For Wednesday, July 19 **YSP problems** **Summer 2006**

You should know about **cardinality** … **bijection** … **countable** … **countably infinite** … **uncountable** …

1. a) $[0,1]$ and $(0,1]$ have the same cardinality. One way: give a bijection between the sets.

b) $[0,1]$ and $(0,1)$ have the same cardinality. One way: give a bijection between the sets.

c) $[0,1]$ and $\mathbb{R}$ (all the real numbers) have the same cardinality. One way: give a bijection between the sets.

2. The sets $[0,1]$ (the unit interval) and $[0,1] \times [0,1]$ (the unit square) have the same cardinality.

Hint Look at the point $.2\boldsymbol{3}4\boldsymbol{5}5\boldsymbol{8}2\boldsymbol{0}9\boldsymbol{9}4\ldots$ in $[0,1]$ which could correspond to the point $(.245294\ldots, .\boldsymbol{35809}\ldots)$ in the square.

3. Create a set analogous to *the* Cantor set by removing the middle one-quarter of the unit interval and then the middle one-quarter of each of the two remaining intervals and then the middle one-quarter of each of the four … What is the probability of a point being in this set? Try to show that this set *does not* contain any interval of positive length (subtle).

4. The countable union of countable sets is countable. If n is a positive integer, define $\mathcal{A}_n$ to be the set of <u>all</u> real solutions of to <u>all</u> polynomial equations with integer coefficients which have degree equal to n. For example, one equation giving some elements of $\mathcal{A}_5$ is $7x^5 - 9x^3 + 44x + 23 = 0$.

a) Show that the set of polynomials with integer coefficients which have degree equal to n is a countably infinite set. Since each equation of degree n has at most n solutions, show that the set $\mathcal{A}_n$ is countable.

b) The union of all the $\mathcal{A}_n$'s is called the set of *algebraic numbers*. Show that $\mathcal{A}$ is countable.

c) Since $\mathbb{R}$ is not countable, and $\mathbb{R}$ contains $\mathcal{A}$, show that there are elements of $\mathbb{R}$ which are not algebraic. These numbers are called *transcendental*.

5. Suppose X is a set. Suppose that $\mathcal{S}(X)$ is the collection of subsets of X. For example, if $X = \{a,b,c\}$, then (tediously!) $\mathcal{S}(X) = \big\{\{\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\big\}$. If $F\colon X \to \mathcal{S}(X)$ is a function, define the subset* NI_F of X to be those x's in X so that x is *not* a member of $F(x)$. Show that there is no x in X so that $F(x) = NI_F$. Therefore the mapping F can't be onto and even more therefore X and $\mathcal{S}(X)$ can't have the same cardinality and even, even more: there is no largest cardinal number!

6. Suppose X is any sample space and P is the probability "measure".

a) How many *disjoint* events $A_1, A_2, \ldots, A_n$ are possible with $P(A_j) > \frac{1}{73}$ for all j?

b) How many *disjoint* events $A_1, A_2, \ldots, A_n$ can there be with $P(A_j) > \frac{1}{206}$ for all j?

c) Show that the cardinality of any collection of disjoint events with positive probability is at most countably infinite.

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* NI could stand for *not in*.

Useful background quote

`Wikipedia` declares that "Johann Wolfgang Goethe (1749-1832) was ... a painter, novelist, dramatist, poet, humanist, scientist, philosopher, and for ten years chief minister of state at the Duchy of Weimar." He was "one of the key figures of German literature" and certainly another certified really, really smart person. His Maxim #1278, appearing in his *Maxims and Reflections*, is:

> *Mathematicians are like a certain type of Frenchman: when you talk to them they translate it into their own language, and then it soon turns into something completely different.*

Isn't that ... nice?

Some (more relevant) history

Although I generally don't want to say too much truth in one session, please realize almost all of the very clever observations during this class and in the homework assignment were due to **Georg Cantor** (1845–1918). His mathematical work met with great opposition in his life, and finally a *paradigm shift* occurred. Historian of science Thomas Kuhn used this phrase to indicate that the same information is applied and understood in an entirely different way. Here's a less kind but perhaps more accurate assessment of what happens during such a change. The quote is from the famous physicist, Max Planck: "A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it." Sigh. I believe this remark was made in connection with some of the controversies surrounding quantum mechanics.

I can recommend two brief biographies of Cantor on the web. One is at `http://www-history.mcs.st-andrews.ac.uk/history/Biographies/Cantor.html` and the other is at `http://en.wikipedia.org/wiki/Georg_Cantor`. The latter webpage has several quotes of Cantor regarding problem #2 of this assignment:

From a letter dated January 5, 1874:

> *Can a surface (say a square that includes the boundary) be uniquely referred to a line (say a straight line segment that includes the end points) so that for every point on the surface there is a corresponding point of the line and, conversely, for every point of the line there is a corresponding point of the surface? I think that answering this question would be no easy job, despite the fact that the answer seems so clearly to be "no" that proof appears almost unnecessary.*

Cantor solved the problem and wrote in 1877:

> *I see it, but I don't believe it!*

That's how this **PARADIGM SHIFT** was perceived even by its inventor/discoverer/author!