Lecture Notes on the Local Equivalence Problems
For Real Submanifolds in Complex Spaces

Xiaojun Huang *
Department of Mathematics, Rutgers University at New Brunswick, USA

§1. Global and Local Equivalence Problems

There is a classical theorem in complex analysis, called the Riemann mapping theorem, which states that any simply connected domain in $\mathbf{C}$ is either holomorphically equivalent to $\mathbf{C}$ or to the unit disk. For more general domains in $\mathbf{C}$, He-Schramm showed [HS] that if ∂D has countably many connected components, then D is holomorphically equivalent to a circle domain whose boundaries are either points or circles. These results give a nice picture on the holomorphic structures for domains in $\mathbf{C}$. When one goes to higher dimensions, a natural question is then to investigate the complex structure for domains in $\mathbf{C}^n$ for $n \geq 2$. More precisely, given two domains in $\mathbf{C}^n$, one would like to know if there is a biholomorphic map between them. This the so-called global equivalence problem in several complex variables. Along these lines of investigations, substantial progress has been made in the past 30 years ([Fe], [CM], [BSW], etc.). However, we are still a certain big distance away from getting a relatively complete picture as in the one complex variable.

An approach to the study of the equivalence problem is to attach holomorphic invariants to each given domain. Since domains in $\mathbf{C^n}$ are open complex manifolds, many (interior) invariants which are crucial for the study of compact complex manifolds are difficult even to define. As already observed by Poincaré about 100 years ago, the interior complex structure of a domain D in $\mathbf{C}^n$ for $n > 1$ is closely related to the partial complex structure in its boundary, which is the so-called CR structure. Hence, the classification of the complex structures for domains in $\mathbf{C}^n$ may be reduced to the equivalence problem for the boundary CR structures. Indeed, this idea has been proved to be fundamental through the work of Cartan, Tanaka, Chern-Moser, etc.. And it indeed led to the solutions to many questions.

---

* Supported in part by NSF-0200689 and a grant from the Rutgers University Research Council

To illustrate what we said above, we give the following classical example of Poincaré:

Proposition 1.1: Let $\mathbf{B}^n = \{z \in \mathbf{C}^n \ : |z| < 1\}$ and $\Delta^n = \Delta \times ... \times \Delta := \{(z_1, \cdots, z_n) : |z_j| < 1\}$. $\mathbf{B}^n$ and Δ^n are diffeomorphic to each other. But $\mathbf{B}^n$ is not biholomorphic to Δ^n.

Proof of Proposition 1.1: Suppose that there is a biholomorphic map f from Δ^n to $\mathbf{B}^n$. Let $p = (p_1, ..., p_n) \in \Delta^n$ be such that $f(p) = 0$. Let $\sigma_j \in Aut(\Delta)$ be such that $\sigma_j(0) = p_j$. Write $\sigma(z_1, ..., z_n) = (\sigma_1(z_1), ..., \sigma(z_n))$ and $F = f \circ \sigma$. Then F is also a biholomorphic map from Δ^n to $\mathbf{B}^n$ with $F(0) = 0$.

Write $F(z) = zA + \sum a_\alpha z^\alpha$. Write τ_θ for the map sending z to $e^{i\theta}z$, and define $F_\theta = \tau_\theta^{-1} \circ F \circ \tau_\theta : \Delta^n \to \mathbf{B}^n$. Then F_θ has the following Taylor expansion at 0: $Az + \sum e^{i(|\alpha|-1)\theta} a_\alpha z^\alpha$. Since $\mathbf{B}^n$ is convex, the map $\frac{1}{2\pi}\int_0^{2\pi} F_\theta(z) d\theta = Az$ still maps Δ^n to $\mathbf{B}^n$. Applying the same argument to F^{-1}, we similarly conclude that the $A^{-1}z$ maps $\mathbf{B}^n$ to Δ^n. Hence, $\mathbf{B}^n$ and Δ^n are holomorphically equivalent through the linear map Az. This yields a contradiction; for $\mathbf{B}^n$ has a smooth boundary, while the boundary of Δ is only Lipschitz continuous. ■

The key step in the proof of the above proposition is to find a better behaved map so that it induces a nice boundary map. Hence the existence of the holomorphic equivalence map imposes the 'match-up' of certain boundary geometry. In the case considered above, the group structure of the domains allows us to get a very rigid map, which can be actually made to be linear. In general, since most domains have trivial automorphism groups (see [GK], for instance), it is unrealistic to conjecture that the holomorphic equivalence of two domains must induce the linear equivalence of their boundary. A fundamental result by the work of C. Fefferman [Fe] and Bochner (see, e.g. [Ho] or [Kr]) asserts that for two bounded smooth strongly pseudoconvex domains, they are holomorphic equivalent if and only if their boundaries are CR equivalent. To state precisely the result of Bochner and Fefferman, we recall the following definition [Kr].

Let $D \subset\subset U$ be a bounded domain in $\mathbf{C}^n$ with defining function $r \in C^\alpha(U)$, where $\alpha \geq 2$. Namely, we assume that $r < 0$ in D, $r > 0$ in $U \setminus \bar{D}$ and $\mathrm{d}r|_{\partial D} \neq 0$. (We call D a domain with C^α-smooth boundary.) Define the Levi form of r by

$$\mathcal{L}_{r,p}(\xi, \xi) = \sum \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}|_p \xi_j \overline{\xi_k}.$$

We call D is pseudoconvex (or strongly pseudoconvex) at $p \in \partial D$ if $\mathcal{L}_{r,p}(\xi, \xi) \geq 0$ (or, $\mathcal{L}_{r,p}(\xi, \bar{\xi}) \geq C|\xi|^2$ with $C > 0$, respectively) for any $\xi = (\xi_1, \cdots, \xi_n)$ with $\sum_{j=1}^n \xi_j r_{z_j}(p) = 0$. D is called a pseudoconvex domain (or, a strongly pseudoconvex domain) if D is pseudoconvex (or, strongly pseudoconvex, respectively) at any boundary point p.

More generally, for a real submanifold $M \subset \mathbf{C}^n$ of real codimension k. We define CT_pM to be the collection of vectors: $\mathcal{L} = \sum (a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \overline{z_j}})|_p$, such that $\mathcal{L}(g) = 0$ for any function g, which is smooth in a neighborhood of M and is constant along M. CT_pM is called the complex tangent vector space of M at p. We define the holomorphic and conjugate holomorphic tangent vector space of M at p to be

$$T_p^{(1,0)}M = CT_pM \cap T_p^{(1,0)}\mathbf{C}^n, \;\; T_p^{(0,1)}M = CT_pM \cap T_p^{(0,1)}\mathbf{C}^n, \text{ respectively.}$$

Write $CR_p(M) = \dim_{\mathbf{C}} T_p^{(1,0)}M$. $CR_p(M)$ is called the CR dimension of M at p. It is easy to see that $CR_p(M)$ is an upper semi-continuous function in p, which is the simplest holomorphic invariant that one can attach to the germ of a real submanifold in $\mathbf{C}^n$. When $CR_p(M)$ is identically 0, we call M a totally real submanifold. When $CR_p(M)$ is a positive constant, we call M a CR submanifold of $\mathbf{C}^n$. Notice that in case M is a real hypersurface, $CR_p(M) \equiv n-1$ and thus M must be a CR submanifold for $n \geq 2$.

Let M and M' be two CR submanifolds of $\mathbf{C}^n$ for $n \geq 2$. We call M and M' to be CR equivalent if there is a smooth diffeomorphism F from M into M' such that $F_*(T_Z^{(1,0)}M) = T_{F(Z)}^{(1,0)}M'$ for any $Z \in M$. Such a map F is called a smooth CR diffeomorphism from M to M'. We call two germs of real submanifolds (M_j, p_j) with $p_j \in M_j$ to be CR equivalent if there is a CR diffeomorphism from a small neighborhood of p_1 in M_1 to a small neighborhood of p_2 in M_2, which maps p_1 to p_2. The following theorem, called the Bochner-Fefferman theorem, is crucial to reduce the equivalence problem for domains to the study of the boundary CR equivalence problem:

Theorem 1.2 (Bochner-Fefferman [Fe] [Ho]): Let D_1 and D_2 be bounded strongly pseudoconvex domains in $\mathbf{C}^n$ with C^∞ boundaries. Then D_1 and D_2 are biholomorphically equivalent if and only if there is a smooth CR equivalence map from ∂D_1 to ∂D_2.

For any real submanifolds M and M' in $\mathbf{C}^n$, we call M and M' to be holomorphically equivalent if there is a biholomorphic map Φ from a neighborhood of M to a neighborhood of M' in $\mathbf{C}^n$ such that $\Phi(M) = M'$. Apparently, when M and M' are holomorphically equivalent, then they are automatically CR equivalent. By the work of many people (see [CM], [Le], [Pi], [BJT], etc.), it is now clear that when M and M' are real analytic CR submanifolds with some extra geometric restrictions, the CR equivalence of M with M' implies their holomorphic equivalence. For instance, the following is a special case of the Baouendi-Jacobowitz- Treves theorem: (For more references on this matter, we refer the reader to the book of Baouendi-Ebenfelt-Rothschild [BER1] or the survey paper [Hu1]):

Theorem 1.3 (Baouendi-Jacobowitz-Treves [BJT]): Let M_1 and M_2 be two real analytic hypersurfaces in $\mathbf{C}^n$. Suppose that M_1 and M_2 do not contain any non-trivial holomorphic curves. Then any smooth CR equivalence map from M_1 to M_2 is actually a holomorphic equivalence map from M_1 to M_2.

We notice that by a result of Diederich-Fornaess [DF], any compact real-analytic submanifold in $\mathbf{C}^n$ does not contain any non-trivial germs of complex analytic curves. We also mention that Theorem 1.3 follows from the more general theory of Chern-Moser, when both M_j are Levi non-degenerate. (See the next section for more notation on this matter.)

Different from the situation in one complex variable, in the $70's$, Pinchuk and Vitushkin first showed that germs of local holomorphic equivalences between strongly pseudoconvex hypersurfaces can be extended to the global holomorphic equivalence maps under certain geometric assumptions for the hypersurfaces. (See [Vit] for references). This gives the evidence that for many important classes of domains, the local CR structures of their boundaries essentially determine their interior global complex structures. There have been many developments along these lines of research. Here, we only state the following theorem recently obtained in [HJ1] and refer the reader to [HJ1] for more references on this matter:

Thereon 1.4 (Huang-Ji [HJ1]): Let D be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$ defined by a real polynomial. If there is a point $p \in \partial D$ such that a small piece of ∂D near p is CR equivalent to a small piece of the unit sphere $\partial \mathbf{B}^n$, then D must be biholomorphic to the unit ball $\mathbf{B}^n$.

With the above discussions, it is also natural to study the local holomorphic equivalence problem for real submanifolds in complex spaces. Namely, one can consider the following two problems:

Question 1.5: Let (M_j, p_j) be real submanifolds in $\mathbf{C}^n$. When is there a biholomorphic map F from a neighborhood U_1 of $p_1 \in M_1$ in $\mathbf{C}^n$ into a neighborhood of $p_2 \in M_2$ in $\mathbf{C}^n$ such that $f(M_1 \cap U_1) \subset M_2$?

Question 1.6: Let (M_j, p_j) be CR submanifolds in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively, with $N \geq n$. Classify all CR embeddings from (M_1, p_1) to (M_2, p_2) up to the CR automorphism groups: $\mathrm{Aut}(M_1)$ and $\mathrm{Aut}(M_2)$.

Question 1.6 is more along the lines of CR rigidity problems, which, unfortunately, we can only briefly touch in §2 of this lecture notes, due to the time limit. In the following sections, we will mainly address some of the recent work on Question 1.5.

Acknowledgment: The present notes grew out of the lectures I gave in the summer graduate school held at Martina Franca, Italy, June-July, 2002. The materials discussed are from the research work carried out in [CM] [Ch] [Mos] [MW] [HJ2] [EHZ1], etc.. Also, we mention to the reader the books by Krantz [Kr], Baouendi-Ebenfelt-Rothschild [BER1] and R. Gardner [Ga], where one can find most of the necessary prerequisites. We thank G. Zampieri and D. Zaitsev for their invitation and hospitality during my pleasant stay in Italy in the summer of 2002. Thanks are also due to S. Ji and D. Zaitsev for their generous help provided during the preparation of the notes.

§ 2. Formal Theory for Levi Non-degenerate Real Hypersurfaces

Let (M, p) be the germ of a real hypersurface in $\mathbf{C}^n$ $(n > 1)$ near p. We will construct the holomorphic invariants of M at p such that we can distinguish the hypersurfaces by reading off their invariants. In this section, we will use the formal power series method. There is a more geometric approach based on the ideas of E. Cartan, that we will address in §5. In the power series method, we will try to find good representation for the hypersurfaces, called their normal form. The invariants are then embedded in the coefficients of their normal form. This section is based on the papers [CM] [EHZ1].

2.1: General theory for formal hypersurfaces: We let (M, p) be a germ of real (formal) hypersurface in the complex n-space with $n \geq 2$. First, after a local change of coordinates, we assume that $p = 0$, $TM = \{v = 0\}, T^{(1,0)}M = \{w = 0\}$, where we use $(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C}$ for the coordinates of $\mathbf{C}^n$ and write $w = u + iv$. Then M near 0 is defined by an equation of the form: $v = \rho$ with $\rho(0) = d\rho(0) = 0$. Notice that ρ is real-valued. Write

$$\rho = \sum a_{k\bar{l}} z_k \overline{z_\ell} + \sum b_{kl} z_k z_l + \sum \overline{b_{kl}} \overline{z_k z_l} + \sum \overline{e_k z_k} u + \sum e_k z_k u + du^2 + O(|(z, w)|^3).$$

Then we have on M:

$$Re\{-iw - 2\sum_{kl} b_{kl} z_k z_l - 2\sum_k c_k z_k u\} = \sum a_{k\bar{l}} z_k \overline{z_l} + du^2 + O(|(z, w)|^3).$$

Define

$$\begin{cases} w' = w - 2i\sum_{kl} b_{kl} z_k z_l - 2i\sum_k c_k z_k w - diw^2, \\ z' = z. \end{cases}$$

In the (z', w') coordinates, M can be expressed as the graph of the following function:

$$v' = \sum a_{k\bar{l}} z'_k \overline{z'_l} + O(3) = z' A \overline{z'}^t + O(3)$$

where $A = \overline{A}^t$ is a matrix. Write

$$A = \Gamma \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \cdots & \cdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_{n-1} \end{pmatrix} \overline{\Gamma}^t = \Gamma \Lambda \overline{\Gamma}^t,$$

Then

$$v' = z' \Gamma \Lambda \overline{(z'\Gamma)}^t + O(3).$$

Let $z'' = z'\Gamma$, $w'' = w'$. We have $v'' = \sum \lambda_j |z''_j|^2 + O(3)$. We say that $p = 0$ is a Levi non-degenerate point of M if $\lambda_j \neq 0$ for each j.

Assume, for the rest of this section, that M is Levi non-degenerate at 0. Then without loss of generality, we can assume that

$$v'' = \sum \epsilon_j \left| \sqrt{|\lambda_j|} z''_j \right|^2 + O(3),$$

where $\epsilon_j = -1$ if $j \leq \ell$; and $\epsilon_j = 1$ if $j > \ell$. With $z'''_j = \sqrt{|\lambda_j|} z''_j$, $w''' = w''$. Then in the (z''', w''') coordinates, M is the graph of the following function:

$$v''' = \sum \epsilon_j |z'''_j|^2 + O(3).$$

Still write z for z''' and w for w'''. Then M is defined by:

$$v = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2 + O(|(z, w)|^3). \tag{2.0}$$

In the above expression and for the rest of this section, when $\ell = 0$, we regard the first term after the equality sign to be zero. Replacing (z, w) by $(z_{\ell+1}, \cdots, z_{n-1}, z_1, \cdots, z_\ell, -w)$ if necessary, we can assume that $\ell \leq \frac{n-1}{2}$. The pair $(\ell, n-1-\ell)$ is called the signature of M at 0. The model of Levi non-degenerate hypersurfaces with signature $(l, n-1-l)$ is the hyperquadric defined as follows:

$$\mathbf{H}^n_\ell = \{v = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\}. \tag{2.0$'$}$$

Notice that the pair $(\ell, n-1-\ell)$ is completely determined by ℓ. Hence, in what follows, for brevity, we call ℓ the signature of the above hypersurface M.

When $\ell = 0$, we call M strongly pseudoconvex. Also, when $\ell = 0$, $\mathbf{H}_0^n = \mathbf{H}^n$ reduces to the classical Heisenberg hypersurface. Let

$$(2.0)'' \qquad \mathbf{S}_\ell^n := \{v > -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\},$$

which is called the Siegel upper half-space and has $\mathbf{H}_\ell^n$ as its real analytic boundary. Let

$$(2.0)''' \qquad \mathbf{B}_\ell^n := \{1 + |z_1|^2 + ... + |z_\ell|^2 \geq |z_{\ell+1}|^2 + ... + |z_{n-1}|^2 + |w|^2\}.$$

Define $\Psi_n = \left(\frac{2z}{i+w}, \frac{i-w}{i+w}\right)$. Then

$$\Phi_n = \Psi_n^{-1} = \left(\frac{2z}{1+w}, \frac{i-iw}{1+w}\right).$$

Both Ψ_n and Φ_n are called the Cayley transformations. It is easy to verify the following properties:

Lemma 2.1: Ψ_n is a bimeromorphic map from $\mathbf{S}_\ell^n$ to $\mathbf{B}_\ell^n$; and Ψ_n bimeromorphically maps $\mathbf{H}_\ell^n = \partial \mathbf{S}_\ell^n$ to $\partial \mathbf{B}_\ell^n$. In particular, Ψ_n is a holomorphic equivalence map from $(\mathbf{H}_\ell^n, 0)$ to $(\partial \mathbf{B}_\ell^n, 0)$.

For convenience of the discussion, we set up some notation to be used for the rest of this section.

For two m-tuples $x = (x_1, \cdots, x_m), y = (y_1, \cdots, y_m)$, we write $< x, y >_\ell = \sum_{j=1}^m \delta_{j,\ell} x_j y_j$, and $|x|_\ell^2 = \sum_{j=1}^n \delta_{j,\ell} |x_j|^2$. Here $\delta_{j,\ell}$ is defined to be -1 for $j \leq \ell$ and to be 1 otherwise. We define the matrix $E_{\ell,n-1}$ to be the diagonal matrix with its first ℓ diagonal elements -1 and the rest 1.

