Here are answers that would earn full credit. Other methods may also be valid.

(14) 1. a) Suppose $f(x) = \frac{1}{3x+5}$. Use the **definition of derivative** to find $f'(x)$.

Answer $f(x+h) = \frac{1}{3(x+h)+5}$ so $\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{3(x+h)+5} - \frac{1}{3x+5}}{h} = \frac{\frac{(3x+5)-\left((3(x+h)+5\right)}{(3(x+h)+5)(3x+5)}}{h} = \frac{3x+5-(3x+3h+5)}{h\big(3(x+h)+5\big)(3x+5)} = \frac{-3h}{h(3(x+h)+5)(3x+5)} = \frac{-3}{\big(3(x+h)+5\big)(3x+5)} =$. Therefore $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-3}{\big(3(x+h)+5\big)(3x+5)} = -\frac{3}{(3x+5)^2}$.

b) Use your answer to a) to write an equation for the line tangent to $y = \frac{1}{3x+5}$ when $x = -1$.

Answer $f(-1) = \frac{1}{3(-1)+5} = \frac{1}{2}$ and $f'(-1) = -\frac{3}{\big(3(-1)+5\big)^2} = -\frac{3}{4}$. An equation is $y - \frac{1}{2} = -\frac{3}{4}(x+1)$.

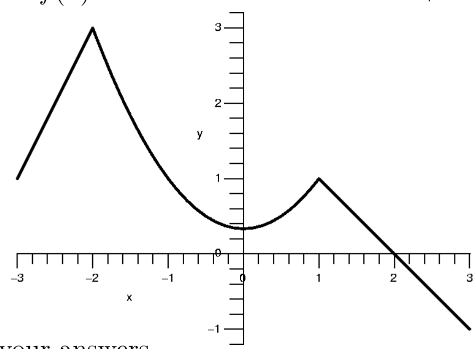
(14) 2. Suppose that the function $f(x)$ is described by $f(x) = \begin{cases} 2x+7 & \text{if } x < -2 \\ Ax^2 + B & \text{if } -2 \le x \le 1 \\ 2-x & \text{if } 1 < x \end{cases}$.

a) Find A and B so that $f(x)$ is continuous for all numbers. Briefly explain your answer.

Answer Away from -2 and 1, $f(x)$ is continuous. As for $x = -2$: $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} 2x+7 = 2(-2)+7 = 3$ and $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} Ax^2+B = A(-2)^2+B = 4A+B$. $f(x)$ is continuous at -2 if $3 = 4A+B$. And $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} Ax^2+B = A+B$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2-x = 1$. $f(x)$ is continuous at 1 if $3 = 4A+B$.

The solutions of $\begin{cases} 3 = 4A+B \\ 1 = A+B \end{cases}$ are gotten by subtracting the second equation from the first. The result is $2 = 3A$ so $A = \frac{2}{3}$ and then the second equation gives $1 = \frac{2}{3} + B$ so $B = \frac{1}{3}$. Therefore $f(x)$ will be continuous for all numbers if $f(x)$ has the formula $\frac{2}{3}x^2 + \frac{1}{3}$ when $-2 \le x \le 1$.

b) Sketch $y = f(x)$ on the axes given for the values of A and B found in a) when x is in the interval $[-3, 3]$.



(20) 3. Evaluate the indicated limits exactly. Give evidence to support your answers.

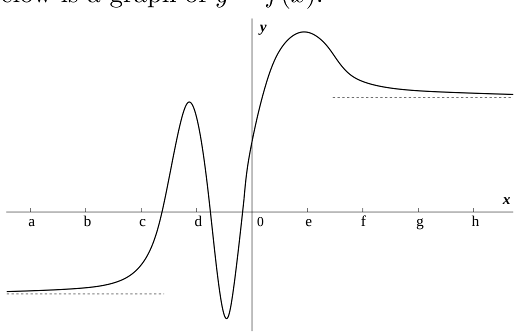
a) $\lim_{x\to 1} \frac{x^2+4x-5}{x-1}$ **Answer** If $x \ne 1$, $\frac{x^2+4x-5}{x-1} = \frac{(x-1)(x+5)}{x-1} = x+5$ so that $\lim_{x\to 1} \frac{x^2+4x-5}{x-1} = \lim_{x\to 1} x+5 = 6$.

b) $\lim_{x\to 0} \frac{x^2}{\sin x}$ **Answer** If $x \ne 0$, $\frac{x^2}{\sin x} = x \cdot \frac{x}{\sin x}$. But $\lim_{x\to 0} \frac{x}{\sin x} = 1$ (much discussed in class) so $\lim_{x\to 0} \frac{x^2}{\sin x} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{x}{\sin x} = 0 \cdot 1 = 0$.

