Problem statement a) **The 1-dimensional wave equation** Suppose that $H(x,t)$ represents the height of a vibrating string over the point x of the real line at time t. Then for small vibrations and for homogeneous strings (think of a guitar string) H satisfies the partial differential equation $\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial t^2} = 0$. Verify that if $f(w)$ is any twice-differentiable function of one variable, then $H(x,t) = f(x-t)$ satisfies the wave equation. Comment on how the wave shape (the graph of f) travels along the string as t changes. If H_1 and H_2 satisfy the wave equation, verify that $H_1 + H_2$ and cH_1 (where c is any constant) also satisfy the wave equation. This is called "the principle of superposition".

b) **The Korteweg-de Vries equation** The following paragraph was written in 1844 by John Scott Russell, a Scottish engineer.

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

This quotation is from `http://www.ma.hw.ac.uk/~chris/scott_russell.html`. See also the `Wikipedia` article about John Scott Russell.

This was the first published record of what is now called a *soliton*. In 1895 the phenomenon was described with a partial differential equation named the *Korteweg-de Vries equation* (KdV): $u_t + u_{xxx} + 6uu_x = 0$ (here $u(x,t)$ is the vertical displacement of the wave at time t and position x). Korteweg and de Vries stated the equation with supporting reasoning. This equation is non-linear and does not satisfy the "principle of superposition". KdV and related equations have turned out to be very important both theoretically and in practical applications (fiber optics, transmission of nerve impulses, some chemical reactions). The following exercises are probably more appropriately done with the help of a tool such as `Maple`.

i) If $K > 0$, show that $u_K = \frac{K}{2}\left(\text{sech}\left(\frac{\sqrt{K}}{2}(x - Kt)\right)\right)^2$ is a solution of KdV. Here "sech" means *hyperbolic secant*.

ii) Verify that $7u_1$ and $u_2 + u_3$ are *not* solutions of KdV.

iii) Describe a connection between the speed of the soliton and its maximum height. (In 1885, Russell wrote "The sound of a cannon travels faster than the command to fire it.")

More information `http://math.cofc.edu/faculty/kasman/SOLITONPICS`