Parameterize $\mathbf{H}_\ell^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|_\ell^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a nonnegative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball M of 0 in $\mathbf{H}_\ell^n$ is said to be of quantity $o_{wt}(m)$, if $\frac{h(tz, t\overline{z}, t^2 u)}{|t|^m} \to 0$ uniformly for (z, u) on any compact subset of U as $t (\in \mathbf{R}) \to 0$. (In this case, we write $h = o_{wt}(m)$). By convention, we write $h = o_{wt}(0)$ if $h \to 0$ as $(z, \overline{z}, u) \to 0$). For a smooth function $h(z, \overline{z}, u)$ defined over U, we use $h^{(k)}(z, \overline{z}, u)$ for the sum of terms of weighted degree k in the weighted expansion of h up to order k. If h is not specified, we use it to denote a weighted homogeneous polynomial of weighted degree k. For a weighted homogeneous holomorphic polynomial of degree k, we use the notation: $(\cdot)^{(k)}(z, w)$, or $(\cdot)^{(k)}(z)$ if it depends only on z.

Next returning to (2.0), we would like to simplify terms in $O(|(z,w)|^3)$ by further changes of coordinates. These changes of coordinates should have the following properties:

(i). Preserves the origin and the real tangent space $\{v=0\}$ of the hypersurfaces at the origin.

(ii). Preserves the complex tangent space $\{w=0\}$ at the origin.

(iii). Preserve the hyperquadric $\mathbf{H}_\ell^n$ up to weighted order 3.

Let $(z',w')=F=(f,g)$ be such a map. Then the general form that F can take, with the properties in (i)-(iii), is as follows:

$$f = zA + \vec{a}w + O(|(z,w)|^2), \quad g = \lambda w + O(|(z,w)|^2) \tag{2.1}$$

with $\lambda \in \mathbf{R}\setminus\{0\}$.

Since F preserves $v=|z|_\ell^2$ up to the third order, it follows that $\lambda>0$ if $\ell<\frac{n-1}{2}$ and $AE_{\ell,n-1}\overline{A}^t=\lambda E_{\ell,n-1}$ in general. The following proposition indicates that we can further limit down our transformation group to make calculations more accessible:

Proposition 2.2: For any transformation F of the form in (2.1), there is a unique $T\in Aut_0(\mathbf{H}_\ell^n)$ such that $F=T\circ F_0$ with $F_0=(f_0,g_0)$ having the following properties:

$$f_0 = z + O(|(z,w))|^2), \quad g_0 = w + O(|(z,w))|^2), \quad Re\left(\frac{\partial^2 g}{\partial w^2}(0)\right) = 0. \tag{2.2}$$

In fact, by a straightforward verification, we can set $T=T_1\circ T_2$. Here,

(a) If $l=\frac{n-1}{2}$ and $\lambda<0$, then $T_2(z,w)=(z_{\ell+1},\cdots,z_{n-1},z_1,...,z_\ell,-w)$. Otherwise T_2 is always set to be the identity.

(b)When $\lambda>0$, we have

$$T_1 = \left(\frac{(z+\vec{a}\cdot w)A}{q(z,w)}, \frac{\lambda w}{q(z,w)}\right)$$

where $q(z,w)=1-2i\langle z,\overline{\vec{a}}\rangle_\ell+(r-i|\vec{a}|_\ell^2)w$. For $\lambda<0$, one can similarly define T_1.

We next normalize $(M,0)$ by transformations satisfying (2.2). Of course, the invariants we get in this way are still subject to the action of $\mathrm{Aut}_0(\mathbf{H}_\ell^n)$, which is a finite dimensional Lie group.

By induction, suppose that we have found a coordinates system: (z,w), in which M has been normalized up to the weighted order s. We then want to see how to choose the new coordinates (z',w') to get the invariant form at the level of weighted order $(s+1)$.

We first mention that for a formal power series $N(z,\overline{z},w,\overline{w})=N(z,\overline{z},u,v)$, we have the decomposition $N=\sum_{s=0}^\infty N^{(s)}(z,\overline{z},u,v)$, where $N^{(s)}(tz,\overline{tz},t^2u,t^2v)=t^sN^{(s)}(z,\overline{z},u,v)$ for $t\in\mathcal{R}$.

Suppose that in (z, w)-coordinates, M is given implicitly by

$$v = |z|_\ell^2 + N_1(z, \overline{z}, u, v) \tag{2.3}$$

and in $(z', w') = (f(z, w), g(z, w))$ coordinates system , M is given by

$$v' = |z'|_\ell^2 + N_2(z', \overline{z}', u', v').$$

By what we mentioned above, we want to keep what we have already achieved. Namely, we want to have $N_2^{(\sigma)}(z, \overline{z}, u, |z|_\ell^2) = N_1^{(\sigma)}(z, \overline{z}, u, |z|_\ell^2)$ for $\sigma \le s$. Since we have assumed that (f, g) satisfies the normalization in (2.2), we have

$$(f - z)^{(1)} = 0, \ (g - w)^{(2)} = 0. \tag{2.4}$$

Write the weighted expansion of (f, g) as follows:

$$f = z + \sum_{\sigma \ge 2} f^{(\sigma)}(z, w), \quad g = w + \sum_{\sigma \ge 3} g^{(\sigma)}(z, w), \tag{2.5}$$

where $f^{(\sigma)}(tz, t^2 w) = t^\sigma f^{(\sigma)}(z, w)$; $g^{(\sigma)}(tz, t^2 w) = t^\sigma g^{(\sigma)}(z, w)$. Then we have

$$\begin{aligned} &\mathrm{Im}\left(w + \sum_{\sigma \ge 3} g^{(\sigma)}(z, w)\right) \\ =&|z|_\ell^2 + 2\mathrm{Re} \sum_{\sigma_1 \ge 2} < \overline{z}, f^{(\sigma_1)}(z, w) >_\ell + \sum_{\sigma_1, \sigma_2 \ge 2} < f^{(\sigma_1)}, \overline{f^{(\sigma_2)}} >_\ell + \\ &+ N_2\left(z + \sum f^{(\sigma)}, \overline{z + \sum f^{(\sigma)}}, u + \mathrm{Re}(\sum_{\sigma \ge 3} g^{(\sigma)}), \mathrm{Im}(w + \sum_{\sigma \ge 3} g^{(\sigma)}(z, w))\right), \end{aligned} \tag{2.6}$$

where (z, w) satisfies (2.3). Suppose that $f^{(\tau-1)}$ and $g^{(\tau)}$ have been determined for $\tau < \sigma \le s+1$, we want to find $f^{(\sigma-1)}$ and $g^{(\sigma)}$ for any $\sigma \le s+1$. In particular, we would like to find $f^{(s)}$, $g^{(s+1)}$, and $N_2^{(s+1)}$. Substituting $w = u + iv$ and $v = |z|_\ell^2 + N_1(z, \overline{z}, u, v)$ in (2.6), we have

$$\begin{aligned} \mathrm{Im}(g^{(\sigma)}(z, u + i|z|_\ell^2)) &= 2\mathrm{Re} < \overline{z}, f^{(\sigma)}(z, u + i|z|_\ell^2) >_\ell \\ +N_2^{(\sigma)}(z, \overline{z}, u, |z|_\ell^2) &- N_1^{(\sigma)}(z, \overline{z}, u, |z|_\ell^2) + G^{(\sigma)}(z, \overline{z}, u), \end{aligned} \tag{2.7}$$

where $G^{(\sigma)}$ is completely determined by $f^{(\tau-1)}$ and $g^{(\tau)}$ for $\tau \le \sigma - 1$ and is zero if $(f - z)^{(\tau-1)}, \ (g - w)^{(\tau)} = 0$ for $\tau \le \sigma - 1$. To proceed further, we make the following definition:

Definition 2.3: Let $f = (f_1, \cdots, f_{n-1})$ and g be (formal) holomorphic functions in a neighborhood of 0 in $\mathbf{C}^n$. Suppose that

$$(f, g) = O(|(z, w)|^2) \quad \text{with } \mathrm{Re}(g''_{ww}(0)) = 0. \tag{2.8}$$

Then the map $\mathcal{L}_\ell$, which sends the above (f, g) to the real-valued (formal) analytic function over $\mathbf{H}^n_\ell$ defined below:

$$\mathcal{L}_\ell(f, g) := \mathrm{Im}\left(g(z, w) - 2i < \overline{z}, f(z, w) >_\ell |_{w=u+i|z|^2_\ell}\right) \tag{2.8$'$}$$

is called the Chern-Moser operator.

Returning to (2.7), we get for $\sigma \geq 3$

$$\mathcal{L}_\ell(f^{(\sigma-1)}, g^{(\sigma)}) = N_2^{(\sigma)}(z, \overline{z}, u, |z|^2_\ell) - N_1^{(\sigma)}(z, \overline{z}, u, |z|^2_\ell) + G^{(\sigma)}(z, \overline{z}, u). \tag{2.9}$$

Since our ℓ is always fixed, we will write $\mathcal{L}$ instead of $\mathcal{L}_\ell$ to simplify the notation. Later we will see that $\mathcal{L}$ has the following uniqueness property: If $\mathcal{L}(f^{(\sigma-1)}, g^{(\sigma)}) = 0$ with $\mathrm{Re}(\frac{\partial^2 g^{(\sigma)}}{\partial w^2})|_0 = 0$ and $\sigma \geq 3$, then it follows that $(f^{(\sigma-1)}, g^{(\sigma)}) \equiv 0$.

Since we assumed that $N_2^{(\sigma)} = N_1^{(\sigma)}$ for $\sigma \leq s$, we conclude that $(f^{(\sigma-1)}, g^{(\sigma)}) \equiv 0$ for $\sigma \leq s$ and thus $G^{(s+1)} \equiv 0$. At the level of weighted degree $s+1$, we have

$$\mathcal{L}(f^{(s)}, g^{(s+1)}) = N_2^{s+1}(z, \overline{z}, u, |z|^2_\ell) - N_1^{(s+1)}(z, \overline{z}, u, |z|^2_\ell). \tag{2.10}$$

Notice that $N_1^{(s+1)}(z, \overline{z}, u, |z|^2_\ell)$ is known from the induction assumption. Our purpose is then to choose

$$f^{(s)}(z, w), \quad g^{(s+1)}(z, w)$$

appropriately so that we can make $N_2^{(s+1)}$ as simple as possible. (2.10) suggests us to pick $N_2^{(s+1)}(z, \overline{z}, u, v)$ so that $N_2^{(s+1)}(z, \overline{z}, u, |z|^2_\ell)$ is in the 'complement' of the range of the Chern-Moser operator $\mathcal{L}$.

Definition 2.4: Let $\mathcal{A}^{(s)}$ be a collection of real-valued polynomials of weighted degree s in (z, w) for $s \geq 4$. Let $\mathcal{A} = \oplus_{s \geq 4} \mathcal{A}^{(s)}$. Assume that $0 \in \mathcal{A}$.
(a) We call $\mathcal{A}$ a uniqueness set for the Chern-Moser operator $\mathcal{L}$ if $\mathcal{L}(f, g) = G|_{w=u+i|z|^2_\ell}$ with $G \in \mathcal{A}$ is only solvable when $G = 0$ and $(f, g) = 0$. (As in Definition 2.3, for (f, g) in the domain of the Chern-Moser operator, we always assume that $(f, g) = O(|(z, w)|^2)$ with $\mathrm{Re}(g''_{ww}(0)) = 0$.)

In (b) and (c), we assume further that $\mathbf{H}_\ell^n$ is a uniqueness set for $\mathcal{A}$ in the sense that for any $G_1, G_2 \in \mathcal{A}$, $G_1 \equiv G_2$ if and only if $G_1|_{w=u+i|z|_\ell^2} = G_2|_{w=u+i|z|_\ell^2}$.
(b) $\mathcal{A}$ is called an admissible space for $\mathcal{L}$ if for any $G_1, G_2 \in \mathcal{A}$, the equation $\mathcal{L}(f,g) = (G_2 - G_1)|_{w=u+i|z|_\ell^2}$ has solution (f,g) only when $(f,g) \equiv 0$ and $G_1 \equiv G_2$.
(c). $\mathcal{A}$ is called a normal space if $\mathcal{A}$ is admissible and for any real-valued polynomial $B^{(s)}(z,\overline{z},u)$ of degree $s \geq 4$, there is a unique $G^{(s)} \in \mathcal{A}^{(s)}$ such that $\mathcal{L}(f^{(s-1)}, g^{(s)}) = G^{(s)}|_{w=u+i|z|_\ell^2} - B^{(s)}(z,\overline{z},u)$ is solvable.

We remark that it can be easily proved that any weighted homogeneous polynomial of weighted degree 3, when restricted to $\mathbf{H}_\ell^n$, is in the range of the Chern-Moser operator. Hence, in Definition 2.4, we take $s \geq 4$.

Summarizing the above, we have the following:

Theorem 2.5: (a) Suppose $\mathcal{A}$ is a normal space for the Chern-Moser operator. Then any formal real hypersurface $(M,0)$ can be transformed by a formal power series to a formal hypersurface defined by $v = |z|_\ell^2 + N$ with $N \in \mathcal{A}$. (b) Suppose that $\mathcal{A}$ is an admissible space. Let $(M_j, 0)$ be formal hypersurfaces which are in the $\mathcal{A}$-normal form, namely, M_j are defined by an equation of the form $v = |z|_\ell^2 + N_j$ with $N_j \in \mathcal{A}$. Let F be a formal holomorphic map from $(M_1, 0)$ to $(M_2, 0)$ satisfying the normalization condition (2.2). Then $F \equiv \mathrm{Id}$ and $N_1 \equiv N_2$.

2.2: $\mathcal{H}_k$ -space and hypersurfaces in the $\mathcal{H}_k$-normal form: To be able to make good use of Theorem 2.5, we need to construct the normal space for the Chern-Moser operator. Apparently, the normal space associated to the Chern-Moser operator is not unique. And it is the case that for different problems, one has to use different normal or admissible spaces. In the following, we present two different admissible spaces for the Chern-Moser operator, following the work in [CM] and [EHZ1]. Unfortunately, the one obtained in [EHZ1] is not a normal space and the normal form obtained in terms of that is in the implicit form. However, it is invariant under the action of the group $\mathrm{Aut}_0(\mathbf{H}_\ell^n)$. This makes it very convenient to use in working on certain problems.

We first discuss the space $\mathcal{S}_k^0$ (The $\mathcal{S}_k$ defined in [EHZ1] is slightly more general than the one defined below):

Definition 2.6: For $s \geq 4$, $\mathcal{S}_k^{0(s)}$ is the collection of all real-valued weighted homogeneous polynomials of degree s in $(z,\overline{z},w,\overline{w})$ with the following property: For each $A(z,\overline{z},w,\overline{w}) \in \mathcal{S}_k^{0(s)}$, there is a set of weighted homogeneous holomorphic polynomials

$$E = \{\phi_j(z,w),\ \psi_j(z,w)\}_{j \leq k^*} \text{ with } k^* < \infty,$$
$$\deg_{wt}(\phi_j) = p_j \leq s/2,\ \deg_{wt}(\psi_j) = q_j \geq s/2 \text{ and } p_j + q_j = s$$

$(j \leq k^*)$ such that
(A): $\phi_j(z,w)$ and $\psi_j(z,w)$ have no linear and constant terms in (z,w) for any j;
(B): for each $\tau \leq s/2$, there are at most k $\phi_j's$ in E with $\deg_{wt}(\phi_j) = p_j = \tau$
(C): $A(z,\overline{z},w,\overline{w})$ is real valued for any $(z,w) \in \mathbf{C}^n$, and has the following decomposition:

$$A(z,\overline{z},w,\overline{w})) = \sum_{q_j=p_j} \phi_j(z,w)\overline{\psi_j(z,w)} + 2\sum_{q_j>p_j} \mathrm{Re}(\phi_j(z,w)\overline{\psi_j(z,w)}). \tag{2.11}$$

We define $\mathcal{S}_k^0 := \oplus_{s=4}^{\infty}\mathcal{S}_k^{0(s)}$

What makes $\mathcal{S}_k^0$ convenient to use is the so-called $\mathcal{H}_k$-class contained in $\mathcal{S}_k^0$, which is defined to be the collection of all real-valued formal power series $A(z,\overline{z},w,\overline{w})$ (for $(z,w) \in \mathbf{C}^{n-1}\times\mathbf{C}$) such that

$$A(z,\overline{z},w,\overline{w}) = \sum_{j=1}^{k} \phi_j(z,w)\overline{\psi_j(z,w)} \tag{2.11$'$}$$

where $\phi_j,\ \psi_j$ are formal holomorphic power series in (z,w) which do not contain any constant and linear terms.

As we will see, the $\mathcal{H}_k$-normal form is invariant under the action of $\mathrm{Aut}_0(\mathbf{H}_\ell^n)$. This makes it very convenient to apply in applications. More precisely, let $T \in \mathrm{Aut}_0(\mathbf{H}_\ell^n)$. Then we can write

$$T(z,w) = (\frac{\lambda(z-aw)U}{q(z,w)}, \frac{\sigma\lambda^2 w}{q(z,w)}),\ \text{with}\ q(z,w) = 1 + 2i < z,\overline{a} >_\ell + (r - i < a,\overline{a} >_\ell)w, \tag{2.12}$$

where λ is a non-zero real number, $a \in \mathbf{C}^{n-1}$ and U is a certain $(n-1)\times(n-1)$ matrix such that

$$UE_{\ell,n-1}\overline{U}^t = \sigma E_{\ell,n-1}, \quad \sigma = \pm 1. \tag{2.13}$$

Let M be a formal real hypersurface which is in the $\mathcal{H}_k$-normal form. Namely, M is defined by an equation of the form:

$$v = |z|_\ell^2 + N(z,\overline{z},w,\overline{w}),\ \text{with}\ N \in \mathcal{H}_k.$$

The following lemma, which can be proved easily, makes the $\mathcal{H}_k$-normal form convenient to apply:

Lemma 2.7: Under the above notation and assumption, $T(M)$ is also in the $\mathcal{H}_k$-normal form. In fact, $T(M)$ is defined by an equation of the form:

$$v = |z|_\ell^2 + N_2(z,\overline{z},w,\overline{w}),\ \text{with}\ N_2(z,\overline{z},w,\overline{w}) = \frac{\sigma\lambda^2}{|q\circ T^{-1}|^2}N_1\circ T^{-1}(z,w) \in \mathcal{H}_k. \tag{2.14}$$

The following result from [EHZ1] is basic for the application of Lemma 2.7 and Theorem 2.5 to work on various local equivalence problems:

Theorem 2.8 (Ebenfelt-Huang-Zaitsev)([EHZ1]) (a): $\mathcal{S}_k^0$ is a uniqueness set for the Chern-Moser operator for $k \leq n-2$. (b). $\mathcal{S}_k^0$ is an admissible space for the Chern-Moser operator for $k \leq \frac{n-2}{2}$.

The $\mathcal{S}_k^0$ (or the $\mathcal{S}_k$ in [EHZ1]) is far from being a normal space. It is an open problem how to complete $\mathcal{H}_k$ or $\mathcal{S}_k^0$ for $k \leq \frac{n-2}{2}$ into a normal space. This problem is closely related to the study of the embeddability problem for real analytic Levi non-degenerate hypersurfaces into hyperquadrics.

