of superexponential growth if we take logs and then take logs again. Consider $A(p, q)$. The sequence is growing, so $A(p, q) > 1$. Therefore $\log(A(p, q)) > 0$ and then $\log(\log(A(p, q))$ has no restriction: it can be any real number. Define $\alpha(p, q)$ to be $\log(\log(A(p, q))$ and $\beta(p, q)$ to be $\log(\log(B(p, q))$. If we take logs twice in the preceding inequality we get $n \log 2 + \alpha(p, q) < \left(n + \frac{1}{2}\right) \log 2 + \beta(p, q) < (n+1) \log 2 + \alpha(p, q)$. Subtract $n \log 2$ from all terms.

Therefore the initial conditions (p, q) in the first quadrant of the plane get changed to numbers whose log(log) values (α, β) are in a diagonal strip of the plane: $\alpha < \frac{1}{2}\log 2 + \beta < \log 2 + \alpha$. $\frac{1}{2}\log 2$ is about .34657. This diagonal strip is shown to the right. The growth constants whose values were previously stated result from the initial condition $(p, q) = (0, 1)$. The (α, β) point gotten from log(log)ing (?) of those numbers is about $(-1.01586, -.98801)$, and is shown in the accompanying graph. This point is certainly inside the strip. We don't know if every point in the strip corresponds to a pair of growth constants.



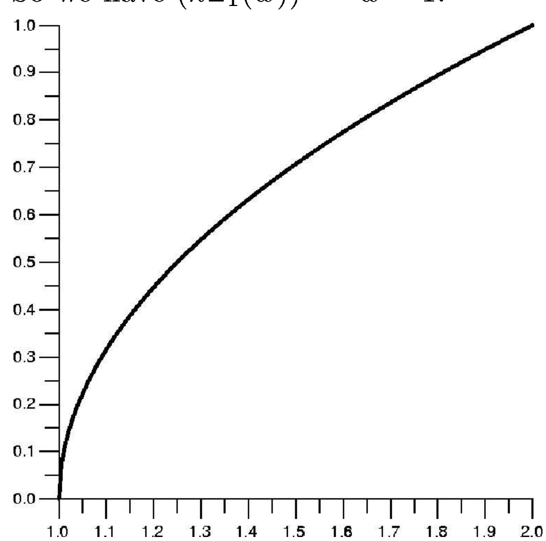
More properties are known about the functions $A(p, q)$ and $B(p, q)$, such as differentiability and some knowledge of the values of their derivatives. We also know various equations that $A(p, q)$ and $B(p, q)$ must satisfy. But they seem to be rather complicated functions, fully reflecting the nonlinearity of the QF recurrence. Verification of the statements needs some sophisticated techniques.

Going backwards

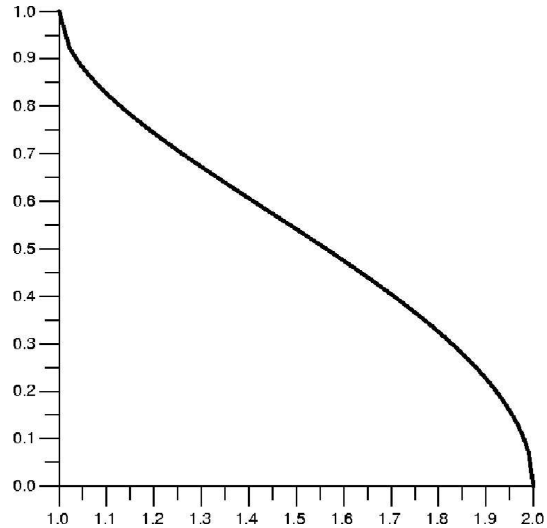
Some recurrences can be run backwards. Let's run the Fibonacci recurrence $F_{n+2} = F_{n+1} + F_n$ with $(F_0, F_1) = (0, 1)$ backwards (reversing "time"): $F_n = F_{n+2} - F_{n+1}$. If $n = -1$, then $F_{-1} = F_1 - F_0 = 1 - 0 = 1$. And $F_{-2} = F_0 - F_{-1} = 0 - 1 = -1$, and $F_{-3} = F_{-1} - F_{-2} = 1 - (-1) = 2$. The backwards sequence looks like this: $\ldots$, $5 = F_{-5}$, $-3 = F_{-4}$, $2 = F_{-3}$, $-1 = F_{-2}$, $1 = F_{-1}$, $0 = F_0$, $1 = F_1$. Proving that $F_n = (-1)^{n+1}F_{-n}$ when n is a negative integer is not difficult. Nothing much new happens when a linear recurrence goes backwards. The explicit formulas are true for negative n's.