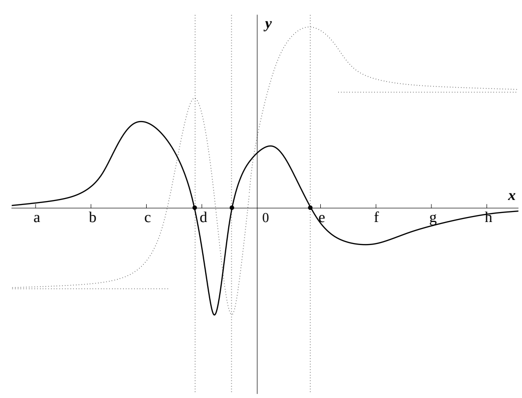
c) $\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3}$ **Answer** If $x \ne \sqrt{3}$, $\frac{\sqrt{x}-\sqrt{3}}{x-3} = \frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})} = \frac{1}{\sqrt{x}+\sqrt{3}}$ so $\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3} \frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$.

d) $\lim_{x\to 1} \frac{3x-2}{\cos(\pi x)}$ **Answer** Since $\cos(\pi) = -1$, the limit is gotten by just "plugging in". Its value is $\frac{3\cdot 1-2}{-1} = -1$.

(12) 4. Below is a graph of $y = f(x)$.



The graph of $y = f(x)$



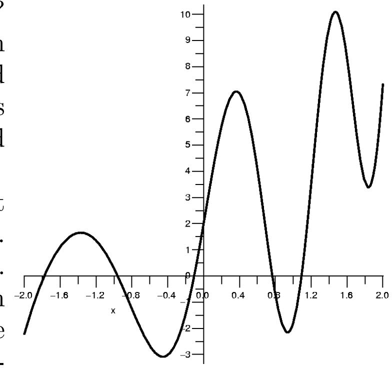
The graph of $y = f'(x)$

Sketch a graph of $y = f'(x)$, the derivative of $f(x)$, on the axes given [below].

Note I copied the $y = f(x)$ graph onto the second graph to help show the relationship between the graphs.

(8) 5. a) Find numbers K and L so that $K \le 2+5\sin\big(x^2+4x\big) \le L$ for all numbers x. Give evidence to support your answer. **Answer** The values of sine are all in the interval $[-1,1]$ so the values of $2+5\sin(\text{anything})$ must be between $2+5(-1) = -3 = K$ and $2+5(1) = 7 = L$.

b) Suppose $f(x) = x^3 + 2 + 5\sin\big(x^2+4x\big)$. What is the sign of $f(2)$? What is the sign of $f(-2)$? You may use your answers to a) to explain your answers here. **Answer** $(-2)^3 = -8$ so $f(-2)$ is the sum of -8 and a number between -3 and 7. Therefore $-11 \le f(-2) \le -1$: $f(-2)$ is negative. $2^3 = -8$ so $f(2)$ is the sum of 8 and a number between -3 and 7. Therefore $5 \le f(2) \le 15$: $f(2)$ is positive.



c) Explain why the equation $f(x) = x^3 + 5\sin\left(x^2+4x\right) = 0$ must have at least one solution. Give an interval in which this solution must be found. You must quote a specific result from this course, explaining its relevance. Your answers to b) may be useful here. **Answer** $f(x)$ is continuous on the interval $[-2,2]$ and the Intermediate Value Theorem applies. Since $f(-2) < 0$ and $f(2) > 0$, the Theorem states that must be some number w in the interval $[-2,2]$ so that $f(w) = 0$.

Note The graph above to the right shows $y = f(x)$ on $[-2,2]$ and gives additional evidence for solutions.

(10) 6. a) If $f(x) = \frac{e^{2x}}{x^3-7}$, what is $f'(x)$? Please do not "simplify" your answer. **Answer** $f'(x) = \frac{e^{2x}2(x^3-7)-e^{2x}(3x^2)}{(x^3-7)^2}$.