We refer the reader to the paper [EHZ1] for a proof of Theorem 2.8. Here, we give a proof of the part that $\mathbf{H}_\ell^n$ is a uniqueness set for $\mathcal{S}_k^0$ when $k \leq \frac{n-2}{2}$. We notice that

$$c_1\mathcal{S}_{k_1}^0 + c_2\mathcal{S}_{k_2}^0 \subset \mathcal{S}_{k_1+k_2}^0$$

for any complex numbers c_1 and c_2.

Proposition 2.8′: Let $A(z,\overline{z},w,\overline{w}) \in \mathcal{S}_k^0$ with $k \leq n-2$. Assume that

$$A^0(z,\overline{z},u) := A(z,\overline{z},u+i<z,\overline{z}>_\ell, u-i<z,\overline{z}>_\ell) \equiv 0$$

as a formal power series in $(z,\overline{z},u)$. Then $A(z,\overline{z},w,\overline{w}) \equiv 0$ as a formal power series in $(z,\overline{z},w,\overline{w})$. In particular, $\mathbf{H}_\ell^n$ is a uniqueness set for $\mathcal{S}_k^0$ with $k \leq \frac{n-2}{2}$.

We first observe that if $A(z,\bar{z},w,\bar{w})$ is weighted homogeneous of degree σ, then $A^0(z,\bar{z},u)$ is weighted homogeneous of degree σ. Hence, if we decompose a formal power series $A(z,\bar{z},w,\bar{w})$ into its weighted homogeneous components

$$A(z,\bar{z},w,\bar{w}) = \sum_\sigma A^{(\sigma)}(z,\bar{z},w,\bar{w})$$

then the decomposition of $A^0(z,\bar{z},u)$ is given by

$$A^0(z,\bar{z},u) = \sum_\sigma (A^0)^{(\sigma)}(z,\bar{z},u),$$

where, in the terminology introduced above,$(A^0)^{(\sigma)} = (A^{(\sigma)})^0$. Moreover, if $A \in \mathcal{S}_k^0$, then $A^{(\sigma)} \in \mathcal{S}_k^0$.

Proof of Proposition 2.8′: It is enough to prove the lemma when A is a weighted homogeneous polynomial of degree $s \geq 4$ and

$$A(z, \overline{z}, w, \overline{w}) = \sum_{j=1}^{k^*} \phi_j(z, w)\overline{\psi_j(z, w)},$$

where ϕ_j, ψ_j are weighted homogeneous holomorphic polynomials, without constant or linear terms, of weighted degree p_j and $q_j = s - p_j$, respectively, and where for each τ there are at most k terms with $p_j = \tau$. Let us expand ϕ_j and ψ_j as follows:

$$\phi_j = \sum_{\nu^j + 2\nu^j_w = p_j} a_j^{(\nu^j)}(z) w^{\nu^j_w}, \ \psi_j = \sum_{\mu^j + 2\mu^j_w = s - p_j} b_j^{\mu^j}(z) w^{\mu^j_w}.$$

Then, if we expand $A(z, \bar{z}, w, \bar{w})$ in powers of w, $\bar{w}$, we can write

$$A(z, \bar{z}, w, \bar{w}) = \sum_{m,l} c_{m,l}(z, \bar{z}) w^m \bar{w}^l,$$

where

$$c_{m,l}(z, \bar{z}) = \sum_{j=1}^{k^*} a_j^{(p_j - 2m)}(z)\overline{b_j^{(q_j - 2l)}(z)}$$

and $p_j + q_j = s$. By isolating the terms in $c_{m,l}(z, \bar{z})$ of bidegree (α, β) in $(z, \bar{z})$ (denoted $c_{m,l,\alpha,\beta}(z, \bar{z})$), we conclude that $A(z, \bar{z}, w, \bar{w}) \equiv 0$ if and only if, for every 4-tuple of nonnegative integers (m, l, α, β),

$$c_{m,l,\alpha,\beta}(z, \bar{\xi}) := \sum_{j \in J(m,l,\alpha,\beta)} a_j^{(\alpha)}(z)\overline{b_j^{(\beta)}(z)} \equiv 0,$$

where the index set $J(m, l, \alpha, \beta)$ consists of those $j \in \{1, \ldots, k^*\}$ for which

$$p_j = \alpha + 2m, \quad q_j = \beta + 2l.$$

Observe that, since $A \in \mathcal{S}^0_k$, there are at most $k \leq n - 2$ indices in the set $J(m, l, \alpha, \beta)$ for each (m, l, α, β).

Now, we use the fact that $A^0(z, \bar{z}, u) \equiv 0$ is equivalent to $A(z, \bar{z}, w, \bar{w})$ vanishing on the quadric $\mathbf{H}^n_\ell$, and the usual complexification argument, to conclude that

$$A(z, \bar{\xi}, w, \bar{\eta}) = 0$$

whenever $w = \eta + 2i < z, \bar{\xi} >_\ell$; or, equivalently,

$$\sum_{m,l} c_{k,l}(z, \bar{\xi})(\eta + 2i < z, \bar{\xi} >_\ell)^m \bar{\eta}^l \equiv 0.$$

Assume, in order to reach a contradiction, that $A(z, \bar{z}, w, \bar{w}) \not\equiv 0$. Then, there is a smallest nonnegative integer l_0 such that $c_{m,l_0,\alpha,\beta}(z, \bar{\xi}) \not\equiv 0$ for some m, α, β. Hence, we can factor out η^{l_0} (of course, if $l_0 = 0$, then we do not need to factor anything) from the identity above and get

$$\sum_{m,l \geq l_0} c_{m,l}(z, \bar{\xi})(\eta + 2i < z, \bar{\xi} >_\ell)^m \bar{\eta}^{l-l_0} \equiv 0.$$

By setting $\eta = 0$, we conclude that

$$\sum_{m} c_{m,l_0}(z, \bar{\xi})(2i < z, \bar{\xi} >_\ell)^m \equiv 0.$$

Isolating the terms of bidegree (α, β) above, we deduce

$$\sum_{m} c_{m,l_0,\alpha-m,\beta-m}(z, \bar{\xi})(2i < z, \bar{\xi} >_\ell)^m \equiv 0.$$

It now follows from [Lemma 3.2, Hu2], [Lemma 3.2, EHZ1] that, for every m, γ, μ,

$$c_{m,l_0,\gamma,\mu}(z, \bar{\xi}) \equiv 0,$$

which contradicts the choice of l_0. This completes the proof of Lemma 3.3. ∎

Making use of Lemma 2.7 and Theorem 2.8, one has the following:

Corollary 2.9 ([EHZ1]): Let $(M_1, 0)$ and $(M_2, 0)$ be two germs of formal real hypersurfaces in the $\mathcal{H}_{k_1}$ and $\mathcal{H}_{k_2}$-normal form defined, respectively, by

$$v = |z|_\ell^2 + N_j.$$

Assume that $k_1 + k_2 \leq n - 2$. Then $(M_1, 0)$ is equivalent to $(M_2, 0)$ by a formal holomorphic map if and only if there is an automorphism $T \in \mathrm{Aut}_0(\mathbf{H}_\ell^n)$ such that

$$N_2 = \frac{\sigma\lambda^2}{|q \circ T^{-1}|^2} N_1 \circ T^{-1}, \text{ or } N_1 = \sigma\lambda^{-2}|q(z, w)|^2 N_2 \circ T(z, w). \tag{2.15}$$

Here, we write $T(z,w) = (\frac{\lambda(z-aw)U}{q(z,w)}, \frac{\sigma\lambda^2 w}{q(z,w)})$, with $q(z,w) = 1 + 2i < z, \overline{a}^t >_\ell + (r - i < a, \overline{a} >_\ell)w$, $UE_{\ell,n-1}\overline{U^t} = \sigma E_{\ell,n-1}$, $\sigma = \pm 1$, $\lambda > 0, r \in \mathbf{R}$, and with a a certain $(n-1)$-tuple. In particular, when M_1 is equivalent to M_2, it must hold that $N_1 \in \mathcal{H}_{k_2}$. And the set of all equivalence maps from $(M_1, 0)$ to $(M_2, 0)$ is precisely the collection of the automorphisms T of $\mathrm{Aut}_0(\mathbf{H}^n_\ell)$ which make (2.15) hold. (Hence, any formal equivalence map from $(M_1, 0)$ to $(M_2, 0)$ is given by a convergent power series.)

Let M be in the $\mathcal{H}_k$-normal form, namely, let M be defined by $v =< z, \overline{z} >_\ell + N$ with $N = \sum_{j=1}^k \phi_j \overline{\psi_j} \in \mathcal{H}_k$, where ϕ_j, ψ_j have no constant and linear terms in (z,w). Let $R(M,N)$ be the minimum k to get such an expression for N. Then as an application of Corollary 2.9, we have the following weak invariant property for $R(M,N)$:

Corollary 2.10: (I). Let M_1 and M_2 be in the $\mathcal{H}_k$-normal form for a certain k. If $R(M_1,N_1) + R(M_2,N_2) \le n-2$ and $(M_1,0)$ is equivalent to $(M_2,0)$, then $R(M_1,N_1) = R(M_2,N_2)$. (II). If in the expression $N = \sum_{j=1}^k \phi_j \overline{\psi_j} \in \mathcal{H}_k$, both $\{\phi_j\}$ and $\{\psi_j\}$ are linearly independent over $\mathbf{C}$, then $R(M,N) = k$, where $M := \{v =< z, \overline{z} >_\ell + N\}$. Moreover, if $N = \sum_{j=1}^k A_j \overline{B_j}$ for A_j, B_j satisfying the same property as ϕ_j, ψ_j do, then there is an invertible constant $k \times k$ matrix C such that $(\phi_1, \cdots, \phi_k) = (A_1, \cdots, A_k)C$, and $(\psi_1, \cdots, \psi_k) = (B_1, \cdots, B_k)\overline{(C^t)^{-1}}$.

Proof of Corollary 2.10: The first part apparently follows from Theorem 2.9 and Equation (2.15). Let $\{\phi_j, \psi_j\}$ be as in Part (II) of the corollary and assume that $\sum_{j=1}^k \phi_j \overline{\psi_j} = \sum_{j=1}^{k'} A_j \overline{B_j}$, where A_j, B_j are holomorphic in their arguments $Z = (z,w)$. Since $\{\phi_j\}$ is a linearly independent finite set, it is easy to see that the set $\{\phi_j^l\}$, where ϕ_j^l are the truncation of ϕ_j up to order l, must be also independent for k sufficiently large. Hence, there exits $\{Z_j\}_{j=1}^k$ such that the matrix $D := \left((\{\phi_j^l(Z_1)\})^t, \cdots, (\{\phi_j^l(Z_k)\})^t\right)$ is invertible. Since $\sum_{j=1}^k \phi_j^l \overline{\psi_j} = \sum_{j=1}^{k'} A_j^l \overline{B_j}$, it follows clearly that $\{\psi_1, \cdots, \psi_k\}$ is a linear combination of $\{B_1, \cdots, B_{k'}\}$. Hence, $k' \ge k$. The last statement can also be similarly seen. ■

Remark 2.11: Corollary 2.10 can be further used to simplify the equation (2.14). To see this, let $M_1 := \{v =< z, \overline{z} >_\ell + \sum_{j=1}^{k_1} \phi_j \overline{\psi_j}\}$ and $M_2 := \{v =< z, \overline{z} >_\ell + \sum_{j=1}^{k_2} \widetilde{\phi}_j \overline{\widetilde{\psi}_j}\}$ be in the $\mathcal{H}_k$-normal form ($k = \max\{k_1, k_2\}$) such that $R(M_1,N_1) = k_1$ and $R(M_2,N_2) = k_2$, where $N_1 = \sum_{j=1}^{k_1} \phi_j \overline{\psi_j}$ and $N_2 = \sum_{j=1}^{k_1} \widetilde{\phi_j} \widetilde{\overline{\psi_j}}$. Assume that $k_1 + k_2 \le n-2$ and $(M_1,0)$ is equivalent to $(M_2,0)$. Then, by Corollary 2.10, it holds that: (I) $k_1 = k_2 = k$; (II) There are a $k \times k$ constant invertible matrix C, an automorphism $T \in \mathrm{Aut}_0(\mathbf{H}^n_\ell)$ with its associated data q, σ, λ as given in

Theorem 2.9 such that

$$
\sigma\lambda^{-2}q(z,w)\cdot(\widetilde{\phi}_1,\cdots,\widetilde{\phi}_{k_0})\circ T=(\phi_1,\cdots,\phi_{k_0})\cdot C,
$$
$$
\sigma\lambda^{-2}q(z,w)\cdot(\widetilde{\psi}_1,\cdots,\widetilde{\psi}_{k_0})\circ T=(\psi_1,\cdots,\psi_{k_0})\cdot\overline{(C^t)^{-1}}. \tag{2.16}
$$

Immediately, we have from (2.16) the following conclusions:
(A). If all $\widetilde{\phi_j}$, $\widetilde{\psi_j}$, ϕ_j ψ_j are polynomials, and at least one of them is not zero, then $q\equiv 1$ and $T=(\lambda(z-aw)U,\sigma\lambda^2 w)$.
(B). If $\{\widetilde{\phi_j},\widetilde{\psi_j}\}$ are rational functions, then so are $\{\phi_j,\psi_j\}$.
(C). If at least one of $\{\widetilde{\phi_j},\widetilde{\psi_j}\}$ is a transcendental function, then at least one of ϕ_j and ψ_j is transcendental, too.

Example 2.12: Let $M_1=\mathbf{H}_0^3$ and let $M_2:=\{(z,w=u+iv)\in\mathbf{C}^3:\ v=|z|^2+|\frac{2z_1^2(1-iw)}{(1+iw)^2}|^2+|\frac{2z_1z_2}{(1+iw)}|^2\}$. Then $R(M_1)=0,\ R(M_2)=2$. Also, M_1 is equivalent to M_2. Notice that $R(M_1)+R(M_2)=2>n-2=1$. Hence, the assumption that $R(M_1)+R(M_2)\le n-2$ in Corollary 2.10 can not be weakened.

2.3. Application to the rigidity and non-embeddability problems: we now first present a discussion on how to apply the materials in §2.2 for the study of the rigidity problem for mappings between the hyperquadrics.

Theorem 2.13 ([EHZ1]): Let $F=(f_1,\cdots,f_{n-1},\phi_1,\cdots,\phi_{N-n},g)$ be a formal holomorphic mapping sending $\mathbf{H}_{\ell_1}^n$ into $\mathbf{H}_{\ell_2}^N$ with $F(0)=0,\ \frac{\partial g}{\partial w}|_0\neq 0$, where g is the normal component of F and $N\ge n>2$. Suppose that $\ell_2\ge\ell_1$ and $\ell_1+\ell_2\le n-1$. Suppose that $N\le 2n-2$. Then there is a linear fractional holomorphic embedding T from $\mathbf{H}_{\ell_2}^n$ to $\mathbf{H}_{\ell_1,\ell_2}^N:=\{(Z,W)\in\mathbf{C}^N:\mathrm{Im}(W)=-\sum_{j\le\ell_1}|Z_j|^2+\sum_{\ell_1<j\le n-1}|Z_j|^2-\sum_{n-1<j\le n-1+\ell_2-\ell_1}|Z_j|^2+\sum_{n-1+\ell_2-\ell_1<j\le N-1}|Z_j|^2\}$ and $T_0\in\mathrm{Aut}_0(\mathbf{H}_{\ell_1}^n)$ such that $T\circ F\circ T_0(z,w)=(z,\phi^*,w)$ with $\phi^*=O(|(z,w)|^2)$. Moreover, when $\ell_2=\ell_1$, T is a self-map and $\phi^*=0$. (For $\ell_1=\ell_2=\ell$, $\mathbf{H}_{\ell_1,\ell_2}^N$ is understood as $\mathbf{H}_\ell^N$.) Also, when $\ell_1<\frac{n-1}{2}$, T_0 can be taken to be the identity map.

Proof of Theorem 2.13: Let $F=(f,\phi,g)=(\widetilde{f},g)$ be a formal holomorphic mapping from $(\mathbf{H}_{\ell_1}^n,0)$ into $(\mathbf{H}_{\ell_2}^N,0)$ with $\frac{\partial g}{\partial w}|_0\neq 0$. Then $\mathrm{Im}(g)=<\widetilde{f},\overline{\widetilde{f}}>_{\ell_2}$ along $\mathbf{H}_{\ell_1}^n$ as a formal power series. Collecting the coefficients of weighted degree 1 and 2, we see that $g=\sigma\lambda^2 w+o_{wt}(2)$, $\widetilde{f}=\lambda zU+\sigma\lambda^2 aw+O(|zw|+|z|^3+|w|^2)$. Here $\sigma=\pm 1,\ \lambda>0$, a is a certain complex vector, $U=(E_1^t,\cdots,E_{n-1}^t)^t$ with $E_j=\frac{1}{\lambda}\frac{\partial\widetilde{f}}{\partial z_j}(0)$, and $<E_j,\overline{E_k}>_{\ell_2}=\sigma\delta_j^k\delta_j$. Since $<E_j,\overline{E_j}>_{\ell_2}\neq 0$ for $j\le n-1$, we can extend $\{E_j\}_{j=1}^{n-1}$ to an orthogonal basis $\{E_j\}_{j=1}^{N-1}$ (with respect to the

Hermitian product $< \cdot, \bar{\cdot} >_{\ell_2}$). Let $\widetilde{U} = (E_1^t, \cdots, E_{N-1}^t)^t$, then $\widetilde{U} E_{\ell_2,N-1} \overline{\widetilde{U}} = \mathrm{diag}(< E_1, \overline{E_1} >_{\ell_2}, \cdots, < E_{N-1}, \overline{E_{N-1}} >_{\ell_2})$, where $E_{\ell_2,N-1}$ is defined, as before, by $< X, \overline{X} >_{\ell_2} = X E_{\ell_2,N-1} \overline{X^t}$. In particular, we see that $< E_j, \overline{\widetilde{E_j}} >_{\ell_2} \neq 0$ for any j. Without loss of generality, we can assume that $< E_j, \overline{\widetilde{E_j}} >_{\ell_2} = c_j = \pm 1$ for $j \geq n$, too. (Notice that $c_j = -\sigma$ for $j \leq \ell_1$ and $c_j = \sigma$ for $\ell_1 < j \leq n-1$.) After changing the position of $E_j's$ for $j > n-1$, we can assume that $\widetilde{U} E_{\ell_2,N-1} \overline{\widetilde{U}^t} = \sigma B^*$, where σB^* is determined by the following Hermitian product:

$$< Z, Z >_{\ell_1,\ell_2} := -\sum_{j=1}^{\ell_1} Z_j \overline{Z_j} + \sum_{\ell_1 < j \leq n-1} Z_j \overline{Z_j} - \sum_{n-1 < j \leq \ell_2 - \ell_1 + n-1} Z_j \overline{Z_j} + \sum_{N-1 > j \geq \ell_2 - \ell_1 + n-1} Z_j \overline{Z_j},$$

Apparently, when $\ell_1 + \ell_2 < \frac{n-1}{2}$, σ must be 1. Otherwise, $\ell_1 = \frac{n-1}{2} = \ell$. In this case, composing F with $T_0(z,w) = (z_{\ell+1}, \cdots, z_{n-1}, z_1, \cdots, z_\ell, -w) \in \mathrm{Aut}_0(\mathbf{H}_\ell^n)$, if necessary, we can also make $\sigma = 1$.