Let's look at QF with the initial conditions $(h_0, h_1) = (1, w)$ when w is a positive number and go backwards. We know $h_{n+2} = h_{n+1} + (h_n)^2$ so $(h_n)^2 = h_{n+2} - h_{n+1}$. If $n = -1$, then $(h_{-1}(w))^2 = h_1(w) - h_0(w)$. We need the w's here because the backwards elements of the recurrence are now functions of w. So we have $(h_{-1}(w))^2 = w - 1$.

Think a bit. We can solve the equation with a square root, but which square root, and should we restrict our consideration to real solutions only? I'll stay away from the complex numbers, although I do love them. So we restrict to $w \geq 1$. We can simplify by only looking at the positive solution of the equation. To the right is a graph of $h_{-1}(w) = \sqrt{w - 1}$ for w in $[1, 2]$.



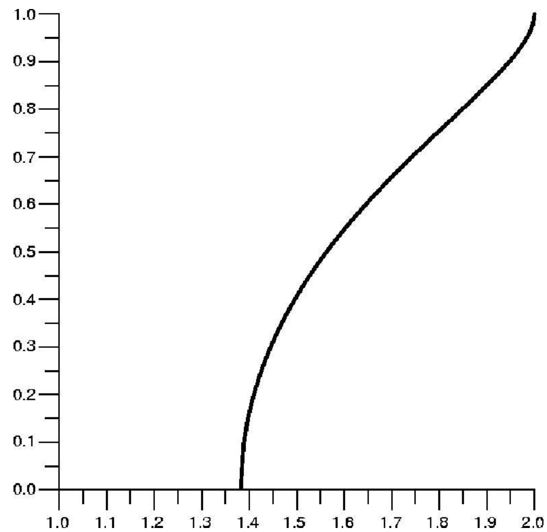
Suppose we go back one more step. We have the equation $(h_{-2}(w))^2 + h_{-1}(w) = 1$. Square root again with the positive choice. The result is $h_{-2}(w) = \sqrt{1-\sqrt{w-1}}$. We need to consider this function on its "natural domain", which is inside the domain of h_{-1}. The domain is $[1,2]$. The maximum value of h_{-2}, at the left end point, is 1.



Now let's consider the third step back, so that $(h_{-3}(w))^2 + h_{-2}(w) = h_{-1}(w)$. The formula for h_{-3} is more elaborate.

$$h_{-3}(w) = \sqrt{\sqrt{w-1} - \sqrt{1-\sqrt{w-1}}}.$$

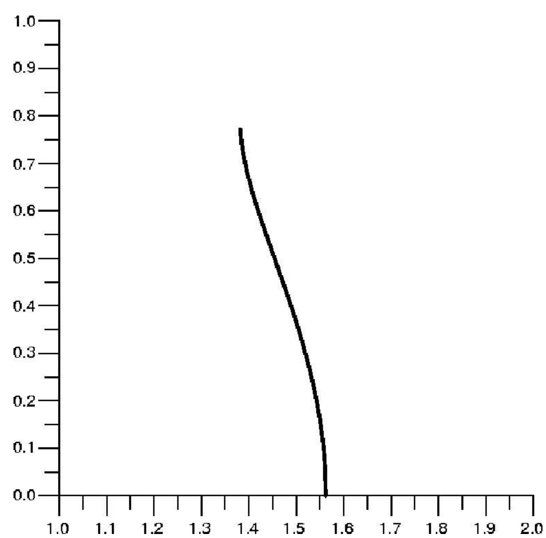
This function's domain is more intricate, but it must be inside $[1,2]$ since h_{-1} and h_{-2} must be defined. The restriction is that w must be larger than 1.38197 (approximately!). The maximum value, at the right end point, is 1.

