640:135:F2 **Answers to the Take-home Semi-exam** **7/28/2006**

Here are answers that would earn full credit. Other methods may also be valid. Graphs of the functions with some commentary are available on the web grading page. Students were not asked to sketch a graph!

1. Information about $f(x) = \frac{x-1}{e^{2x}}$.

The FUNC level

The domain is all x. The range is more problematic. Let's consider $\lim_{x\to\infty} \frac{x-1}{e^{2x}}$. Here we have polynomial growth (degree 1 in the top) and exponential growth (since the exponential constant, 2, is positive) in the bottom. Exponential growth wins so the limit is 0. Thus the x-axis, the graph of $y = 0$, is a horizontal asymptote. As for $\lim_{x\to\infty} \frac{x-1}{e^{2x}}$, the limit on the "other side", the top $\to -\infty$ and the bottom $\to 0$ and is positive. So I think the limit is $-\infty$. The range is likely to be something like $(-\infty, \text{MAX VALUE}]$ and the MAX VALUE will be identified later. There are *no* vertical asymptotes.

The FUNC′ level

$f'(x) = \frac{3-2x}{e^{2x}}$ after some algebra. The only critical number is $x = \frac{3}{2}$. The critical point is $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)$ and this is $\left(\frac{3}{2}, \frac{\frac{1}{2}}{e^3}\right) = \left(\frac{3}{2}, \frac{1}{2e^3}\right)$. The second coordinate of this point is *positive*. Since the right infinite limit is 0 and the left infinite limit is $-\infty$, this critical point is the absolute maximum. I am essentially using the idea of the "Zeroth Derivative Test" mentioned in class. Of course I could also check the signs of $f'(x)$ to the left and right of $\frac{3}{2}$. The function is increasing in the interval $\left(-\infty, \frac{3}{2}\right]$ and is decreasing in the interval $\left[\frac{3}{2}, +\infty\right)$. The MAX VALUE mentioned earlier is $\frac{1}{2e^3}$ so the range of $f(x)$ is $\left(-\infty, \frac{1}{2e^3}\right]$.

The FUNC″ level

After more algebra, we get $f''(x) = \frac{4(x-2)}{e^{2x}}$. This is 0 only at $x = 2$. Since the exponential function is always positive, the sign of $f''(x)$ is the sign of its top. If $x > 2$, $f''(x)$ is positive and $f(x)$ is concave up. If $x < 2$, $f''(x)$ is negative and $f(x)$ is concave down. The concavity changes at $x = 2$, so $(2, f(2)) = \left(2, \frac{1}{e^4}\right) = (2, e^{-4})$ is the only inflection point.

2. Information about $f(x) = \frac{x^3+1}{x}$.

The FUNC level

Here the domain is all x except 0, so the domain is $x \neq 0$. There are four limits to investigate. $\lim_{x\to\infty} \frac{x^3+1}{x}$ compares cubic growth (x^3) with linear growth (x). I think the cubic growth wins, and the answer is $+\infty$. You could also think this way: $\frac{x^3+1}{x} = x^2 + \frac{1}{x}$, and, as $x \to \infty$, the first term $\to \infty$ and the second term is, at least, positive. Now for $\lim_{x\to-\infty} \frac{x^3+1}{x}$: again (for me, anyway) this is essentially x^2 behavior (x^3 [neglecting the +1] divided by x) for x's very large and negative, so again the limit is $+\infty$. We need to consider $\lim_{x\to0^-} f(x)$. As $x \to 0$ (on either side) the top, $x^3 + 1$, must $\to 1$. The bottom is a small negative number. Therefore we consider the quotient $\frac{1}{\text{SMALL NEGATIVE NUMBER}}$. This is a large negative number. The limit must be $-\infty$. A similar analysis shows that $\lim_{x\to 0^-} f(x) = +\infty$. The y-axis is a vertical asymptote ($x = 0$). There are no horizontal asymptotes. Consideration of the "left" piece of $f(x)$ alone shows that the range of $f(x)$ is all numbers.