In the following, we assume that $\sigma = 1$. Letting

$$T(z,w) = \left(\frac{\lambda^{-1}(Z - aW)\widetilde{U}^{-1}}{q(Z,W)}, \frac{\lambda^{-2} W}{q(Z,W)}\right), \tag{2.17}$$

where $q(Z,W) = 1 + 2iZ E_{\ell_2,N-1} \overline{a}^t + (r - i < a, \overline{a} >_{\ell_2})W$, with $r = \frac{1}{2}\lambda^{-2}\mathrm{Re}(\frac{\partial^2 g}{\partial w^2}|_0)$. Write $F^* = T \circ F := (f^*, \phi^*, g^*)$. Then (f^*, g^*) satisfies the normalization condition (2.2), and $\phi^* = O(|(z,w)|^2)$. Notice that T biholomorphically maps $\mathbf{H}_{\ell_2}^n$ to $\mathbf{H}_{\ell_1,\ell_2}^N$. Namely, $\mathrm{Im}(g^*) = \sigma f^* B^* \overline{f^*}^t$ along $\mathbf{H}_{\ell_1}^n$. Now, we can inductively apply Theorem 2.8 to prove that $f^* = z,\ g^* = w$. Indeed, we first notice that by collecting terms of weighted degree ≤ 4 in the equation $\mathrm{Im}(g^*) = f^* B^* \overline{f^{*t}}$, we see by Theorem 2.8 and the normalization condition that $f^{*(j-1)} = 0,\ g^{*(j)} = 0$, for $3 \leq j \leq 4$. Suppose that $f^{*(\tau-1)},\ g^{*(\tau)} = 0$ for $\tau \leq K_0$. Collecting terms of weighted degree $K_0 + 1$,

$$\mathcal{L}(f^{*(K_0)}, g^{*(K_0+1)}) = 2\sum_{\kappa=1}^{k} \sum_{j=2}^{[K_0/2]} \epsilon_\kappa \mathrm{Re}(\phi^{*(j)}_{\kappa} \overline{\phi^{*(K_0-j)}_{\kappa}}), \tag{2.17$'$}$$

where ϵ_j is the $(n-1+j)$-th element in the diagonal matrix σB^*. Since $k \leq n-2$, the right hand side of (2.17$'$) is in $\mathcal{S}_{n-2}^0$. Hence, it follows from Theorem 2.8, that $f^{*(K_0)} = 0,\ g^{*(K_0+1)} = 0$. By induction, we see that $f^* = 0,\ g^* = 0$.

Returning to ϕ^*, we get $\sum_{j=1}^k \epsilon_j |\phi^*{}_j|^2 \equiv 0$. Assume that $\ell_2 = \ell_1$. Since we assumed that $\ell_1 \leq (n-1)/2$, all ϵ_j then must have fixed sign. Hence, $\phi^* \equiv 0$. as remarked above, σ must be 1 when $\ell_1 = \ell_2 < (n-1)/2$; and σ can be made to be 1 by replacing F with $F(z_{\ell+1}, \cdots, z_{n-1}, z_1, \cdots, z_\ell, -w)$, if necessary, when $\ell_1 = \ell_2 = (n-1)/2$.

When $\ell_2 > \ell_1$, write

$$\Phi_I = (\phi_1^*, \cdots, \phi_\kappa^*) \text{ with } \kappa = \ell_2 - \ell_1 \text{ and } \Phi_{II} = (\phi_{\kappa+1}^*, \cdots, \phi_{N-1}^*). \tag{2.18.}$$

We also see $\|\Phi_I\|^2 = \|\Phi_{II}\|^2$ over $\mathbf{H}_{\ell_1}^n$. ∎

We give some applications of Corollary 2.9 to the problem of embedding a non-degenerate formal hypersurface $M \subset \mathbf{C}^n$ of signature ℓ into $\mathbf{H}_{\ell'}^N$ with $N \leq 2n-2$ ($\frac{N-1}{2} \geq \ell' \geq \ell$, $\ell+\ell' \leq n-1$).

Let $M = \{v =< z, \overline{z} >_\ell + N\}$ be a formal non-degenerate hypersurface of signature ℓ with $N = o_{wt}(3)$. Assume that F is a formal holomorphic embedding from $(M, 0)$ into $(\mathbf{H}_{\ell'}^N, 0)$. As we see above, after replacing F by $F \circ T_0$, if necessary, and then composing it with a certain holomorphic linear fractional map from $(\mathbf{H}_{\ell'}^N, 0)$ to $(\mathbf{H}_{\ell,\ell'}^N, 0)$, we can write $F = (f, \Phi_I, \Phi_{II}, g)$, where (f, g) satisfies the normalization condition (2.2) and $\Phi_I, \Phi_{II} = O(|(z,w)|^2)$ as defined in (2.18). Applying the implicit function theorem, we conclude that M is equivalent through $F_0 = (f, g)$ to the following hypersurface:

$$\widetilde{M} = \{v =< z, z >_\ell - \|\Phi_I \circ F_0^{-1}\|^2 + \|\Phi_{II} \circ F_0^{-1}\|^2 =< z, z >_\ell + H_{N-n}\}.$$

Notice that $H_{N-n} \in \mathcal{H}_{N-n}$ and F_0 satisfies (2.2). Conversely, by Corollary 2.9, we have the following

Proposition 2.14: Let $M := \{v =< z, z >_\ell + N\}$ where $N \in \mathcal{H}_k$. Suppose that $N \leq 2n - 2 - k$. Then $(M, 0)$ can be formally embedded into $\mathbf{H}_{\ell,\ell'}^N$ if and only if there are vector valued holomorphic functions $\Phi_I(z, w)$, $\Phi_{II}(z, w) = O(|(z, w)|^2)$ with $\ell' - \ell$ and $N - n - \ell' + \ell$ components, respectively, such that

$$\sigma^* N(z, \overline{z}, w, \overline{w}) = -\|\Phi_I(z, w)\|^2 + \|\Phi_{II}(z, w)\|^2, \tag{2.19}$$

where σ^* is either identically 1 or identically -1. In particular, when $\ell = \ell'$, then M can be embedded into $\mathbf{H}_\ell^N$ with $N \leq 2n - 2 - k$ if and only there are $(N - n)$ formal holomorphic functions $\{\phi_j\}_{j=1}^{N-n}$ such that

$$N(z, \overline{z}, w, \overline{w}) = \sigma^* \sum_{j=1}^{N-n} |\phi_j(z, w)|^2. \tag{2.20}$$

where σ^* must be 1 when $\ell < \frac{n-1}{2}$.

More generally, assume that $\ell' = \ell$ and let M be given by $M := \{v =< z, \overline{z} >_\ell + N^{(s)} + o_{wt}(s)\}$ with $N^{(s)}(\not\equiv 0) \in \mathcal{S}_k^{0(s)}$. Let $F = (f, \phi, g)$ be a formal embedding of M into $\mathbf{H}_\ell^N$ with (f, g) satisfying (2.2) and $\phi = O(|(z, w)|^2)$. When $N \le 2n - 2$, an inductive use of Theorem 2.8 shows that $(f, g) = (z + f^{(s-1)} + o_{wt}(s), w + g^{(s)} + o_{wt}(s+1))$ and $\phi = (\phi_1^{(\sigma)}, \cdots, \phi_{N-n}^{(\sigma)}) = 0$ for $2\sigma < s$. In particular, it follows from Theorem 2.8 that $s = 2s'$ must be even if $N \le 2n - 2$ and $k \le n - 2$. Assume this. For terms of weighted degree s, we have $\mathcal{L}(f^{(s-1)}, g^{(s)}) = \|\phi^{(s')}\|^2 - N^{(s)}$. Since $\|\phi^{(s')}(z, u + i < z, \overline{z} >_{\ell_1})\|^2 \in \mathcal{S}_{N-n}^0$, it follows that if $k + N - n \le n - 2$ then

$$N^{(s)}(z, \overline{z}, u, v)|_{v=|z|_\ell^2} \equiv \|\phi^{(s')}(z, u + i < z, \overline{z} >_\ell)\|^2. \tag{2.21}$$

Therefore, we have

Corollary 2.15: Let $M = \{v =< z, \overline{z} >_\ell + N^{(s)} + o_{wt}(s)\}$ be a formal non-degenerate hypersurface of signature ℓ with $N^{(s)}(\not\equiv 0) \in \mathcal{S}_k^{0(s)}$, $k \le n - 2$, $s \ge 4$. Assume that $k \le n - 2$ and $N \le 2n - 2 - \delta_s^e k$ with $\delta_s^e = 0$ for s odd and equal to 1 otherwise. Suppose that there is no holomorphic solution $\phi^{(s')}$ to (2.21). Then $(M, 0)$ cannot be formally embedded into $\mathbf{H}_\ell^N$, when $\ell < \frac{n-1}{2}$. For $\ell = \frac{n-1}{2}$, if there is no solution to

$$N^{(s)}(z, \overline{z}, u, v)|_{v=|z|_\ell^2} \equiv \pm\|\phi^{(s')}(z, u + i < z, \overline{z} >_\ell)\|^2.$$

Then $(M, 0)$ cannot be formally embedded into $\mathbf{H}_\ell^N$.

Example 2.16: Let $M(\subset \mathbf{C}^n) := \{v = |z|^2 + \mathrm{Re}(w^{s-1}\overline{h(z)}) + o_{wt}(2s)\}$ be the germ of a formal non-degenerate hypersurface of signature 0, where $s > 2$ and $h(z)$ is a non-zero homogeneous polynomial of degree 2. Then there is no vector valued weighted holomorphic polynomial $\phi^{(s)}$ of weighted degree s such that $\mathrm{Re}((u+i|z|^2)^{s-1}\overline{h(z)}) = \|\phi^{(s+1)}(z, u+i|z|^2)\|^2 \ge 0$ over $w = u+i|z|^2$. Notice that $k = 2$. Hence, when $N \le 2n-4$, $(M, 0)$ can never be formally holomorphically embedded into $\mathbf{H}^N$. Also notice that $M_0(\subset \mathbf{C}^n) := \{v = |z|^2 + \mathrm{Re}(w^{s-1}\overline{h(z)})\}$ can be holomorphically embedded into $\mathbf{H}_1^{n+2}$ through the map $F = (\frac{1}{2}(w^{s-1} - h(z)), z, \frac{1}{2}(w^{s-1} + h(z)), w)$.

To conclude this subsection, we present one more application to the study of a rigidity problem, which asks if two CR embeddings of a strongly pseudoconvex hypersurface M in $\mathbf{C}^n$ into the Heisenberg hypersurface $\mathbf{H}^N$ are the rigid motion of each other. Namely, if F, Ψ are two C^l-smooth CR embeddings from M into $\mathbf{H}^N$, is there a $T \in \mathrm{Aut}(\mathbf{H}^N)$ such that $T \circ F = \Psi$? Here l is a certain positive number. This problem has been answered in the work of Webster [We2] when $N = n + 1 \ge 4$. The reader can find a geometric approach along the lines of Webster

[We2] on this problem in [EHZ2] when $N-n\le\frac{n-2}{2}$. The arguments here are essentially those in [EHZ1].

Let $M=\{v=<z,\overline{z}>_\ell+N\}$ be a formal non-degenerate hypersurface of signature ℓ with $N=o_{wt}(3)$. Assume that F,Ψ are formal holomorphic embeddings from $(M,0)$ into $(\mathbf{H}^N_{\ell'},0)$ and $(\mathbf{H}^{N'}_{\ell''},0)$, respectively. (Assume that $N'\ge N$. Also, for simplicity, assume that $\ell+\ell'<n-1$). After composing F, Ψ with certain holomorphic linear fractional maps from $(\mathbf{H}^N_{\ell'},0)$ to $(\mathbf{H}^N_{\ell,\ell'},0)$ and from $(\mathbf{H}^{N'}_{\ell'},0)$ to $(\mathbf{H}^{N'}_{\ell,\ell'},0)$, respectively, we can write $F=(f,\Phi_I,\Phi_{II},g)$ and $\Psi=(f^*,\Phi^*_I,\Phi^*_{II},g^*)$ where (f,g) (f^*,g^*) satisfy the normalization condition (2.2) and $\Phi_I,\Phi_{II},\Phi^*_I,\Phi^*_{II}=O(|(z,w)|^2)$ as defined in (2.18). Therefore, M is equivalent through $F_0=(f,g)$ or $\Psi_0=(f^*,g^*)$ to the following hypersurfaces $\widetilde{M}$, M^*, defined, respectively by:

$$v=<z,z>_\ell-\|\Phi_I\circ F_0^{-1}\|^2+\|\Phi_{II}\circ F_0^{-1}\|^2,\quad v=<z,z>_\ell-\|\Phi^*_I\circ\Psi_0^{-1}\|^2+\|\Phi^*_{II}\circ\Psi_0^{-1}\|^2.$$

Notice that $F_0\circ\Psi_0^{-1}$ is a normalized formal biholomorphic map from $(M^*,0)$ to $(\widetilde{M},0)$ satisfying (2.2), and $\widetilde{M}$, M^* are in the $\mathcal{H}_{N-n}$, $\mathcal{H}_{N'-n}$-normal form, respectively. By Theorem 2.8, we see that when $N+N'\le 4n-2$, $F_0\equiv\Psi_0$ and $-\|\Phi_I\|^2+\|\Phi_{II}\|^2\equiv-\|\Phi^*_I\|^2+\|\Phi^*_{II}\|^2$ along M as formal power series. In particular, when $\ell'=\ell''=\ell$, there is a constant matrix U with $U\cdot\overline{U^t}=\mathrm{Id}$ such that $\Phi^*_{II}=\Phi_{II}\cdot U$ by a result of D'Angelo [Da] and by noting that $\Phi^*_I=\Phi_I=0$. Hence, after applying another $T\in\mathrm{Aut}_0(\mathcal{H}^{N'}_l)$ to Ψ, we see that the new F and Ψ satisfy the relation: $\Psi=(F,0)$.

2.4. Chern-Moser normal Space $\mathcal{N}_{CH}$: The space $\mathcal{H}_k$ we presented in the above subsections is indeed very convenient to apply due to its invariant property under the action of $\mathrm{Aut}_0(\mathbf{H}^n_\ell)$. However, it is not a normal space and thus can only be used to model a very limited class of germs of real hypersurfaces. For the study of general Levi non-degenerate hypersurfaces, we need to make use of the normal space $\mathcal{N}_{CM}$ discovered by Chern-Moser in [CH]. The Chern-Moser normal space is not invariant under the action of $\mathrm{Aut}_0(\mathbf{H}^n_\ell)$. Thus a hypersurface which is in the $\mathcal{N}_{CH}$-normal form is still subject to the action of this group. However, it can be used to model any germ of hypersurface.

Since the discussion on the Chern-Moser normal form is available in many nice expositions ([Vit], [BER2], etc.), we here just give a brief account on this theory. Define

$$\begin{aligned}&\Delta_\ell:=-\sum_{j\le\ell}\frac{\partial^2}{\partial z_j\overline{z_j}}+\sum_{j\ge\ell+1}^{n-1}\frac{\partial^2}{\partial z_j\partial\overline{z}_j},\\ &\mathcal{N}_{CH}:=\{h=\sum_{k,l\ge2}F_{k\bar{l}}(z,\overline{z},u),\ \text{with}\ F_{k\bar{l}}=\sum_{|\alpha|=k,|\beta|=l}a_{\alpha\overline{\beta}}(u)z^\alpha\overline{z^\beta}\cdot\\ &\qquad\overline{h}=h,\ \Delta_\ell F_{2\overline{2}}=\Delta_\ell^2F_{2\overline{3}}=\Delta_\ell^3F_{3\overline{3}}=0\}\end{aligned}\tag{2.23}$$

The following is a fundamental result of Chern-Moser in this subject:

Theorem 2.17(Chern-Moser [CM]): Assume the above definition and notation. Then (a). $\mathcal{N}_{CM}$ is a normal space. (b). Any germ of Levi non-degenerate real analytic hypersurface $(M,0)$ with signature ℓ can be transformed by the germ of a biholomorphic map to a convergent Chern-Moser normal form. (c). Let $(M_j,0)$ be germs of formal real hypersurfaces at 0 defined by $v=|z|_\ell^2+N_j$ with $N_j\in\mathcal{N}_{CM}$. Then $(M_1,0)$ and $(M_2,0)$ are equivalent by a formal holomorphic map F, satisfying the normalization (2.2), if and only if $F\equiv\mathrm{Id}$ and $N_1\equiv N_2$.

The proof of Theorem 2.17 can be found in [§3, 4, CM], which we skip here. However, we mention that one of the significant features in the above theorem is that a convergent germ of hypersurface has a convergent Chern-Moser normal form.

In terms of the above theorem, the general procedure to see if two germs $(M_j,0)$, which are already in the Chern-Moser normal form, are equivalent to each other, is as follows: First apply $T\in\mathrm{Aut}_0(\mathbf{H}_\ell^n)$ to M_2 to obtain $T(M_2)$. Then by solving infinitely many times the Chern-Moser equation (2.10) to find a new normal form for $T(M_2)\,:\,v=|z|_\ell^2+N_{2,T}$. Finally, $(M_1,0)$ is equivalent to $(M_2,0)$ if and only if $N_1\equiv N_{2,T}$ for a certain T. The major difficulty here is that it is extremely difficult in general to find $N_{2,T}$ from the defining equation of $T(M_2)$. Indeed, it is the purpose to get rid of this difficulty that motivated us to find an invariant normal form (with respect to $\mathrm{Aut}_0(\mathbf{H}_\ell^n)$) in [EHZ1]. Unfortunately, the admissible space we obtained in [EHZ1] only works for a very small class of real hypersurfaces, which are actually those which can be formally embedded into the hyperquadrics with restricted codimension. The interested reader is referred to the paper [EHZ1] for more on this matter.

We notice that $\mathcal{S}_k^0$ is not a subclass of the Chern-Moser normal space. For instance, for $\sigma>1$, $h=\mathrm{Re}(z_1^{2\sigma}\overline{w^\sigma})$ contains a term of the form $u^\sigma z_1^{2\sigma}$. While h is in $\mathcal{H}_2$, it is not in the Chern-Moser normal space.