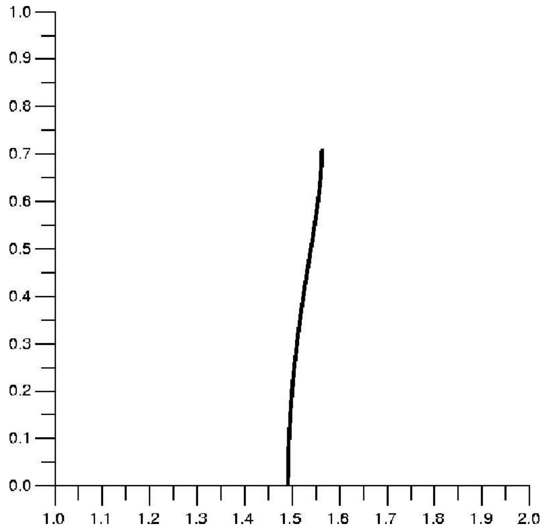


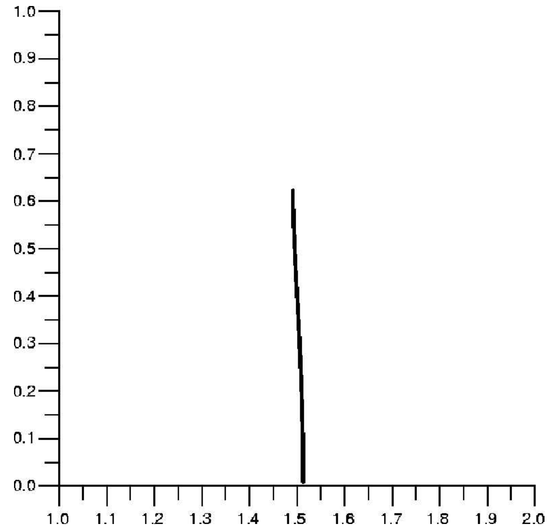
The fourth backwards function, $h_{-4}(w)$, is given by the formula

$$\sqrt{\sqrt{1-\sqrt{w-1}} - \sqrt{\sqrt{w-1} - \sqrt{1-\sqrt{w-1}}}}$$

The domain of this function is an interval whose right end point is 1.56250. Since h_{-3} is evaluated inside this formula, the formula only makes sense for $w \geq 1.38197$, because otherwise $h_{-3}(w)$ won't be defined. The maximum value, at the left end point, is about .73.

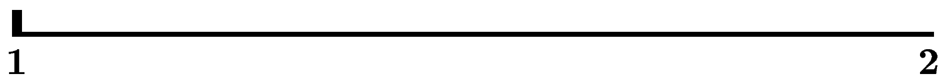






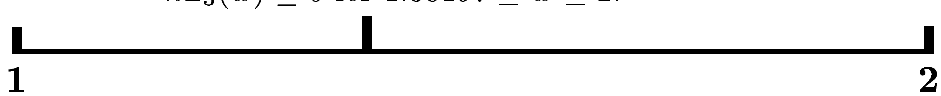
The graphs of $h_{-5}(w)$ and $h_{-6}(w)$ are above. The left graph is 0 at about 1.49139 and has a maximum value of about .67. The right graph is 0 at about 1.51282 and has a maximum value of about .5. Let me summarize with the following:

$h_{-1}(w) \geq 0$ for $1 \leq w$.

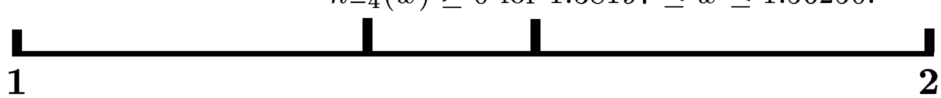


$h_{-2}(w) \geq 0$ for $1 \leq w \leq 2$.