The FUNC$'$ level

After algebra, $f'(x) = \frac{2x^3-1}{x^2}$. We can ignore the bottom (!) because 0 is not in the domain of $f(x)$. Also, the bottom doesn't influence the sign of $f'(x)$ since it is squared and always positive. Where is $f'(x) = 0$? $2x^3 - 1 = 0$ becomes $2x^3 = 1$ becomes $x^3 = \frac{1}{2}$ becomes $x = \frac{1}{2^{1/3}}$. There is exactly one number where $f'(x) = 0$, and that number is $\frac{1}{2^{1/3}}$. Since $f\left(\frac{1}{2^{1/3}}\right) = \frac{\frac{1}{2}+1}{\frac{1}{2^{1/3}}} = 3 \cdot 2^{-2/3}$, the only critical point is $\left(\frac{1}{2^{1/3}}, 3\cdot 2^{-2/3}\right)$. What type of critical point is it? For $x > \frac{1}{2^{1/3}}$, $f'(x) > 0$ so $f(x)$ is increasing in the interval $\left(\frac{1}{2^{1/3}}, \infty\right)$. Similarly, for $x < \frac{1}{2^{1/3}}$, $f'(x) < 0$. $f(x)$ is decreasing in the interval $(-\infty, 0)$ and in the interval $\left(0, \frac{1}{2^{1/3}}\right)$. Please note that 0 is not in the domain of $f(x)$, so $f(x)$ can't be increasing in the interval of all x's to the left of the critical point! The only critical point must be a relative minimum. Since as $x \to 0^-$, we already saw that $f(x) \to -\infty$, this is *not* an absolute minimum.

The FUNC$''$ level

More work gets $f''(x) = \frac{2(x^3+1)}{x^3}$. This takes more analysis than $f'(x)$ because the x^3 in the bottom does influence the sign. The only solution of $f''(x) = 0$ is $x = -1$. For $x < -1$, the top is negative, *but* the bottom is negative also (!). So $f''(x)$ is positive in that interval, and $f(x)$ is concave up for $x < -1$. When x is between -1 and 0, the top is positive but the bottom is negative, so $f''(x)$ is negative. In the interval $(-1, 0)$, $f(x)$ is concave down. Therefore $(-1, f(-1)) = (-1, 0)$ is an inflection point. Finally, if $x > 0$, the top and bottom of $f''(x)$ are both positive, so $f(x)$ is concave up in the interval $(0, \infty)$.

3. Information about $f(x) = x + \sin x$.

The FUNC level

The domain of $f(x)$ is all x. What about the range? Realize that $-1 \le \sin x \le +1$. As $x \to \infty$, we have $f(x)$ can be thought of as REALLY LARGE+SOMETHING BETWEEN -1 AND $+1$. So $\lim_{x\to+\infty} f(x) = +\infty$. Similar logic gives $\lim_{x\to-\infty} f(x) = -\infty$. Then, using the Intermediate Value Theorem since this function is continuous everywhere, we conclude that the range of $f(x)$ is all numbers: $(-\infty, +\infty)$. There are no horizontal or vertical asymptotes.