§ 3. Bishop Surfaces with Vanishing Bishop Invariants

In this section, we study the holomorphic equivalence problem for submanifolds in $\mathbf{C}^n$ with higher codimension. There have been many generalizations of the Chern-Moser theory to the so-called generic strongly pseudoconvex CR submanifolds. (See the survey paper [BER2] for some references in this regard and the recent paper [BRZ] for some other related studies). In this notes, we would like to focus on the normal form problem for Bishop surfaces [Bis] in $\mathbf{C}^2$. The study of Bishop surfaces has attracted considerable attention since the work of E. Bishop in 1965. (See [BG], [KW], [Mos], [MW] [HK]). These surfaces are interesting, due to the following reasons: First, from the point of view of complex analysis, they can be viewed as the simplest higher codimensional analogy of strongly pseudoconvex hypersurfaces; secondly, they have a rich complex structure at the complex tangent and have trivial complex structure elsewhere, namely they can also be viewed as the simplest models where one sees the CR singularity; thirdly, from the work of Moser-Webster [MW], which we will discuss in the next section, one sees a tremendous interaction of complex analysis with the classical dynamics problems encountered in Mechanics [SM]– An understanding of such a problem may provide useful information and motivation to many converge problems in Mechanics. The basic references to this section include the papers [MOS] [MW] and [HK].

To be more specific, we let M be a real surface in $\mathbf{C}^2$. Then for any $p \in M$, $CR_M(p)$ can be only $0, 1$. When $CR_M(p) = 0$, we say M is totally real at p. By the semi-continuity of the CR dimension function, we conclude that M must be totally real in a neighborhood of p in M. When M is further real analytic, then an easy application of the complexification shows that (M, p) is holomorphically equivalent to $(\mathbf{R}^2, 0)$, where $\mathbf{R}^2 := \{(x, y) \in \mathbf{C}^2,\ x, y \in \mathbf{R}\}$. On the other hand, if $CR_M(q) \equiv 1$ for $q \approx p$, then apparently $(M, p) \approx (\mathbf{C} \times \{0\}, 0)$. Hence, from the equivalence point of view, only points with CR dimension 1 but not constantly 1 nearby are interesting. Among such points, only those which have CR dimension 1 but 0 nearby are stable under small perturbation. Such points are called isolated CR singular points.

Now, let $p \in M$ be a point with a non-trivial complex tangent. Namely, we assume that $CR_M(p) = 1$. After a holomorphic change of coordinates, we can assume that $p = 0$ and $T^{(1,0)} = CT_pM = \{w = 0\}$, where we use (z, w) for the coordinates of $\mathbf{C}^2$. Then M near 0 can be defined by an equation of the form: $w = h(z, \overline{z}) + o(|z|^2)$. Here $h(z, \overline{z}) = az^2 + bz\overline{z} + c\overline{z}^2$. Replacing w by $w - (a - c)z^2$, if necessary, we can assume that $a = c$. Assume that $b \neq 0$. Replacing w by w/b and replacing z by $ze^{i\theta}$ for a suitable θ, we can assume that $h = z\overline{z} + \lambda(z^2 + \overline{z^2})$ with $\lambda \geq 0$. By a straightforward verification, one can see that λ is a biholomorphic invariant, called the Bishop invariant.(See Lemma 3.2 below). When $\lambda < \frac{1}{2}$, we call $p = 0$ an elliptic complex tangent of M. When $\lambda > \frac{1}{2}$, we call $p = 0$ a hyperbolic complex tangent point of M. When $\lambda = 1/2$ or when $b = 0$ but $c \neq 0$, we say $p = 0$ is a parabolic complex tangent. An elliptic, parabolic or hyperbolic complex tangent point is called a non-degenerate complex tangent point. In the other case, we say 0 is a degenerate complex tangent point. A real surface M is called a Bishop surface if all of its complex tangents are non-degenerate. In this notes, we are mainly concerned with the equivalence problem of M at an elliptic complex tangent point. Hence, we have $\lambda \in [0, 1/2)$. In this section, we discuss the formal theory of Moser [Mos] when the surface is formally equivalent to the model surface $M_0 := \{w = |z|^2\}$. In the next section, we discuss the

Moser-Webster theory for Bishop surfaces with non-vanishing Bishop invariants.

The understanding to the general Bishop surfaces with vanishing Bishop invariant is still not complete. It is an open question to get a complete set of invariants for analytic Bishop surfaces with vanishing Bishop invariant.

We first state a general result along these lines proved in [HK]:

Theorem 3.1 (Huang-Krantz [HK]): Let M be a real analytic Bishop surface with vanishing Bishop invariant at 0. Then $(M, 0)$ can be flattened in the sense that there is a biholomorphic change of coordinates such that in the new coordinates, it holds that $M \subset \mathbf{C} \times \mathbf{R}$. More precisely, in the new coordinates, M near 0 can be defined by an equation of the form:

$$w = |z|^2 + E(z, \overline{z}), \ E(z, \overline{z}) = \overline{E(z, \overline{z})}. \tag{3.1}$$

We start with the following statement on invariance of the Bishop invariant.

Lemma 3.2: Suppose that M_j for $j = 1, 2$ are Bishop surfaces with only CR singular point at p_j, respectively. Then the Bishop invariant of M_1 at p_1 is the same as the Bishop invariant of M_2 at p_2, if M_1 is biholomorphically equivalent to M_2.

Proof of Lemma 3.2: Without loss of generality, we can assume that $p_j = 0$. Let $F = (f, g)$ be a biholomorphic map from M_1 to M_2. Then $F(0) = 0$, for F preserves the CR dimension. After a change of coordinates, we can assume that

$$M_j : w = z\overline{z} + \lambda_j(z^2 + \overline{z}^2) + O(|z|^3), \quad 0 \leq \lambda_j \leq \infty.$$

When $\lambda_j = \infty$, we regard M_j as a surface defined by an equation of the form: $w = z^2 + \overline{z}^2 + o(|z|^2)$. For simplicity of calculation, we assume, in the following, that $\lambda_j < \infty$.

Notice that F must preserve the complex tangent space of M_j at 0. We can write $F = (f, g)$ with $f = az + bw + O(|(z, w)|^2)$ and $g = cw + d^{(2)}(z) + O(|w|^2 + |zw| + |z|^3)$. Using the equation of M_2, we get

$$c(z\overline{z} + \lambda_1(z^2 + \overline{z}^2)) + d^{(2)}(z) = |az + bw|^2 + \lambda_2 2Re(az + bw)^2 + O(|z|^3),$$

where $(z, w) \in M_1$. Collecting the coefficients of $z\overline{z}, z^2, \overline{z}^2$, we get

$$c = |a|^2, \ d^{(2)} = d_2 z^2, \ c\lambda_1 + d_2 = \lambda_2 a^2, \ c\lambda_1 = \overline{a}^2 \lambda_2. \tag{3.2}$$

Hence it follows that

$$c > 0, \lambda_1 = \lambda_2, a \in \mathbf{R},\ d^{(2)} = 0 \tag{3.3}$$

This completes the proof of Lemma 3.2. ∎

3.1. Formal theory for Bishop surfaces with vanishing Bishop invariant. We now focus on the case $\lambda = 0$ and present the formal theory of Moser [Mos]. Let M be a real analytic Bishop surface with vanishing Bishop invariant at 0. By Theorem 3.1, after a change of coordinates, we can assume that M is defined by an equation of the form:

$$w = |z|^2 + E(z, \overline{z}) \ \text{ with } \overline{E(z, \overline{z})} = \overline{E}(\overline{z}, z) = E(z, \overline{z}).$$

We notice that M near 0 bounds a family of holomorphic disks defined by

$$\{(z, w) :\ v = 0, u = r^2, r^2 \geq |z|^2 + E(z, \overline{z})\}.$$

Namely, let σ_r be a Riemann mapping from the unit disk in $\mathbf{C}$ to the domain

$$D_r := \{z \in \mathbf{C} : r^2 > |z|^2 + R(z, \overline{z})\}. \tag{3.4}$$

Then the map ϕ_r from the unit disk $\Delta := \{z \in \mathbf{C} : |z| < 1\}$, which sends z to $(\sigma_r(z), r^2)$, is holomorphic in Δ, real analytic up to the unit circle and maps the unit circle to M. Such a ϕ_r is called a holomorphic disk attached to M.

Conversely, for any holomorphic map $\phi = (\phi_1, \phi_2)$ from the unit disk to $\mathbf{C}^2$, which is continuous up to $\partial\Delta$, if it is attached to M (namely, $\phi(\partial\Delta) \subset M$) and if $\|\phi\| << 1$, then $\phi(\Delta) = D_r$ for a certain r. This can be seen easily by noticing that for such a map, ϕ_2 must be constant; for its imaginary part has boundary value 0.

Next, let $(M_j, 0)$ $(j = 1, 2)$ be two real analytic surfaces defined, respectively, by an equation of the form:

$$w = |z|^2 + E_j(z, \overline{z}) \ \text{ with } \overline{E_j(z, \overline{z})} = \overline{E_j}(\overline{z}, z) = E_j(z, \overline{z}).$$

And let $F = (f, g)$ be a biholomorphic map from $(M_1, 0)$ to $(M_2, 0)$. Then F must send a holomorphic disk attached to M_1 to a holomorphic disk attached to M_2. From this, it follows easily that $g(z, w) = g(w)$ with $g(r^2) > 0$ for $0 < r << 1$. Also, $f(z, r^2)$ for each fixed r must be a conformal map from the disk $|z|^2 + E_1(z, \overline{z}) \leq r^2$ to the disk $|z|^2 + E_2(z, \overline{z}) \leq g(r^2)$.

In particular, when both $M_1 = M_2 = M_\lambda = \{w = |z|^2 + \lambda(z^2 + \overline{z}^2)\}$ with $\lambda = 0$, $f(z, r^2)$ must be a conformal map from $|z|^2 \le r^2$ to $|z| \le \sqrt{g(r^2)}$. Hence

$$f(z, r^2) = \sqrt{g(r^2)} e^{i\theta(r)} \frac{z - ra(r)}{r - \overline{a(r)} z}$$

for certain $\theta(r), a(r)$.

Since f is analytic in (z, w), we can conclude that $f(0, u) = -\sqrt{g(u)} e^{i\theta(r)} a(r)$ is real analytic in u. Write $g(w) = w(g^*(w))^2$ with $g^*(r^2) > 0$, $g^*(0) > 0$. Then

$$f(0, u) = -\sqrt{u} g^*(u) e^{i\theta(\sqrt{u})} a(\sqrt{u}),$$

we see that $\sqrt{u} a(\sqrt{u}) e^{i\theta(\sqrt{u})}$ is analytic.

In particular, we see that $u|a(\sqrt{u})|^2$ and thus $|a(\sqrt{u})|^2$ is analytic.

Next, $\frac{\partial f}{\partial z}(0, u) = g^*(u) e^{i\theta(\sqrt{u})}(1 - |a(\sqrt{u})|^2)$. We conclude that $e^{i\theta(\sqrt{u})}$ is also analytic, and thus $\sqrt{u} a(\sqrt{u})$ is analytic too. In this manner, we can write

$$f(z, u) = g^*(u) \Lambda(u) \frac{z - c(u)u}{1 - \overline{c}(u) z}$$

where $c(u) = \frac{a(\sqrt{u})}{\sqrt{u}}$ is analytic in u with $|c(u)| \le \frac{1}{\sqrt{u}}$,or $|c(u)u| < \sqrt{u}$; $g^*(u)$ and $\Lambda(u)$ are analytic in u with $g^*(0) > 0$. Summarizing what we did and with a further straightforward verification, we have

Proposition 3.3([MW] [Mos]): $Aut_0(M_\lambda)$ with $\lambda = 0$ consists of the following transformations:

$$\text{(3.5)} \qquad \begin{cases} w' = wa(w)\overline{a}(w), \\ z' = a(w) \frac{z - wb(w)}{1 - \overline{b}(w) z} \end{cases}$$

with $a(0) \ne 0$, $a(w), b(w)$ holomorphic functions in w.

Still let M be defined by $w = |z|^2 + E(z, \overline{z})$ with $E(z, \overline{z} = \overline{E(z, \overline{z})} = O(|z|^3)$ real analytic in z. We subject to M a transformation of the form: $F = (f, g)$ where $f = az + bw + O(|z, w|^2)$, $g = g(w)$ with $g(r^2) > 0$ for $r > 0$.

Lemma 3.4: There is a unique $T \in \mathrm{Aut}_0(M_\lambda)$ such that $T \circ F = (\widetilde{f}, \widetilde{g})$ satisfies the following normalization condition:

$$\text{(3.6)} \qquad \widetilde{f} = \sum_{j=0}^{\infty} z^j f_j(w) \;\; \textit{with} \;\; f_0 = 0, \;\; f_1(u) > 0 \;\; f_1(0) = 1 \;, \widetilde{g} = w.$$

Proof of Lemma 3.4: First, we can easily make $F = (f, g) = (z + O(w + |z|^2), w + O(w^2))$. Choose $T_0 \in \mathrm{Aut}_0(M_0)$: $T_0(z, w) = (a(w)z, a^2(w)w)$, with $a(0) \neq 0$, $a(u) > 0$ for $u \geq 0$. We like to have $T_0 \circ F = (\cdot, w)$. For this, we need the function relation: $a^2(g(w))g(w) = w$. Hence, $a(g(w)) = \frac{1}{g^*(w)}$, where, as before, $g(w) = wg^*(w)$ with $g^*(0) \neq 0$. Apparently, such an $a(w)$ can be uniquely solved.

Still write F for $T_0 \circ F$. Let

$$T_1 = \left(\Lambda(w)\frac{z - c(w)w}{1 - \overline{c}(w)z}, w\right) \in \mathrm{Aut}_0(M_0).$$

Write $F = (\sum_{j=0}^{\infty} f_j(w)z^j, w)$ and letting $c(w) = \frac{f_0(w)}{w}$. Then

$$T_1 \circ F = \left(\Lambda(w)\frac{\sum_{j=1}^{\infty} f_j(w)z^j}{1 - \overline{c}(w)f(z, w)}, w\right) = (\sum_{j=1}^{\infty} \widetilde{f}_j(w)z^j, w).$$

Then

$$\widetilde{f}_1(w) = \frac{\Lambda(w)f_1(w)}{(1 - \overline{c}(w)wc(w))}.$$

Notice that $f_1(0) = 1$. Also, we can apparently choose Λ such that $\widetilde{f}_1(u) > 0$ for $u \geq 0$. We proved the existence of $T \in Aut_0(M_\lambda)$ such that the $T \circ F$ satisfies the normalization (3.6) in the lemma.

We next prove the uniqueness of T. Suppose that there are $T_1 = (\phi_1, \psi_1)$ and $T_2 = (\phi_2, \psi_2)$ such that both $T_1 \circ F$ and $T_2 \circ F$ satisfy the normalization condition in (3.6). Then one can see easily that it must hold $\psi_1 = \psi_2$ when restricted to M_0. We leave it to the reader to verify that $\phi_1 = \phi_2$ along M_0.

There is another normalization used in [Mos] for F:

Lemma 3.4′([Mos]): There is a unique $T \in \mathrm{Aut}_0(M_\lambda)$ with $\lambda = 0$ such that

$$T \circ F = \left(\sum_{j=0}^{\infty} z^j f_j(w), g(w)\right) \text{ with } f_0(w) = 0,\ f_1(w) \equiv 1,\ g(w) = w + o(|w|). \tag{3.7}$$

Proof of Lemma 3.4′: We choose T_1 to be of the form $\left(\frac{z - c(w)w}{1 - \overline{c}(w)z}, w\right)$. Let $c(w) = \frac{f_0(w)}{w}$. Then $T_1 \circ F = \left(\sum_{j=1}^{\infty} f_j(w)z^j, g(w)\right)$.

Next we take $T_2 = (g^*(w)\Lambda(w)z, w(g^*(w))^2)$ with $g^*(u) > 0$ if $u \geq 0$. Then, we can choose $\Lambda(u)$ with $|\Lambda(u)| \equiv 1$ such that $g^*(u)\Lambda(u)f_1(u) \equiv 1$. Then $T_2 \circ T_1 \circ F$ has the normalization as in Lemma 3.4$'$. The uniqueness part can also be done easily. ∎

We now derive the Moser pseudo-formal norm for $(M, 0)$, where M is defined as in (3.1). We will subject to M the transformation of the following form:

$$\left\{\begin{array}{l} z' = F = z + f(z,w), \;\; w' = w \text{ with} \\ f(z,w) = \sum_{l=1}^{\infty} f_l(w)z^l, \;\; f_1(u) > 0, \;\; f_1(0) = 1. \end{array}\right. \tag{3.8}$$

Proposition 3.5: With the above notation, there is a unique formal holomorphic transformation $(z', w') = F(z, w)$ as in (3.8) such that in the (z', w') coordinates, $F(M)$ is given by the following pseudo-normal form:

$$w' = z'\overline{z'} + \phi(z') + \overline{\phi(z')} \tag{3.9}$$

where $\phi(z') = \sum_{j=s\geq 3}^{\infty} a_j(z')$.

In the above lemma, if all $a_j = 0$, then M is formally equivalent to the model M_0. Otherwise, we can assume that $a_s \neq 0$. In fact, replacing (z', w') by $(\kappa z, \kappa^2 w)$ for a suitable κ, we can further make $a_s = 1$. It can be verified that s is then also a biholomorphic invariant of $(M, 0)$, which we call the s-invariant. When $(M, 0)$ is formally equivalent to the model, we say the s-invariant of $(M, 0)$ is ∞.

Proof of Proposition 3.5: Substituting (3.8) into (3.9), we have

$$w = (z + f(z,w))(\overline{z} + \overline{f}(\overline{z}, \overline{w})) + \phi(z + f(z,w)) + \overline{\phi}(\overline{z} + \overline{\phi(z,w)}),$$

for $w = |z|^2 + E(z, \overline{z})$. Collecting terms of degree s in $(z, \overline{z})$, we get

$$E^{(s)} = z\overline{f^{(s-1)}}(z, z\overline{z}) + f^{(s-1)}(z, z\overline{z})\overline{z} + \phi^{(s)}(z) + \overline{\phi^{(s)}(z)} + G^{(s)}(z, \overline{z})$$

where $G^{(s)}$ is completely determined by $f^{(\sigma-1)}(z, z\overline{z}), g^{(\sigma)}(z, z\overline{z})$ and $\phi^{(\sigma)}$ for $\sigma < s$. Moreover, $G^{(s)}$ is 0 when $f^{(\sigma-1)}(z, z\overline{z}) = g^{(\sigma)}(z, z\overline{z}) = \phi^{(\sigma)}(z) = 0$ for $\sigma < s$. We will also assign the weight of u to be 2.