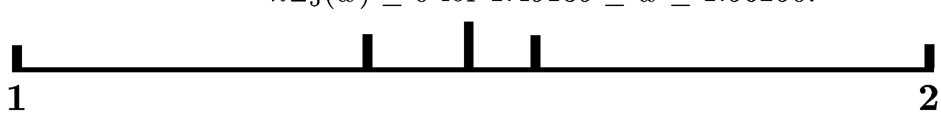
$h_{-3}(w) \geq 0$ for $1.38197 \leq w \leq 2$.



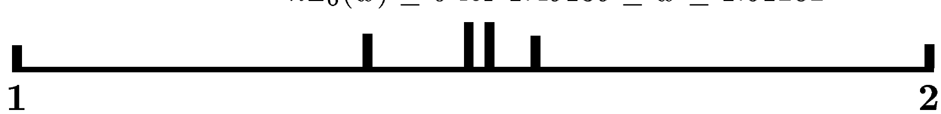
$h_{-4}(w) \geq 0$ for $1.38197 \leq w \leq 1.56250$.



$h_{-5}(w) \geq 0$ for $1.49139 \leq w \leq 1.56250$.



$h_{-6}(w) \geq 0$ for $1.49139 \leq w \leq 1.51282$



I hope that the pictures or the numbers or the diagrams suggest something. The computations going backwards get increasingly delicate, because the complexity of the nested square roots increases and the numbers involved get very close. The interval of existence is getting narrower, and its width actually $\to 0$. This assertion is *not* obvious!

Theorem There is a unique number, $L(1) \approx 1.50787\,47553\,92775\,47766\,\ldots$, so that the initial conditions $(h_0, h_1) = (1, L(1))$ give a *doubly infinite* sequence of positive real numbers satisfying the QF recurrence.

In fact, as the notation $L(1)$ might suggest, given any positive real number x, there's a unique positive real number $L(x)$ so that $(x, L(x))$ is part of a doubly infinite QF sequence. The function $L(x)$ is differentiable and has interesting properties. Due to the nonlinearity of the QF recurrence, verifying these statements is not straightforward.

Does computation always help?

Here is another anecdote about the tendency (desire?) of human beings to see patterns. Since the initial conditions $(1, 1.50787\ldots)$ are *special* maybe we should compute the growth constants of the resulting QF sequence. The growth constants are <u>approximately</u>

$$A(1, 1.50787\ldots) \approx 1.88695\,859\ldots \quad \text{and} \quad B(1, 1.50787\ldots) \approx 1.88695\,854\ldots$$

Much intricate floating point computation produces these numbers and everyone *knows* that such arithmetic accumulates errors. Therefore a natural conjecture is certainly that a doubly infinite QF sequence occurs exactly when the growth constants are equal. That result would certainly be pretty, and would make all the numbers seem more interesting.

Motivated by these numbers, "ONE" can spend several weeks trying to prove this result. Remember that the ecstasy of discovery is merely the overrated climax of the joy of investigation.* I don't agree with that entirely, especially when the ONE involved is ME and the result wanted is *FALSE*. Very careful computation shows that the last digits (the 9 and the 4) of the numbers displayed above are <u>correct</u>, and the two growth constants are not equal. Actually the growth constants for doubly infinite QF sequences are always *almost* equal (the quotient of their logs always is 1 to about 6 decimal places!) but they are rarely exactly equal to 1. WE DON'T KNOW A NICE CHARACTERIZATION OF DOUBLY INFINITE QF SEQUENCES IN TERMS OF THEIR GROWTH CONSTANTS.

That's the story (and, now, the closing credits!)

