For Friday, July 21

YSP problems

Summer 2006

You need to know about **pigeon hole principle** … **complete graph (K_n)** … **Ramsey number** … **Boole's inequality (the easiest Bonferroni inequality)** … **greed*** … **random search** …

1. An *integer lattice point* is a point whose coordinates are integers (positive or negative). So $(9, 3)$ and $(-4, 2)$ and $(7, 0)$ are integer lattice points in the plane.

a) If you have 5 integer lattice points in the plane, then the midpoint of the line segment joining at least one pair of them must also be an integer lattice point.

b) A similar statement is true for integer lattice points in space, where three numbers form the coordinates *if* 5 is changed to another number. Find that number and verify the result.

2. Fifty-one points are scattered inside a square with a side of 1 meter. Prove that some set of three of these points can be covered by a square with side 20 cm.

3. One million trees grow in a forest. It is known that no tree has more than 600,000 leaves. Show that at any moment there are two trees in the forest that have exactly the same number of leaves.

4. Find an overestimate of R_5 using the method of Thursday's lecture.

Note In fact, it isn't too hard to get an overestimate of R_n for any positive integer n using that method. This also shows that R_n *exists* which is not immediately obvious. So this provides some justification of the statement that any sufficiently large *random* structure must have small scale ordered behavior.

5. Take K_n and *three-color* the graph: make each edge red or green or blue. How large should n be to guarantee that there is a monochromatic K_3? This gets an over-estimate of the appropriate Ramsey number. Can you get an underestimate: that is, find a specific three-colored K_n (n as large as possible) with no monochromatic K_3?
How about red or green or blue or yellow (four-coloring)?

6. Find an underestimate of R_6 using the *probabilistic method* of Friday's lecture.

***** In the algorithmic sense!

What's known?

I showed that $9 \leq R_4 \leq 63$. R_4 is equal to 18. That R_4 is *exactly* 18 is not obvious! This is what's known now about the next two Ramsey numbers: $43 \leq R_5 \leq 49$ and $102 \leq R_6 \leq 165$.

Useful background information and quote

Joel Spencer tells this anecdote about Paul Erdős (1913–1996), who was one of the greatest and strangest mathematicians of the twentieth century. Erdős essentially invented the probabilistic method and applied it to many problems.

> *Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R_5 or they will destroy our planet. In that case, he claims, we should marshall all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R_6. In that case, he believes, we should attempt to destroy the aliens.*

This quotation is from Joel Spencer's book, *Ten Lectures on the Probabilistic Method*, 1987, Society for Industrial and Applied Mathematics (a paperback, 78 pages, $29). Much of this book can be read and enjoyed by most YSP participants.

Here are two books about Paul Erdős and his highly unusual life:

Paul Hoffman, *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*, 1999, Little Brown (a paperback, 320 pages, $10).

Bruce Schechter, *My Brain Is Open: The Mathematical Journeys of Paul Erdős*, 2000, Touchstone Books (a paperback, 224 pages, $10).

The following remark of Groucho Marx (1890–1977) could apply to almost everything we have discussed:

> *A child of five would understand this. Send someone to fetch a child of five.*