We will inductively determine F and ϕ. Suppose $F^{(\sigma)}$ and $\phi^{(\sigma)}$ have been solved for $\sigma < s$. Write $\Gamma(z, \overline{z}) = E^{(s)} - G^{(s)}$. We then see that $\phi^{(s)} = \Gamma(z, 0)$. Write $\Gamma(z, \overline{z}) - \Gamma(z, 0) - \Gamma(0, \overline{z}) = \Gamma_0(z\overline{z}) + \sum_{l=1}^{\infty}(z^l\Gamma_l(z\overline{z}) + \overline{z^l}\Gamma_l(z\overline{z}))$ with $\overline{\Gamma_l} = \Gamma_l$, $\Gamma_0 = \overline{\Gamma_0}$. Since $f_1(u) > 0, f_0 = 0$, we obtain

$$\left\{\begin{array}{l} f_1^{(2s')}(u) = \frac{\Gamma_0^{(2s'+2)}(u) - \Gamma_0^{(2s'+2)}(0)}{2u}, \\ f_l^{(2s')}(u)u = \Gamma_{l-1}^{(2s'+2)}(u) \textit{ or } f_l^{(2s')}(u) = \frac{\Gamma_{l-1}^{(2s'+2)}(u) - \Gamma_{l-1}^{(2s'+2)}(0)}{u}, \quad l > 1. \end{array}\right. \tag{3.10}$$

Let $f^{(s)}, g^{(s)}$ be the unique solutions given as above. Let $F^{(s)} = (z + \sum f^{(s)}, w)$. Then F satisfies the normalization as in (3.6). Now, the composition of such a map formally transforms $(M, 0)$ into a special form as in (3.9).

Similarly, one also has the following:

Proposition 3.5$'$ ([Mos]): Let $(M, 0)$ be given as in (3.1). Then there is a unique formal holomorphic transformation $(z', w') = F(z, w)$, that satisfies the normalization in (3.7), such that in the (z', w') coordinates, $F(M)$ is given by a pseudo-normal as in (3.9).

A surface defined by an equation of the form in (3.9) is said to be presented in the Moser pseudo-normal form. It should be mentioned that the coefficients embedded in the Moser pseudo-normal form are far from being holomorphic invariants. Indeed, the Moser pseudo-normal form is still subject to the action of a huge group: $\mathrm{Aut}_0(M_0)$, which, different from the real hypersurface case, is of infinite dimension. It has been an open question how to simplify the Moser peudo-normal form further to get a more invariant representation for Bishop surfaces with vanishing Bishop invariant. It is also an open question if a real analytic $(M, 0)$ can be transformed into a convergent Moser pseudo-normal form through a convergent power series. In the following subsection, we will show that if M is formally equivalent to the model M_0, then it is biholomorphic to M_0. We will follow essentially the argument in [Mos] for this purpose.

3.3. Bishop surfaces which are formally equivalent to $(M_\lambda, 0)$ with $\lambda = 0$: In this section, we give the proof of the following theorem of Moser:

Theorem 3.6(Moser)[Mos]: Suppose $(M, 0)$ is formally equivalent to $(M_\lambda, 0)$ with $\lambda = 0$. Then $(M, 0)$ is biholomorphic equivalent to $(M_\lambda, 0)$.

Let $M : w = z\overline{z} + E(z, \overline{z})$ with $E = O(|z|^3)$ real valued be formally equivalent to M_λ with $\lambda = 0$. $E(z, \xi)$ can be assumed to be holomorphic in the polydisc $|z|, |\xi| \leq 1$, $\sup_{|z|,|\xi|<1} |E(z, \overline{z})| < \eta_0$. Replacing (z, w) by $(\epsilon, \epsilon^2 w)$ for $\epsilon << 1$, we can always make η_0 sufficiently small.

We will seek the transformation of the form $z' = z + f(z, w)$, $w' = w$ as in (3.6), such that

$$w = (z + f(z, w))(\overline{z} + \overline{f}(\overline{z}, \overline{w})), \text{ or}$$

$$\overline{z} f(z, w) + z\overline{f}(\overline{z}, \overline{w}) = E - |f(z, w)|^2, \ w = z\overline{z} + E(z, \overline{z}).$$

Consider its lineariztion:

$$\overline{z} f(z, z\overline{z}) + z\overline{f}(\overline{z}, z\overline{z}) = E$$

which may not be solvable in general. However, as what we did above, we can solve the following

$$\overline{z}f(z,z\overline{z}) + z\overline{f}(\overline{z},z\overline{z}) + \phi(z) + \overline{\phi(z)} = E(z,\overline{z})$$

where $f(z,w) = \sum_{j=1}^{\infty} f_j(w)z^j$ with $f_0 = 0$, $f_1(u) > 0$ and $f_1(0) = 1$. Still write

$$E = E_0(z\overline{z}) + \sum_{l=1}^{\infty}(E_\ell(z\overline{z})z^l + E_l(z\overline{z})\overline{z}^l) + E(z,0) + E(0,\overline{z}). \tag{3.11}$$

Then as in (3.10), we have the following

$$\begin{cases} \phi(z) = E(z,0), \\ f_1(u) = \frac{E_0(u)-E_0(0)}{2u} \\ f_\ell(u) = \frac{E_{l-1}(u)-E_{l-1}(0)}{u}, \quad l = 2,3,\cdots. \end{cases} \tag{3.12}$$

For the rest of this section, for $1/2 < r < 1$, we write

$$\Delta_r = \{(z,w):\ |z| < r,\ |w| < r^2\}, \quad D_r = \{(z,w):\ |z| < r,\ |w| < r\}. \tag{3.12$'$}$$

We will also use c_j, c_j' to denote certain absolute constant.

Proposition 3.7: Suppose that $E(z,\xi) \in \mathrm{Hol}(D_r)$. Let $\rho \in (1/2, r)$. Write

$$\|E\|_r = \sup_{|z|<r,|\xi|<r} |E(z,\xi)|,\ |f|_r = \sup_{|z|<r,|w|<r^2} |f(z,w)|.$$

Then $f,\ \phi$ are holomorphic over D_r with following estimates:

$$\begin{cases} |f|_\rho < c_1(r-\rho)^{-1}\|E\|_r; \\ |f_z|_\rho + |f_w|_\rho \le c_1(r-\rho)^{-2}\|E\|_r, \\ \sup_{|z|<r} |\phi(z)| \le \|E\|_r. \end{cases} \tag{3.13}$$

Proof: Note that $z^\ell E_\ell(z\xi) = \frac{1}{2\pi i}\int_0^{2\pi} E(e^{i\phi}z, e^{-i\phi}\xi)e^{-il\theta}d\theta$. By the maximum principle,

$$\sup_{|w|\le r^2} |E_\ell(w)| \le \sup_{|w|=r^2} |E_\ell(w)| = r^{-l}\|E\|_r,\ \sup_{|w|\le r^2} |f_\ell(w)| \le 2r^{-|\ell|-2}\|E\|_r.$$

Hence,

$$|f|_\rho \le \sum_{\ell=1}^{\infty} \rho^\ell \sup_{|w|\le\rho^2} |f_\ell(w)| \le \sum_{\ell=1}^{\infty} \rho^\ell 2r^{-|\ell|-2}\|E\|_r = \frac{c_0}{r-\rho}\|E\|_r,$$

$$\sup_{|z|<r} |\phi(z)| \leq \|E\|_r.$$

This, in particular, shows that f is holomorphic in any D_ρ for $\rho < r$. Thus we see $f, \phi \in \mathrm{Hol}(D_r)$.

To get the estimates for derivatives, we set $\tau = \frac{r+\rho}{2}$. By the Cauchy estimates, we obtain:

$$|f|_\tau \leq \frac{c_1'}{r-\tau}\|E\|_r,\ |f_z|_\rho \leq \frac{|f|_\tau}{\tau-\rho} \leq \frac{c'{}_1\|E\|_r}{(r-\rho)^2},\ |f_w|_\rho \leq \frac{c_1'\|E\|_r}{(r-\rho|^2}.$$

This completes the proof of the proposition. ∎

The following is basic for applying the rapidly convergent power series method to prove Theorem 3.6.

Lemma 3.8: Suppose that $M : w = z\overline{z} + E(z, \overline{z})$ with the s-invariant $s = \infty$. (Namely, suppose that M is formally convergent to the model M_0). Assume that $\mathrm{ord}(E) \geq d$. Then the transformed surface $F(M) :\ w' = zz' + E'$ obtained above has $\mathrm{ord}(E') \geq 2d - 2$.

Proof of Lemma 3.8: We have the equation:

$$\overline{z}f(z,w) + z\overline{f}(\overline{z},\overline{w}) = E - |f(z,w)|^2 - E'(z+f, \overline{z} + \overline{f(z,w)}). \tag{3.14}$$

Apparently, when $\mathrm{ord}(E) = d$, by (3.12), we have $\mathrm{ord}(f) = d-1$ and thus $\mathrm{ord}(|f(z,w)|^2) \geq 2d-2$. Notice that $\mathrm{ord}(f(z,w) - f(z, z\overline{z})) \geq 2d - 3$. Since we assumed that $s = \infty$, it must hold that $\mathrm{ord}(\phi(z)) \geq 2d - 2$. (Otherwise, $E'(z', \overline{z'}) = \mathrm{Re}(b_{s_0} z'^{s_0}) + o(|z'^{s_0}|)$ with $2 < s_0 < 2d - 2$ and $b_{s_0} \neq 0$.) Therefore it is easy to conclude that $\mathrm{ord}(E') \geq 2d-2$ by the way f, ϕ were constructed. (See (3.12)). ∎

Now let $M' = F(M)$ be as above defined by: $w' = |z'|^2 + E'(z', \overline{z'})$. We will estimate E'. After complexification, namely, after replacing $\overline{z}$ by a new variable ξ, we have

$$E'(z',\xi') = -\xi(f(z,w) - f(z,z\xi)) - z(\overline{f}(\xi,w) - \overline{f}(\xi,z\xi)) - f(z,w)\overline{f}(\xi,w) + \phi(z) + \overline{\phi}(\xi) \tag{3.15}$$

where $z' = z + f(z,w)$, $\xi' = \xi + \overline{f}(\xi, w)$, $w = z\xi + E(z,\xi)$.

Let r' and r be such that $\frac{1}{2} < r' < r < 1$ and choose $\sigma, \rho \in (r', r)$ such that $r - \rho = \rho - \sigma = \sigma - r' = \frac{1}{3}(r - r')$.

Lemma 3.9: Let M be as above with $\mathrm{ord}(E) \geq d$. Then, there exists an absolute constant $1 > \delta > 0$ such that if $\|E\|_r < \delta(r - r')^2$, the above defined mapping $F : (z,w) \to (z', w') =$

$(z+f(z,w),w)$ takes every value in Δ_σ exactly once from Δ_ρ, and takes M into $M' = F(M)$ with $E'(z,\overline{z}')$ holomorphic in $z', \xi' \in \overline{D_{r'}}$ and

$$\|E'\|_{r'} \le c_2 \|E\|_r \left\{ \frac{\|E\|_r}{(r-r')^2} + \left(\frac{r'}{r}\right)^{\frac{d}{2}} \right\}. \tag{3.16}$$

Proof of Lemma 3.9: Write $\Psi(z',w') = F^{-1}(z',w') = (\psi(z',w'),w')$. We need to show that for each fixed w with $|w| < \sigma^2$ and z' with $|z'| < \sigma$, we can solve uniquely the equation $z' = z + f(z,w)$ with $|z| < \rho$. For this purpose, we let δ be sufficiently small so that $|f_z|_\tau + |f_w|_\tau < \frac{1}{20}$ and $|f|_\rho \le \frac{1}{20}(r-r')$ with $\tau = \frac{r+r'}{2}$. Let $z_1 = z'$ and $z_{j+1} = z' - f(z_j,w)$ for $j = 2, \cdots$. By the standard argument on the Picard iteration procedure, one can verify that $|z_j| < \rho$ and $z_j \to z$ with $|z| < \rho$, too. Apparently, z is the solution that we want.

This proves that Ψ biholomorphically maps Δ_σ into its image contained in Δ_ρ. Notice that for $(z,\xi) \in D_\sigma$, $|w(z,\xi)| = |z\xi + E(z,\xi)| \le |\sigma|^2 + \|E\|_r < \rho^2$ provided that $\|E\|_r < \rho^2 - \sigma^2$, which holds automatically by the way we choose δ above. Hence, we conclude that E' is holomorphic in D_σ. Moreover,

$$\|E'\|_{r'} \le \|Q\|_\sigma,$$

where

$$\begin{aligned} Q(z,\xi) = &-\xi(f(z,w) - f(z,z\xi)) - \\ &-z(\overline{f}(\xi,w) - \overline{f}(\xi,z\xi)) - f(z,w)\overline{f}(\xi,w) + \phi(z) + \overline{\phi}(\xi) \end{aligned}. \tag{3.17}$$

To estimate $\|Q\|_\sigma$, recall that for $(z,\xi) \in D_\sigma$, $|w| \le \sigma^2 + \|E\|_r < \rho^2$. Hence,

$$|f(z,w) - f(z,z\xi)| \le \sup_{\Delta_\rho} |f_w| \|E\|_\sigma \le c_1 (r-\rho)^{-2} \|E\|_r^2,$$

$|\phi(z)| \le \|E\|_r$ for $|z| < r$. Also, by the Schwarz Lemma, $|\phi(z)| \le \left(\frac{\sigma}{r}\right)^d \|E\|_r$ for $|z| < \sigma$. Notice that

$$|f(z,w)\overline{f}(\xi,w)| \le c_1^2 (r-\rho)^{-2} \|E\|_r^2$$

Hence, $\|Q\|_\sigma \le c_2' \{(r-\rho)^{-2}\|E\|_r^2 + \left(\frac{\sigma}{r}\right)^d \|E\|_r\}$. To complete the proof of the lemma, we just need to notice that $r - \rho = \frac{r-r'}{3}$ and thus $(\frac{\sigma}{r})^2 \le \frac{r'}{r}$. ∎.

Proof of Theorem 3.6: We start with $M : w = z\overline{z} + E(z,\overline{z})$ with $\mathrm{ord}(E) \ge 3$ and assume that the s-invariant of M is ∞. Choose $\{r_v\}_{v=1}^\infty$ with $r_v = \frac{1}{2}(1 + \frac{1}{v+1})$,

$$\rho_v = r_v - \frac{1}{3}(r_v - r_{v+1}), \quad \sigma_v = r_v - \frac{2}{3}(r_v - r_{v+1}).$$

We mentioned that we can a priori make $\epsilon' := \|E\|_{r_1}$ arbitrarily small. Our goal will be proving that when ϵ' is chosen to be sufficiently small, then the $E_v(z,\xi)$ obtained successively will be biholomorphic in D_{r_v} and $\|E_v\|_{r_v} \to 0$ as $v \to \infty$. Moreover, $\Phi_v = \Psi_1 \circ \Psi_2 \circ \cdots \Psi_v$ converges uniformly in $\Delta_{1/2}$. Hence, it follows that $\Phi_v^{-1}(M) = M_v$ converges to $w = z\overline{z}$. Namely, the inverse of the limit of $\{\Phi_v\}$ biholomorphically maps $(M,0)$ into $(M_0,0)$. In details, we explain as follows:

Note that $\mathrm{ord}(E_v) \geq d_v = 2^v + 2$ for $v \geq 1$. Set

$$\epsilon_v = (r_v - r_{v+1})^{-2}\|E_v\|_{r_v}.$$

Suppose ϵ_v is smaller than the δ required in Lemma 3.9. Then by (3.16),

$$\epsilon_{v+1} \leq \left(\frac{r_v - r_{v+1}}{r_{v+1} - r_{v+2}}\right)^2 c_2\epsilon_v\left(\epsilon_v + (\frac{r_{v+1}}{r_v})^{\frac{d_v}{2}}\right). \tag{3.18}$$

Hence

$$\epsilon_{v+1} \leq c_3\epsilon_v(\epsilon_v + \lambda_v). \tag{3.19}$$

Here

$$\lambda_v = \left(1 - \frac{1}{(v+2)^2}\right)^{\frac{d_v}{2}} \to 0.$$

Now, we have the following:

Lemma 3.10: Suppose a positive sequence $\{\epsilon_v\}$ with $\epsilon_1 << 1$ satisfies (3.19). Then when ϵ_1 is sufficiently small, $\epsilon_v \leq 2^{-v}$. Moreover, for any $c' < 1$, by making ϵ_1 sufficiently small, one also has $\epsilon_v < c'$

Proof of Lemma 3.10: Notice that $\lambda_v < e^{-(v+2)^2 2^{v-1}} < c_5' e^{-v^2}$. We first choose $N >> 1$ and $\epsilon_1 << 1$ such that $\lambda_v < 2^{-v}$ for $v \geq N$, $\epsilon_N < 2^{-N} < (4c)^{-1}$. Then $\epsilon_{N+1} \leq 2^{-N-1}$. By an induction, one sees that $\epsilon_v < 2^{-v}$ for any $v \geq N$. The rest of the proof is apparent. ■

Hence, once we start with $\epsilon' << 1$, then Lemma 3.10 says that Proposition 3.9 can always be applied. We see that $\|E_v\|_{r_v} \leq \epsilon_v \leq 2^{-v} \to 0$. The reader can easily verify the uniform convergence of $\{\Phi_v\}$ as $v \to \infty$ over $\Delta_{1/2}$.

Now the mapping $\Phi = \lim \Phi_v$ defines a biholomorphic mapping from $(\mathbf{C}^2, 0)$ to $\mathbf{C}^2, 0)$. Its inverse maps M into the model $w' = |z'|^2$. By Lemma 3.4 or Lemma 3.4$'$, we can also make Φ^{-1} satisfy the normalization in (3.6) or (3.7), respectively. ■

§ 4. Moser-Webster's Theory on Bishop surfaces with Non-Exceptional Bishop invariants

Now we turn to real analytic elliptic Bishop surfaces with non-vanishing Bishop invariant. Let M be defined by

$$w = q_\lambda(z, \overline{z}) + o(|z|^2), \tag{4.1}$$

where $q_\lambda(z, \overline{z}) = z\overline{z} + \lambda(z^2 + \overline{z}^2)$ with $0 < \lambda < \frac{1}{2}$. Write $M_\lambda = \{w = q_\lambda(z, \overline{z})\}$. Then M_λ is still foliated near 0 by closed analytic curves which bound holomorphic curves. Hence, a similar argument as in §3 can be used to show that $\mathrm{Aut}_0(M_\lambda)$ consists of precisely the maps of the form:

$$z' = a(w)z,\ w' = a^2(w)w \ \ \textit{with } a = \overline{a}, \quad a(0) \neq 0. \tag{4.2}$$

More generally, we call $0 \in M$ a non-exceptional complex tangent if $0 \in M$ is a non-degenerate complex tangent with Bishop invariant $\lambda \neq 0, \frac{1}{2}, \infty$ and if the quadratic equation: $\lambda\gamma^2 - \gamma + \lambda = 0$ has no roots of unity. It is shown in [MW] that for the general model M_λ with λ non-exceptional, $\mathrm{Aut}_0(M_\lambda)$ also consists precisely of the maps of the form in (4.2). (See [Corollary 3.5, MW]).

One might also want to use the methods in the previous sections to normalize Bishop surfaces near general non-degenerate Bishop complex tangents. However, one would find out that even the linear algebra involved for the linearized equation will immediately become a lot of more complicated. Up to now, no one seems to have succeeded in obtaining a complete set of invariants in this way. In the paper of Moser-Wester [MW], they reduced the normalization problem to the normalization problem for a pair of involutions intertwined by a conjugate holomorphic involution. This reduction enables them to completely settle the local equivalence problem for elliptic Bishop surfaces with non-vanishing Bishop invariant. In the following, we present a quick discussion on the theory of Moser-Webster. The reader is referred to their original paper [MW] for more details.