You've met three new numbers, $1.45109\ldots$, $1.43633\ldots$, and $1.50787\ldots$. I tried the *Inverse Symbolic Calculator* web page, which reported no connections with known mathematical constants. I don't know much else about these numbers, but I guess that all three of them are transcendental (they are *not* solutions of polynomial equations of any degree with integer coefficients). For the first two numbers, a proof can probably be given using a result due to Liouville (numbers that are very well approximated by rational numbers are transcendental –see the next page, please). I DON'T KNOW FOR SURE, THOUGH.

The methods used to prove the statements in this talk vary from traditional techniques such as continued fractions (several centuries old) to relatively new ideas which were created in the last 10 years to study nonlinear differential equations.

These results were discovered by the author together with many people. The following faculty members made major contributions: Bill Duke (now at UCLA), Roger Nussbaum, Mike Saks, and Gene Speer. Tom Peters, a Rutgers undergraduate (now at Columbia as a grad student), studied a cubic analog of QF sequences.

* Huh?

Just a little about numbers ...

There are lots and lots of numbers. In fact, there are lots and lots of different *kinds* of numbers: real, complex, quaternionic, numbers used in signal processing which are strings of bits, and many others.

Here I'll only talk about *real* numbers.

We start with the integers. Actually, let's start with the *counting numbers*: 1, 2, 3, and so on. Already there are interesting questions. There are infinitely many positive integers. It's not totally obvious that there are infinitely many *prime* numbers although convincing arguments (o.k., PROOFS!) have been known for several thousand years. It is not known if there are infinitely many *twin primes*, prime numbers separated by 2. So 5 and 7 are twin primes. Our ignorance is embarrassing!

Then we build up: 0 and negative integers permit us to add and subtract freely. If we want to allow division, we need to be careful about the rules (no matter how much you'd like it, there is no multiplicative inverse for 0, not if you want other rules to still be true!). We get the *rational numbers*. Here we run into new difficulties. One number can have different names: $\frac{4}{7} = \frac{8}{14} = \frac{40}{70} = \frac{-12}{-21}$. This is quite weird.

The *real numbers* are all possible decimal expansions. You can choose a sign ($+$ or $-$), a finite number of digits to the left of the decimal point, and then an infinite sequence of digits to the right of the decimal point. More confusion can occur. A real number may have more than one decimal "address": $43.26999\ldots$ is the same as $43.27000\ldots$ but this shouldn't be distressing, since rational numbers already have a similar address deficiency (an even worse deficiency, because there are infinitely many different ways to write rationals). Real number arithmetic is so hard that real computers can't handle all real numbers!

Rational numbers always must have a repeating decimal representation. Other numbers are *irrational*, and can't be written as quotients of integers. The square root of 2 is such a number (again, this was known a long time ago). What is not obvious, and was relatively recently observed, is that there are many more irrationals than rationals. Here is one way of restating this assertion: if you throw all the real numbers in a bag and pull one out "at random", then the probability is 1 that the number will be irrational. This isn't obvious, and the idea of "probability" needs to be carefully investigated.

A number is *algebraic* if it is a root of a polynomial with integer coefficients. So $\sqrt{2}$ is algebraic ($x^2 - 2$) and so is $\frac{5-\sqrt{17}}{6^{1/4}-9\cdot 77^{3/17}}$. I don't know the polynomial for that number!

Some numbers are not algebraic (π and e are not algebraic). Numbers which are not algebraic are called *transcendental*. Now an even more startling fact: take the bag of real numbers again, and pick a number "at random". As mentioned before, the number will be irrational with probability 1. Moreover, the number picked will be transcendental with probability 1. I don't think human beings can understand random real numbers.

$$\text{!}\quad \{\text{positive integers}\} \subset \{\text{integers}\} \subset \{\text{rational numbers}\} \subset \{\text{real numbers}\} \quad \text{!}$$

$$\begin{aligned} \text{!}\quad \{\text{real numbers}\} &= \{\text{irrationals}\} \cup \{\text{negligible stuff}\} \quad \text{!} \\ \text{!}\quad &= \{\text{transcendentals}\} \cup \{\text{other negligible stuff}\} \quad \text{!} \end{aligned}$$