For Tuesday, July 18 **YSP problems** **Summer 2006**

You should know about **sample space** ... **event** ... **probability** ... **the rules of the game** ... **independent** ... **random variable** ... **expectation** ...

1. Historians of mathematics generally agree that the systematic study of probability began with an exchange of letters between Fermat and Pascal in 1654. They considered some gambling problems. One was discussed in class. Another is called *the problem of points*. Here is a (fictionalized) version of the problem taken from a web page of `mathforum.org`:

> Pascal and Fermat are sitting in a cafe in Paris and decide, after many arduous hours discussing more difficult scenarios, to play the simplest of all games, flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get 10 points wins. Knowing that they'll just end up taking each other out to dinner anyway, they each ante up a generous 50 francs, making the total pot worth 100. They are, of course, playing 'winner takes all'. But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that a friend is sick, and he must rush to his home town of Toulouse. The carriage man who has delivered the message offers to take him, but only if they leave immediately. Of course Pascal understands, but later, in correspondence, the problem arises: how should the 100 francs be divided?

Solve this problem by computing the probability that each participant will win (assume the coin is fair). Then divide the sum according to these numbers.

2. Pick two points at random from the unit interval, $[0, 1]$. What is the chance that the distance between them is less than $\frac{1}{2}$? (The sample space here is really the unit square!)

3. **Problem JvN** We saw how to "solve" the problem of using an unfair (biased) coin to simulate the tosses of a fair coin. How efficient is the procedure? Description: we are given a coin which shows heads with probability p and tails with probability $q = 1 - p$. We toss the coin two times. If HT shows, we report heads, and if TH shows, we report tails. Otherwise, repeat. *On average* how many tosses or flips will the method take to report either a head or a tail?

Note Your answer should be symmetric in p and q (so the result is the same if they are interchanged). Also, if the coin is very biased (so either p or q is close to 1) then the expected number of tosses should be very large.

4. **Problem NvJ** Suppose you have a fair coin. Describe how to simulate a sequence of H's and T's so that the probability of H is $\frac{2}{3}$ and the probability of T is $\frac{1}{3}$. Can you find the expectation (as in the previous problem) for the number of coin flips needed *on average* to report one "biased" flip?

5. a) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. b) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$.

Useful background quote

Carl Jung (1875-1961) was a founder of modern psychiatry. Here is a quote from his autobiographical book, *Memories, Dreams, Reflections*:

> *. . . All my life it remained a puzzle to me why it was that I never managed to get my bearings in mathematics when there was no doubt whatever that I could calculate properly. Least of all did I understand my own* moral *doubts concerning mathematics.*
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> *Equations I could comprehend only by inserting specific numerical values in place of the letters and verifying the meaning of the operation by actual calculation. As we went on in mathematics I was able to get along, more or less, by copying out algebraic formulas whose meaning I did not understand, and by memorizing where a particular combination of letters had stood on the blackboard. I could no longer make headway by substituting numbers, for from time to time the teacher would say, "Here we put the expression so-and-so," and then he would scribble a few letters on the blackboard. I had no idea where he got them and why he did it–the only reason I could see was that it enabled him to bring the procedure to what he felt was a satisfactory conclusion. I was so intimidated by my incomprehension that I did not dare to ask any questions.*
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> *Mathematics classes became sheer terror and torture to me. . . .*

Spoilers: 68.75 and 31.25 .75 6 and 26