The FUNC$'$ level

Certainly $f'(x) = 1 + \cos x$. This is 0 when $\cos x = -1$. I know where this occurs, at least in the interval $[0, 2\pi]$: at $x = \pi$. But cosine repeats every 2π, so $f'(x) = 0$ at π and $\pi + 2\pi = 3\pi$ and $\pi + 2\cdot 2\pi = 5\pi$ and ... and don't forget $\pi - 2\pi = -\pi$ and $\pi - 2\cdot 2\pi = -3\pi$ and ... Every odd integer (positive or negative) multiple of π is a critical number. At those critical numbers, sine is equal to 0 (since $\sin^2 + \cos^2 = 1$), so at those numbers, $f(x) = x$. The critical points are ((ANY ODD INTEGER)π, (ANY ODD INTEGER)π). What kind of critical points are these? Considering the sign of $f'(x)$ will allow us to answer that. But remember $-1 \leq \cos x \leq 1$ so $1+\cos x$ is always ≥ 0. Therefore $f'(x)$ must be positive where it is *not* 0. $f(x)$ is increasing on both sides of each critical number, so $f(x)$ is *neither* any kind of maximum *nor* any kind of minimum. Most directly: $f(x)$ is increasing for all numbers: $(-\infty, +\infty)$. The only problem can be at the critical numbers, but think: $f(x)$ increases to the left and to the right of each critical number, and $f(x)$ is continuous, so $f(x)$'s values at those numbers must exactly fit "in between", and $f(x)$ is increasing everywhere.

The FUNC″ level

Surely $f''(x) = -\sin x$. First look at $[0, 2\pi]$ and see what happens. The complete behavior as far as concavity is concerned will just repeat every 2π. In $[0, 2\pi]$, $f''(x) > 0$ where sine is negative. That's $(\pi, 2\pi)$. So $f(x)$ is concave up in that interval, and, similarly, $f(x)$ is concave down in $(0, \pi)$. So, generally, there are inflection points at *every* multiple of π and intervals of reversing concavity, each π long, alternate. Between odd and even multiples of π, $f(x)$ is concave up, and between even and odd multiples of π, $f(x)$ is concave down. Students were *not* requested to sketch the graph, but it actually wiggles around the line $y = x$. One collection of inflection points happen also to be critical points (at the odd multiples of π) and another collection of inflection points are at the even multiples of π.

4. Information about $f(x) = \frac{2x^2+9}{x^2+x+3}$.

The FUNC level

The quadratic formula applied to x^2+x+3 gives $\frac{-1\pm\sqrt{1^2-4\cdot 1\cdot 3}}{2\cdot 1}$, but notice that $1^2-4\cdot 1\cdot 3 = -11$ is negative. The bottom of the formula for $f(x)$ has no real roots and therefore the domain of $f(x)$ is all x. The bottom and the top of the formula for $f(x)$ are polynomials of equal degree, 2 (both quadratics). Therefore I conclude that $\lim_{x\to\pm\infty} f(x) = 2$ (*both* limits!). I see this because the top is close to $2x^2$ and the bottom is close to x^2 when x is large in magnitude (either + or −). You can also divide the top and the bottom of the formula defining $f(x)$ by x^2 and get $f(x) = \frac{2+\frac{9}{x^2}}{1+\frac{1}{x}+\frac{3}{x^2}}$, and then as $x \to \infty$ and $x \to -\infty$ (either), a limiting value of 3 should be apparent. There are no vertical asymptotes, but $y = 2$ is a horizontal asymptote. The derivative will give information allowing the range to be determined.

The FUNC′ level

After some work we get $f'(x) = \frac{2x^2-6x-9}{(x^2+x+3)^2}$. The bottom is never 0 and is always positive becase of the square. The roots of the top are $\frac{+6\pm\sqrt{6^2-4\cdot 2\cdot(-9)}}{2\cdot 2}$. This simplifies to $\frac{3}{2}\pm\frac{3}{2}\sqrt{3}$. There are two critical numbers. The function $f'(x)$ is continuous for all x, and can't change sign away from the critical numbers, so its sign in each interval can be determined by just checking it at one (convenient!) point. And I don't need the value of the derivative, only its sign. In $\left(-\infty, \frac{3}{2}-\frac{3}{2}\sqrt{3}\right)$, I'll consider $f'(-1{,}000)$. The x^2 terms are much large than all the other terms, so this is positive. I conclude that $f(x)$ is *increasing* in $\left(-\infty, \frac{3}{2}-\frac{3}{2}\sqrt{3}\right)$. What about the interval $\left(\frac{3}{2}-\frac{3}{2}\sqrt{3}, \frac{3}{2}+\frac{3}{2}\sqrt{3}\right)$? One convenient x is 0, and $f'(0) = -\frac{9}{3}$, negative. $f(x)$ is decreasing in this interval. Similar consideration of $f'(1{,}000)$ (it is positive) verifies that $f(x)$ is increasing in $\left(\frac{3}{2}+\frac{3}{2}\sqrt{3},\infty\right)$. Now we can conclude that $f(x)$ has an absolute max at $x=\frac{3}{2}-\frac{3}{2}\sqrt{3}$ and an absolute min at $x=\frac{3}{2}+\frac{3}{2}\sqrt{3}$ because I know the increasing/decreasing behavior *and* I also know the asymptotic behavior as $x\to\pm\infty$. The range will be the closed interval between $f(x)'s$ values at those two points. This is too darn ugly and I won't do it. I will wait and grade papers and write the nicest solution on the grading page.