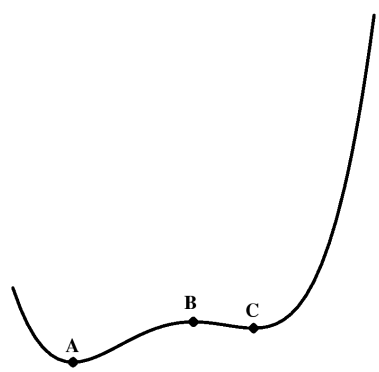
b) If $f(x) = (x^7+\cos x)(4x^4+3x^3)$, what is $f'(x)$? Please do not "simplify" your answer. **Answer** $f'(x) = (7x^6 - \sin x)(4x^4+3x^3) + (x^7+\cos x)(16x^3+9x^2)$.

c) Suppose that $f(x)$ is a differentiable function and that $f(1) = -2$, $f'(1) = 6$, $f(2) = 8$, and $f'(2) = 7$. Suppose that $g(x)$ is a differentiable function and that $g(1) = 2$, $g'(1) = 4$, $g(2) = 5$, and $g'(2) = -3$.

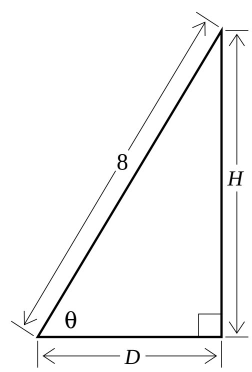
Suppose $F(x) = x^5 f(x)$. Compute $F'(1)$. **Answer** $F'(x) = 5x^4 f(x) + x^5 f'(x)$ so that $F'(x) = 5\cdot 1^4 f(1) + 1^5 f'(1) = 5(-2) + 1(6) = -4$.

Suppose $G(x) = f(g(x))$. Compute $G'(1)$. **Answer** $G'(x) = f'(g(x))g'(x)$ so $G'(1) = f'(g(1))g'(1) = f'(2)(4) = 7\cdot 4 = 28$.

(12) 7. The function $f(x)$ is defined by the formula $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 + 1$ and a graph of the function, without any quantitative information or axes included, is [below] to the right. Use calculus to find exact coordinates of the points **A** and **B** and **C**. **Answer** The points **A** and **B** and **C** are where the tangent line is horizontal. Locate the appropriate x values by finding the roots of the derivative. So $f'(x) = x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1)$. The roots are -2, 0, and 1. Since $f(-2) = \frac{1}{4}(-2)^4 + \frac{1}{3}(-2)^3 - (-2)^2 + 1 = \frac{16}{4} - \frac{8}{3} - 4 + 1 = -\frac{5}{3}$, $f(0) = 1$, and $f(1) = \frac{1}{4}1^4 + \frac{1}{3}1^3 - 1^2 + 1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$, **A** is $\left(-2, -\frac{5}{3}\right)$, **B** is $(0,1)$, and **C** is $\left(1, \frac{7}{12}\right)$.



(10) 8. A ladder which is 8 feet long has one end on flat ground and the other end on the vertical wall of a building. H is the height from the ground to the point at which the ladder touches the building, and D is the distance between the bottom of the ladder and the bottom of the wall. θ is the acute angle between the ladder and the ground.



a) Write H as a function of D. That is, give a formula for H involving D and no other variable. What is the domain of this function when used to describe this problem? (The answer should be related to the problem's geometry.)

Answer By Pythagoras, $D^2 + H^2 = 8^2$ so $H = \sqrt{8^2 - D^2}$. Here $0 \le D \le 8$ is the appropriate domain of the function.

b) Write H as a function of θ. That is, give a formula for H involving θ and no other variable. What is the domain of this function when used to describe this problem? (The answer should be related to the problem's geometry.)

Answer Since $\sin\theta = \frac{H}{8}$, $H = 8\sin\theta$. The appropriate domain is $0 \le \theta \le \frac{\pi}{2}$.

Note The "natural domains" of $\sqrt{8^2 - D^2}$ and $8\sin\theta$ are different from the answers above. The natural domains would be $[-8,8]$ and "all real numbers", respectively. But the triangular diagram only makes sense for the domains given in the answers.