4.1 Complexification $\mathcal{M}$ of M and a pair of involutions associated with $\mathcal{M}$: Assume that M is defined by an equation of the form

$$w = z\overline{z} + \lambda(z^2 + \overline{z}^2) + H(z, \overline{z}) \text{ with } \ H(z, \overline{z}) = o(|z|^2).$$

Replacing $\overline{z}$ by ξ and $\overline{w}$ by η, we obtain a complex surface $\mathcal{M}$ in $\mathbf{C}^4$ near the origin defined by

$$(w, \eta) = \Psi_0(z, \xi) := \begin{cases} w = z\xi + \lambda(z^2 + \xi^2) + H(z, \xi), \\ \eta = z\xi + \lambda(z^2 + \xi^2) + \overline{H}(\xi, z) \end{cases}. \tag{4.3}$$

Consider the projections π_1 and π_2 from $\mathcal{M}$ to the (z, w) and (ξ, η) spaces, respectively. Then π_j are two-to-one branched covering maps. Write $\hat{\tau_j}$ for the deck transformations of π_j. Namely, for

$p, q \in \mathcal{M}$ $\hat{\tau}_j(p) = q$ if and only if $\pi_j(p) = \pi_j(q)$. One sees that $\hat{\tau}_j$ extend to biholomorphic self-maps of $(\mathcal{M}, 0)$. Also, write $\hat{\rho}$ for the conjugate holomorphic self-map of $(\mathcal{M}, 0)$: $\hat{\rho}(z, w, \xi, \eta) = (\overline{\xi}, \overline{\eta}, \overline{z}, \overline{w})$. Then, the following relations are fundamental:

$$\hat{\tau_j}^2 = \hat{\tau}_j, \quad \hat{\rho}^2 = \hat{\rho}, \quad \hat{\tau}_2 = \hat{\rho} \circ \hat{\tau}_1 \circ \hat{\rho}, \tag{4.4}$$

In what follows, we call $(\hat{\tau}_j, \hat{\rho})$ the Moser-Webster triplet. Notice that $\mathcal{M}$ is parameterized by (z, ξ) by (4.3). We can define the following self-maps of $(\mathbf{C}^2, 0)$:
(i): $\tau_1(z, \xi) := (z, \xi')$ if and only if $\pi_1(z, w, \xi, \eta) = \pi_1(z, w, \xi', \eta')$ for a certain $(z, w, \xi', \eta') \in \mathcal{M}$.
(ii): $\tau_2(z, \xi) := (z', \xi)$ if and only if $\pi_1(z, w, \xi, \eta) = \pi_1(z', w', \xi, \eta)$ for a certain $(z', w', \xi, \eta) \in \mathcal{M}$.
(iii): $\rho(z, \xi) := (\overline{\xi}, \overline{z})$

A direct computation shows that τ_2 and τ_1 are given, respectively, by:

$$\begin{cases} z' = -z - \frac{1}{\lambda}\xi + h_1(z, \xi), \\ \xi' = \xi, \end{cases} \tag{4.5}$$

$$\begin{cases} z' = z, \\ \xi' = -\frac{1}{\lambda}z - \xi + h_2(z, \xi) \end{cases} \tag{4.6}$$

where $h_j(z, \xi) = o(|(z, \xi)|)$.

τ_j are naturally associated to $\hat{\tau}_j$ by (4.3):

$$\tau_j = \Psi_0^{-1} \circ \hat{\tau}_j \circ \Psi_0, \;\; \rho = \Psi_0^{-1} \circ \hat{\rho} \circ \Psi_0. \tag{4.7}$$

The following lemma can be proved by a direct construction:

Lemma 4.1: Bishop surfaces $(M, 0)$ and $(\widetilde{M}, 0)$ with Bishop invariant $\lambda \neq 0, \frac{1}{2}, \infty$ are holomorphic equivalent if and only if there is a biholomorphic map Ψ from $(\mathcal{M}, 0)$ to $(\widetilde{\mathcal{M}}, 0)$ such that $\Psi \circ \hat{\tau}_j = \hat{\tilde{\tau}}_j \circ \Psi$ and $\Psi \circ \hat{\rho} = \hat{\tilde{\rho}} \circ \Psi$.

Suppose that we have a general pair of holomorphic involutions τ_1 and τ_2, together with a conjugate holomorphic involution from $(\mathbf{C}^2, 0)$ to $(\mathbf{C}^2, 0)$. Let $\mathcal{M}$ be the complexification of the Bishop surface in (4.1). Suppose that there is a biholomorphic map Φ from $(\mathbf{C}^2, 0)$ to $(\mathcal{M}, 0)$ such that

$$\Phi \circ \tau_j = \hat{\tau}_j \circ \Phi, \quad \text{and } \Phi \circ \rho = \hat{\rho} \circ \Phi.$$

Then we say $\{(\mathcal{M}, 0), \hat{\tau}_j, \hat{\rho}\}$ is parameterized by $\{(\mathbf{C}^2, 0), \tau_j, \rho\}$ through Φ. Notice that it then always holds that $\tau_2 = \rho \circ \tau_1 \circ \rho$.

The following is a fundamental fact in the theory of Moser-Webster, whose proof can be reduced to the proof of Lemma 4.1

Proposition 4.2: Let $(M,0)$ and $(\widetilde{M},0)$ be two Bishop surfaces with Bishop invariant $\lambda \neq 0, \frac{1}{2}, \infty$. Suppose that the Moser-Webster triplet of their complexifications are parameterized by $\{\tau_j, \rho\}$ and $\{\widetilde{\tau_j}, \widetilde{\rho}\}$, respectively. Then $(M,0)$ is holomorphically equivalent to $(\widetilde{M},0)$ if and only if there is a biholomorphic map ψ from $(\mathbf{C}^2,0)$ to $(\mathbf{C}^2,0)$ such that

$$\widetilde{\tau_j} = \psi\tau_j\psi^{-1}, \quad \widetilde{\rho} = \psi\rho\psi^{-1}.$$

4.2: Linear theory of a pair of involutions intertwined by a conjugate holomorphic involution: Assume that we have two involutions $\tau_j :\ (\mathbf{C}^2,0)$ to $(\mathbf{C}^2,0)$ and an anti-holomorphic involution: $\rho : (\mathbf{C^2},\mathbf{0}) \rightarrow (\mathbf{C^2},\mathbf{0})$ such that $\tau_2 = \rho\tau_1\rho$. We always assume that the linear parts T_j of τ_j satisfy the following properties:

$$\begin{cases} T_1,\ T_2 \text{ have no common non-trivial eigenvectors;} \\ \det(T_j) + 1 = trT_j = 0. \end{cases} \tag{4.8}$$

Notice that this is always the case for the holomorphic involutions obtained from Bishop surfaces with Bishop invariant $\lambda \neq 0, \frac{1}{2}, \infty$. Indeed, for such involutions,

$$T_2 = \begin{pmatrix} -1 & -\frac{1}{\lambda} \\ 0 & 1 \end{pmatrix},\ T_1 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\lambda} & -1 \end{pmatrix}. \tag{4.9}$$

We first present the linear theory for these involutions.

Assume τ_j, ρ are linear. Let $\phi = \tau_1 \circ \tau_2$. By studying the normalization of ϕ, one can find a new coordinates system (see [Lemma 2.2, MW]) (x,y) in which

$$\begin{cases} \tau_1(x,y) = (\gamma y, \gamma^{-1}x),\ \tau_2(x,y) = (\gamma^{-1}y, \gamma x), \\ \phi(x,y) = (\mu x, \mu y^{-1}) \text{ with } \mu = \gamma^2, \mu^2 \neq 1. \end{cases} \tag{4.10}$$

Also, it holds either

$$\begin{cases} \rho(x,y) = (\overline{y}, \overline{x}) \ \text{ and } \ \gamma = \overline{\gamma} > 1, \text{ or} \\ \rho(x,y) = (\overline{x}, \overline{y}) \text{ and } |\gamma| = 1, \quad 0 < \arg(\gamma) < \frac{\pi}{2}. \end{cases} \tag{4.11}$$

The coordinates system which put τ_j, ρ into the above normalization is unique up to the scaling map, which maps (x,y) to (ax,ay) with $a = \overline{a}$.

We now discuss how to construct a Bishop surface M such that $(\mathcal{M}, \hat{\tau_j}, \hat{\rho})$ is parameterized through a certain biholomorphic map Φ by the above mentioned set of involutions.

First, we let $\xi = b(\gamma x + y)$, $z = a(x + \gamma y)$, $a, b \in \mathbf{C}$. Then z is invariant under the action of τ_1 and ξ is invariant under the action of τ_2. We will also so construct w and η that they are invariant under the action of both τ_1 and τ_2. We need to choose a, b such that ρ will be associated to the mapping $(z, w, \xi, \eta) \to (\overline{\xi}, \overline{\eta}, \overline{z}, \overline{w})$ in the complexification of the surface. Hence, when $\gamma = \overline{\gamma} > 1$, we need to choose $a = \overline{b}$. When $\gamma\overline{\gamma} = 1$ with $0 < arg(\gamma) < \frac{\pi}{2}$, we chose $a\gamma = \overline{b}$. Hence for ρ in (4.11), we have, respectively, the following expressions:

$$
\begin{cases} (i): \ \xi = b(\gamma x + y), \ z = \overline{b}(x + \gamma y); \\ (ii): \ \xi = b(\gamma x + y), \ z = \overline{b\lambda}(x + \gamma y). \end{cases} \tag{4.12}
$$

We only consider how to construct Bishop surfaces in Case (ii). The simplest quadratic polynomials that are invariant under the action of both τ_j are cxy with $c \in \mathbf{C}^1$. We then want to choose b so that $q_\lambda(x, y)$ is a multiple of xy. In fact,

$$
\begin{aligned}
w =& q_\lambda(z, \xi) = z\xi + \lambda(z^2 + \xi^2) \\
&= |b|^2\overline{\gamma}(\gamma x^2 + \gamma^2 xy + xy + \gamma y^2) \\
&+\lambda\left(b^2(\gamma^2 x^2 + 2\gamma xy + y^2) + \overline{b^2\gamma^2}(x^2 + 2\gamma xy + \gamma y^2)\right)
\end{aligned}
$$

Hence we obtain

$$
\begin{cases} |b|^2 + \lambda b^2\gamma^2 + \overline{b}^2\overline{\gamma^2}\lambda = 0 \\ |b|^2 + \lambda b^2 + \overline{b}^2\lambda = 0. \end{cases} \tag{4.13}
$$

Therefore, we can choose b be such that $|b| = 1$ and $b^2 = -\gamma^{-1}$. For such a choice of b, we have $\lambda = (\gamma + \gamma^{-1})^{-1} > 0$ and $q = z\xi + \lambda(z^2 + \xi^2) = \lambda^{-1}(1 - 4\lambda^2)xy$

Now, it is straightforward to verify that $\{\tau_j, \rho\}$ is a parameterization for the Moser-Webster triplet on the complexification of $w = q_\lambda(z, \overline{z})$ through the map

$$
\Phi(x, y) = (z(x, y), w(z, y) = \lambda^{-1}(1 - 4\lambda^2)xy, \xi(x, y), \eta(x, y) = \overline{w}(x, y) = w(x, y)), \tag{4.14}
$$

where $z(x, y)$ and $\xi(x, y)$ are given by the second formula in (4.12).

Notice that $\lambda = (\gamma + \gamma^{-1})^{-1}$ or $\lambda\gamma^2 - \gamma + \lambda = 0$. Hence when $\gamma\overline{\gamma} = 1$ with $0 < \arg(\gamma) < \pi/2$, the Bishop invariant of the quadric $\lambda > \frac{1}{2}$. Namely, M has a hyperbolic complex tangent at 0. The reader can verify that in the case of $\gamma = \overline{\gamma}$, the involutions studied above parameterize the Moser-Webster triplet for elliptic Bishop quadrics with $\lambda \neq 0$.

4.3. General theory on the involutions and the Moser-Webster normal form: We now study the non-linear involutions τ_j and ρ with $\tau_2 = \rho \circ \tau_1 \circ \rho$, whose linear parts satisfy the property in (4.8). For the purpose of studying Bishop surfaces, one can assume that ρ is conjugate linear. By §3, after a linear change of coordinates, we also assume that in (x, y)-coordinates,

$$(4.15) \qquad \tau_j := \begin{cases} x' = \gamma_j y + p_j(x,y) \\ y' = \gamma_j^{-1} x + q_j(x,y) \end{cases}$$

$$(4.15)' \qquad \rho(x,y) = (\overline{y}, \overline{x}) \text{ and } \gamma = \overline{\gamma} > 1; \quad \text{or } \rho(x,y) = (\overline{x}, \overline{y}) \ \text{ and } |\gamma| = 1, \ \arg(\gamma) \in (0, \pi/2).$$

$$(4.16) \qquad \phi = \tau_2 \circ \tau_1 := \begin{cases} x' = \mu x + f(x,y) \\ y' = \mu^{-1} y + g(x,y) \end{cases}$$

where $\gamma_1 = \gamma_2^{-1} = \gamma, \ \mu = \gamma^2, \ \mu^2 \neq 1, \ p_j, q_j, f, g = o(|(x,y)|)$. We will subject to $\tau_j, \ \phi$ a transformation of the following form:

$$(4.17) \qquad \psi := \begin{cases} x = t + u(t,T) \\ y = T + v(t,T) \end{cases}$$

For any formal power series $p(t,T)$, we can write it as

$$p(t,T) = \sum_{-\infty}^{\infty} p_s(t,T),$$

with $p_s(\tau t, \tau^{-1} T) = \tau^s p_s(t,T)$, for any $\tau \in \mathbf{R}$. We call p_s is of type s. We impose the normalization condition for the transformation in (4.17): $u_1 = v_{-1} = 0$. A fact is that for any ψ as in (4.17), there is a unique factorization: $\psi = \psi_0 \circ \delta$ where ψ_0 is normalized and $\delta(t,T) = (\alpha(tT)t, \beta(tT)T)$ for certain α, β with $\alpha(0) = \beta(0) = 1$.

Theorem 4.3 (Moser-Webster [MW]): Let $\tau_j, \ \rho, \ \phi$ be in (4.13)-(4.14) with $\mu = \gamma^2$ not a root of unity. Then there is a unique normalized transformation ψ of the form (4.17) such that

$$\psi^{-1}\tau_1\psi(t,T) = (\Lambda T, \Lambda^{-1}t), \psi^{-1}\tau_2\psi(t,T) = (\Lambda^{-1}T, \Lambda t),$$
$$\psi^{-1}\phi\psi(t,T) = (Mt, MT), \ \psi^{-1}\rho\psi(t,T) = \rho(t,T)$$

where $\Lambda = \gamma + \sum_{\alpha=1}^{\infty} \Lambda_\alpha(tT) = \gamma + o(1), M = \Lambda^2(tT)$. The most general transformation that makes τ_j into the above normal form is of the form: $\psi \circ \sigma$ with $\sigma(t,T) = (r(tT)t, r(tT)T)$. Here $r(tT) = \overline{r}(tT)$ and $r(0) \neq 0$. Also in these coordinates, Λ satisfies either the property $\Lambda(tT) = \overline{\Lambda}(tT)$ or $\Lambda(tT)\overline{\Lambda}(tT) = 1$, according to the first form or the second form ρ takes in $(4.15)'$, respectively.

Idea of the Proof of Theorem 4.3: The proof is done by an induction argument. Here, we only sketch the the proof how to construct the unique normalized map ψ which puts τ_j, ϕ into their normal form. The reader can find the detailed proof for statements in the theorem, in the original paper of Moser-Webster [§3, MW].

Assume that there is a ψ whose terms of order less than m can be chosen uniquely so that $\psi^{-1}\tau_j\psi$ has the property in the theorem up to order $m-1$. Thus we assume that τ_j take the following form:

$$\tau_j : \begin{cases} x' = \Lambda_j y + p_j + \cdots, \\ y' = \Lambda_j^{-1} x + q_j + \cdots \end{cases}$$

where $\Lambda_j(xy) = \Lambda_j(xy)$ are polynomial of degree $< m-1$, p_j, q_j are holomorphic polynomials of degree $m \geq 2$. The dots denote terms of order at least $m+1$. Using $\tau_j^2 = Id$ and noting that $\Lambda_j(\tau_j) = \Lambda_j + O(m)$, we get

$$\gamma_j q_j(x, y) + p_j(\gamma_j x, \gamma_j^{-1} y) = 0. \tag{4.17$'$}$$

It then follows that

$$\phi : \begin{cases} x' = Mx + a + ..., \\ y' = M^{-1} y + b + ..., \end{cases}$$

where $M = \Lambda_1\Lambda_2^{-1}$ and $a = \gamma_1 q_2 + p_1(\gamma_2 y, \gamma_2^{-1} x)$, $b(x, y) = \gamma_1^{-1} p_2(x, y) + q_1(\gamma_2 y, \gamma_2^{-1} x)$.

We want to choose u, v so that $\widetilde{\phi} = \psi^{-1}\phi\psi$ has the form given in Theorem 4.3, modifying terms of order at least $(m+1)$. Then one can see that it forces $\psi^{-1}\tau_j\psi$ also to have the form as in Theorem 4.3 modifying terms of order at least (m+1). Let $\widetilde{\phi}$ be in Theorem 4.3. Since $\psi\widetilde{\phi} = \phi\psi$, we have:

$$\begin{cases} u(\mu t, \mu^{-1} T) - \mu u(t, T) = (a - \widetilde{a})(t, T) \\ v(\mu t, \mu^{-1} T) - \mu^{-1} v(t, T) = (b - \widetilde{b})(t, T). \end{cases} \tag{4.18}$$

We want to make $\widetilde{a_s} = 0$ for $s \neq 1$ and $\widetilde{b_s} = 0$ for $s \neq -1$. This leads to the equation:

$$\begin{cases} (\mu^s - \mu) u_s = a_s, \ s \neq 1 \\ (\mu^s - \mu^{-1}) v_s = b_s, \ s \neq -1, \end{cases} \tag{4.19}$$

which clearly can be solved by the assumption.

Then, $\widetilde{a}_1 = a_1 = A(tT)t$, $\widetilde{b}_{-1} = b_{-1} = B(tT)T$, and ψ is uniquely determined up to order m.

We next show that $p_j(x, y) = P_j(xy)y, q_j(x, y) = Q_j(xy)x$. By (4.17)$'$, we have

$$\gamma q_2 + p_1\tau_2 = tA, \ \gamma^{-1} p_2 + q_1\tau_2 = TB.$$

By (4.17), $q_1 = -\gamma^{-1} p_1\tau_1, \ q_2 = -\gamma p_2\tau_2$ up to order m. Therefore, $p_1 - \mu p_2 = \gamma^{-1} yA$, $p_2 - p_1\phi = \gamma_1 xB$. This then leads to $p_1 - \mu p_1\phi = y(\gamma^{-1}A + \mu\gamma B), \ p_2 - \mu p_2\phi = y(\gamma^{-1}\mu A + \gamma B)$. Since μ is not a root of unity, this implies that p_j are of type $s = -1$. Similarly, we can get q_j are of type $s = +1$.