The FUNC$''$ level

Much work later a formula for $f''(x)$ appears, and it is $\frac{(2x)(-2x^2+9x+27)}{(x^2+x+3)^3}$. The roots turn out to be 0 and $\frac{9}{4}\pm\frac{3}{4}\sqrt{33}$ (quadratic formula). I'll call the second and third roots r_+ and r_- corresponding to the choice of sign. Notice that $\frac{3}{4}\sqrt{33}$ is about three-quarters of 6 or so, so it is certainly bigger than 3, So $r_\pm$ are $\frac{9}{4}\pm$ SOMETHING BIGGER THAN 3. So $r_-<0<r_+$. The second derivative does change sign at each one of those roots. Why? Because the top is a third degree polynomial, and it has three distinct roots. The bottom is always positive and doesn't influence the sign of $f''(x)$. This much is easy: when x is large positive, $f''(x)$ is negative and when x is large negative, $f''(x)$ is positive. You can get this by examining the coefficients of the highest degree terms in the top ($\underline{-4}x^3$) and bottom ($\underline{1}x^6$). Now I can tell you what happens: in $(-\infty,r_-)$, $f''(x)$ is positive; in $(r_-,0)$, $f''(x)$ is negative; in $(0,r_+)$, $f''(x)$ is positive; in $(r_+,+\infty)$, $f''(x)$ is negative. Notice that the signs are consistent in that the predicted sign in the left–most interval gives later the predicted sign in the right–most interval. Therefore: in $(-\infty,r_-)$, $f(x)$ is concave up; in $(r_-,0)$, $f(x)$ is concave down; in $(0,r_+)$, $f(x)$ is concave up; and, finally, in $(r_+,+\infty)$, $f(x)$ is concave down. $f(x)$ does have changes in concavity at r_- and 0 and r_+, and these are the first coordinates of inflection points. The second coordinates of the inflection points would be gotten by substituting these numbers into $f(x)$ (I know one of them: $f(0)=3$) and for $r_\pm$, this is too darn ugly again and I won't do it. I'll wait and grade papers and write the nicest solution on the grading page.

5. Suppose you know that $y=f(x)$ is a differentiable function, and that $y=5x+7$ is tangent to $y=f(x)$ when $x=2$.

a) What is $f(2)$? What is $f'(2)$?

Answer If $y=5x+7$ is tangent to $y=f(x)$ at $x=2$ then the line must pass through $(2,f(2))$. When $x=2$ on the line, $y=17$. So $f(2)=17$. The slope of the tangent line must be the value of $f'(x)$ at $x=2$. The slope is 5 so $f'(2)=5$.

b) Suppose you additionally know that $f''(2)=8$. If $x=2.03$, then $y=5(2.03)+7=17.15$ (this came from formula for the line given above). Do you suspect that $f(2.03)$ is larger or smaller than 17.15? Explain your answer.

Answer If $f''(2)=8$, the graph is concave *up* near $x=2$. At least a small piece of the graph near $x=2$ is above a small piece of the tangent line near $x=2$. It is likely that $f(2.03)$ will be *greater than* 17.15.