Returning to τ_j, ϕ, we may write

$$\tau_j : \begin{cases} x' = (\Lambda_j + P_j)y + ... \\ T' = (\Lambda_j^{-1} + Q_j)x + ... \end{cases}$$

with $\Lambda_2 = \Lambda_1^{-1}$. One can also verifies that $(\Lambda_j + P_j)(\Lambda_j^{-1} + Q_j) = 1 + O(m)$. By induction, we proved the existence of ϕ, which normalizes τ_j and ϕ.

The rest of the proof is similarly done, which we refer the reader to [§3, MW]) ∎.

The following result of Moser-Webster provides a convergence result when γ does not have norm 1. The proof is based on a majorant argument, motivated by the study of the normalization problem for area preserving mappings in mechanics. (See [SM]). The proof can be found in [§4, MW].

Theorem 4.4: Let $\{\tau_j,\ \rho\}$ be as in Theorem 4.3. Assume that $|\gamma| \neq 1$. Then the normalization ψ in Theorem 4.3 and the normal forms for τ_j, ϕ are convergent near the origin.

Making using of Theorem 4.2, Theorem 4.3 and a similar way for constructing Bishop surfaces from the involutions as discussed in §4.2, Moser-Webster obtained the following Theorem. (See [§5, MW])

Theorem 4.5 (Moser-Webster): Let $(M, 0)$ be a real analytic Bishop surface with Bishop invariant $\lambda \in (0, 1/2)$. Then there is a holomorphic change of coordinates, such that in the new coordinates, M is represented by an equation of the form:

$$w = z\overline{z} + (\lambda + \epsilon w^s)(z^2 + \overline{z^2}), \quad \epsilon = 0, \pm 1. \tag{4.20}$$

§5. Geometric Method to the Study of Local Equivalence Problems

The method we discussed in the previous sections is fundamentally based on the understanding of the associated power series. Results obtained in such a manner are usually easy to apply; and invariants obtained so are relatively easy to computer. However, it mainly applies to real analytic submanifolds. The convergence issue may also be very difficult to handle in certain cases. In this section, we introduce to the reader a geometric approach for the study of the equivalence problem, initiated from the work of E. Cartan. This method applies to smooth CR generic submanifolds. The invariants are the so-called curvature functions and their covariant derivatives.

Notice that the invariants from the power series method are usually embedded into the coefficients of the normal forms. There are many references related to the topics discussed here. We mention [Ga] [CM] [Ch] [CJ] [Ja] [HJY] [HJ2], to name a few.

5.1 Cartan's theory on the equivalent problem: Let $V, \widetilde{V} \subset \mathbf{R}^n$ be open subsets with $p \in V$ and $\widetilde{p} \in \widetilde{V}$. Let $\theta_V = (\theta_V^1, ..., \theta_V^n)^t$ and $\widetilde{\theta}_{\widetilde{V}} = (\widetilde{\theta}_{\widetilde{V}}^1, ..., \widetilde{\theta}_{\widetilde{V}}^n)^t$ be co-frames on V and $\widetilde{V}$, respectively. Let $G \subset GL(n, \mathbf{R})$ be a connected linear subgroup. We would like to understand the following question: When does there exist a diffeomorphism Φ from V to $\widetilde{V}$ with $\Phi(p) = q$ such that $\Phi^*(\widetilde{\theta}_{\widetilde{V}})(p) = \gamma_{V\widetilde{V}}(p)\theta_V$, where $\gamma_{V\widetilde{V}}(p) \in G$ for each p?

To answer the question, we construct its G-co-frame bundle (Y, π, V), where $\forall p \in V$, $\pi^{-1}(p) = \{g \cdot \theta_V(p) : g \in G\}$. (Since we only consider the local problem, we can identify Y as the product manifold $V \times G$.)

Notice that G acts smoothly from the left on Y, which is defined as follows: $\forall C \in G$, and $P = g \cdot \theta_V(p) \in \pi^{-1}(p)$,

$$(C, P) = (Cg) \cdot \theta_V(p) \in \pi^{-1}(p).$$

This action makes Y into a so-called G-structure bundle over V

Now, θ_V can be lifted naturally to globally defined 1-forms $\omega = (\omega_1, \cdots, \omega_n)^t$: $\omega_l|_{g\theta_V(p)} = g\pi_V^*(\theta_V^l(p))$.

Similarly, we can define a G-structure co-frame bundle $(\widetilde{Y}, \widetilde{\pi}, \widetilde{V})$

In what follows, when there is no confusion, we identify the space Y with $V \times G$ through a manner, which should be obvious from the context. For instance, in the following lemma, Y is identified with $V \times G$. through the map $g\theta_V(p) \mapsto (p, g)$.

Lemma 5.1 is simple but important for Cartan's theory.

Lemma 5.1: There exists a diffeomorphism $\Phi : V \to \widetilde{V}$ with $\Phi(p) = \widetilde{p}$ satisfying $\Phi^*(\widetilde{\theta}_{\widetilde{V}}) = \gamma_{V\widetilde{V}}\theta_V$, where $\gamma_{V\widetilde{V}}$ smoothly maps V into G, if and only if there exits a diffeomorphism $\Phi^1 : V \times G \to \widetilde{V} \times G$ such that

$$\Phi^{1*}\widetilde{\omega} = \omega, \quad with\ \Phi^1(P) = \widetilde{P},$$

where $P \in \pi_V^{-1}(p)$ and $\widetilde{P} \in \pi_{\widetilde{V}}^{-1}(\widetilde{p})$.

Proof of Lemma 5.1: We need only to show that the existence of Φ^1 gives the required Φ. (This is because if we know Φ, we can set $\Phi^1(u, S) = (\Phi(u), S\gamma_{V\widetilde{V}}^{-1}(u))$.)

Assume that $\Phi^1 : Y \to \widetilde{Y}$ is a diffeomorphism such that $\Phi^{1*}(\widetilde{\omega}) = \omega$. Write $\Phi^1(u, S) = (\Phi(u, S), T(u, S))$ with $u \in V$, $\Phi(u, S) \in \widetilde{V}$ and $T(u, S) \in G$.

The assumption that $\Phi^*(\widetilde{\omega}) = \omega$ gives that

$$\Phi^{1*}(T\pi^*_{\widetilde{V}}(\widetilde{\theta_{\widetilde{V}}})) = S\pi^*_V(\theta_V), \quad \text{or}$$
$$(\pi_{\widetilde{V}} \circ \Phi^1)^*(\widetilde{\theta_{\widetilde{V}}}) = (T(u,S))^{-1}S\pi^*_V(\theta_V), \quad \text{or}$$
$$(\Phi(u,S))^*(\widetilde{\theta}_{\widetilde{V}}) = (T(u,S))^{-1}S\theta_V.$$

Since $\{\widetilde{\theta}_{\widetilde{V}}\}$ is a co-frame fro $T^*(\widetilde{V})$ and $\{\theta_V\}$ is a co-frame for T^*V, we conclude that the partial derivatives of $\Phi(u,S)$ with respect to the group variables must be zero. Hence $\Phi(u,S) = \Phi(u)$. In particular, $T(u,S)^{-1}S = \gamma_{V\widetilde{V}}(u)$. The proves the existence of the required map from V to $\widetilde{V}$. ∎

Let dim $G = r$. Then $\dim(Y) = n+r$. With the forms $\omega_1, ..., \omega_n$, we would like to add r more 1-forms $\omega^{n+1}, ..., \omega^{n+r}$ on $V \times G$ to form a co-frame Ω over Y such that $\Phi^{1*}(\widetilde{\omega}) = \omega$ if and only if $\Phi^{1*}(\widetilde{\Omega}) = \Omega$. If this is the case, we call such an equivalence problem an e-equivalence problem. Whether we can extend ω to Ω to reduce to an e-equivalence problem depends strongly on the property of the group G. Fortunately, for the CR equivalence problem for Levi non-degenerate hypersurfaces, we do have such a reduction which is the content of the Cartan-Chern-Moser theory ([CM]).

Suppose that ω has an extension to Ω such that there is a diffeomorphism Φ from Y to $\widetilde{Y}$ with $\Phi^*(\omega_j) = \omega_j$ ($j \leq n$) if and only if $\Phi^*(\widetilde{\Omega}) = \Omega$, namely, $\Phi^*(\widetilde{\omega}^j) = \omega^j$ for any $j \in \{1, ..., n+r\}$. The forms $\{\omega^j\}$ for $j \geq n+1$ are called the connection forms.

Next we introduce Cartan's method for the study of the e-equivalence problem, by introducing a new type of invariant functions from what presented in the above sections.

Let $\Omega = \{\omega_j\}$ be a coframe over a domain $V \subset \mathbf{R}^n$. For any differentiable function γ over V, we define its *covariant partial derivative*:

$$d\gamma = \sum \gamma_{|i}\omega^i. \tag{5.1}$$

Since $\{\omega_j\}$ is a co-frame, we can uniquely write $d\omega^i = \sum C^i_{jk}\omega^j \wedge \omega^k$ with $C^i_{jk} = -C^i_{kj}$. Let $\widetilde{\Omega} = \{\widetilde{\omega}_j\}$ be a co-frame over another domain $\widetilde{V} \subset \mathbf{R}^n$. Apparently, if there is a Φ with $\Phi^*(\widetilde{\Omega}) = \Omega$, then it mus hold

$$\widetilde{C}^i_{jk|l} \circ \Phi = \Phi^*(\widetilde{C}^i_{jk|l}) = C^i_{jk|l}.$$

Hence $\{C^i_{jk|l}\}$ are the simplest invariant functions attached to the e-equivalence problem for the co-frame Ω. Now, we can inductively take the covariant derivatives of the obtained invariants to get new invariant functions. More precisely, for each integer s with $s \geq 1$, we define

$$\Gamma_s(\Omega, V) := \left\{C^i_{jk}, C^i_{jk|l_1}, \cdots, C^i_{jk|l_1 \ldots l_s} \mid 1 \leq i,j,k,l_1,...,l_s \leq n+r\right\}, \tag{5.2}$$

which is written as a lexicographically ordered set. We define

$$k_s(p) := \mathrm{rank}\{d\ \Gamma_s(\Omega, V)\}(p), \quad p \in V, \tag{5.3}$$

to be the dimension of the span of the differentials which occur in the ordered set $\Gamma_s(\Omega, V)$. The *order* of the e-structure Ω at $p \in V$ is the smallest $j_0 = j_0(p)$ such that

$$k_{j_0}(p) = k_{j_0+1}(p).$$

In this case, the *rank* of the e-structure Ω at p is defined as

$$\rho_0 = \rho_0(p) := k_{j_0}(p).$$

We say that the e-structure Ω is *regular* of order j_0 and of rank ρ_0 at $p \in V$ if there exists a neighborhood U_p of p in V such that the order $j_0(q) \equiv$ *constant* and rank $\rho_0(q) \equiv$ *constant*, $\forall q \in U_p$. Then we can find ρ_0- functions $\{g_1, ..., g_{\rho_0}\} \subset \Gamma_{j_0}(\Omega, V)$, and a certain neighborhood U_p of p in Y so that $d\ g_1 \wedge \cdots \wedge d\ g_{\rho_0} \neq 0$,

$$d\ g \wedge d\ g_1 \wedge \cdots \wedge dg_{\rho_0} \equiv 0 \text{ on } U_p, \quad \text{for any } g \in \Gamma_{j_0+1}(\Omega, V). \tag{5.4}$$

Notice $0 \leq j_0 \leq n + r - 1$. The case $j_0 = 0$ occurs when the functions $C^i_{jk} \equiv$ *constant* for all i, j and k. And the case $j_0 = n + r - 1$ occurs if and only if one invariant function is added at each jet level. Notice that $0 \leq \rho_0 \leq n + r - 1$. When $\rho_0 = n + r - 1$, we say that $\Gamma(\Omega, V)$ is of the *maximal rank.*

Next, for each $g \in \Gamma_{j_0+1}(\Omega, Y)$, since $dg \wedge dg_1 \wedge ... \wedge dg_{\rho_0} \equiv 0$, we conclude that there is a unique function A_g such that

$$g = A_g(g_1, ..., g_{\rho_0})$$

where A_g is defined near a neighborhood of $(g_1(p), ..., g_p(p))$ which is called the *the relation function* of g with respect to $\{g_1, ..., g_{\rho_0}\}$.

The following fundamental theorem is due to Cartan.

Theorem 5.2 (E. Cartan [Ga]): Let Ω and $\widetilde{\Omega}$ be two smooth regular e-structures of order j_0 and rank ρ_0. Let $g_1, ..., g_{\rho_0}$ be as in (5.14). Let $\widetilde{g}_1, ..., \widetilde{g}_{\rho_0}$ be such that they have the identical lexicographic indices as for $g_1, ..., g_{\rho_0}$. Then the following statements are equivalent:
(i) There exists a C^∞ diffeomorphism $\Phi : (V, p) \to (\widetilde{V}, \widetilde{p})$ with $\Phi^*\widetilde{\Omega} = \Omega$.
(ii) $\widetilde{g}_j(\widetilde{p}) = g_j(p)$ holds for $1 \leq j \leq \rho_0$, and for any function $g \in \Gamma_{j_0+1}(\Omega, V)$, and $\widetilde{g} \in \Gamma_{j_0+1}(\widetilde{\Omega}, \widetilde{V})$ with the same lexicographic order, it holds that $A_g = A_{\widetilde{g}}$ near $(g_1(p), ..., g_p(0))$.

Suppose that Ω is a real analytic co-frame and V is connected. Then, there is a proper real analytic subset E such that any point in $V-E$ is a regular point. Also, from the uniqueness property of real analytic functions, the order and the rank of Ω are all the same in $V-E$. We define the order and the rank of Ω in V to be the order and the rank of Ω at any point in $V-E$.

We call Ω an algebraic co-frame if $\omega^j = \sum h_l^j \mathrm{d}x^l$ with h_l^j Nash algebraic smooth functions. We define the algebraic degree of ω^j to be the maximum degree of the algebraic functions h_l^j. Here, we recall that for a Nash algebraic smooth function $h \not\equiv 0$, there is an irreducible polynomial $P(x,X)$ in (x,X) such that $P(x,h)\equiv 0$. Then we define the degree of h to be the degree of the polynomial $P(x,X)$. It is apparent that when Ω is an algebraic co-frame, then any curvature functions and relation functions are algebraic, too. Suppose that the order of Ω is j_0. We set

$$\ell(\Omega) = \max_{g\in\Gamma_{j_0+1}}\deg(g). \tag{5.5}$$

Then we have the following versions of the Cartan theorem in the analytic category and algebraic category, which are a lot of more convenient to apply:

Theorem 5.3: Let Ω and $\widetilde{\Omega}$ be analytic e- structures at p and $\widetilde{p}$, respectively, with p a regular point of Ω. Then the following are equivalent:
(i). There exists a C^ω diffeomorphism $\Phi : (V,p) \to (\widetilde{V},\widetilde{p})$ such that

$$\Phi^*\widetilde{\Omega} = \Omega.$$

(ii). $\Gamma_k(\widetilde{\Omega},V)(\widetilde{p}) = \Gamma_k(\Omega,V)(p)$ holds for all k.
(iii). Suppose that Ω and $\widetilde{\Omega}$ have order j_0 and rank ρ_0 at p and $\widetilde{p}$, respectively. Also assume that $\widetilde{p}$ is a regular point for $\widetilde{\Omega}$. Let $g_1,...,g_{\rho_0}$ be as above, and let $\widetilde{g}_1,...,\widetilde{g}_{\rho_0}$ be the corresponding relation functions with the same lexicographic order as for $g_1,...,g_{\rho_0}$. Then $\widetilde{g}_j(\widetilde{p}) = g_j(p)$ holds for $1\le j\le \rho_0$, and for any function $g\in\Gamma_{j_0+1}(\Omega,V)$, and $\widetilde{g}\in\Gamma_{j_0+1}(\widetilde{\Omega},\widetilde{V})$ with the same lexicographic order, it holds that $A_g = A_{\widetilde{g}}$ near $(g_1(p),...,g_p(0))$.

Theorem 5.4([HJ2]): Suppose that Ω and $\widetilde{\Omega}$ are algebraic co-frames with

$$l_0 = \max\{\ell(\Omega),\ell(\widetilde{\Omega})\}.$$

Assume that $p\in V$ is a regular point for Ω. (See (5.5) for the definition of $\ell(\Omega)$ and $\ell(\widetilde{\Omega})$). Let $\widetilde{p}\in\widetilde{V}$. Then the following statements are equivalent:
(i) $\Gamma_{2l_0^3}(\Omega,V)(p) = \Gamma_{2l_0^3}(\widetilde{\Omega},\widetilde{V})(\widetilde{p})$.

(ii) There is a real analytic diffeomorphism Φ^1 from a neighborhood of $p \in V$ to a neighborhood of $\widetilde{p}$ in$\widetilde{V}$ such that $\Phi^{1*}(\widetilde{\Omega}) = \Omega$.

We mention that both Theorem 5.3 and Theorem 5.4 can be stated in the holomorphic category when Ω, $\widetilde{\Omega}$ are holomorphic or holomorphically algebraic co-frames. For instance, we have the following:

Theorem 5.3$'$: Let Ω and $\widetilde{\Omega}$ be holomorphic co-frames at $p \in V$ and $\widetilde{p} \in \widetilde{V}$, respectively. Here V (or, $\widetilde{V}$) is a neighborhood of p (or, $\widetilde{p}$, respectively) in $\mathbf{C}^n$. Then the following are equivalent:
(i). There exists a biholomorphic map $\Phi : (V, p) \to (\widetilde{V}, \widetilde{p})$ such that

$$\Phi^*\widetilde{\Omega} = \Omega, \quad \Phi(p) = \widetilde{p}.$$

(ii). $\Gamma_k(\widetilde{\Omega}, V)(\widetilde{p}) = \Gamma_k(\Omega, V)(p)$ holds for all k.

The proof of these results are based on the Frobenius Theorem. We first prove Theorem 5.2. Apparently, we need only to show that $(ii) \Rightarrow (i)$. ($(i) \Rightarrow (ii)$ can be seen by the basic fact that if Φ is a C^∞ diffeomorphism from (V, p) to $(\widetilde{V}, \widetilde{p})$ with $\Phi^*(\widetilde{\Omega}) = \Omega$, then $\Gamma_j(\Omega, V) = \Gamma_j(\widetilde{\Omega}, \widetilde{V}) \circ \Phi$.)

Proof of Theorem 5.2: Consider the manifold $M \subset V \times \widetilde{V}$ defined by $g_j(x) = \widetilde{g}_j(\widetilde{x})$ for $(x, \widetilde{x}) \approx (p, \widetilde{p})$. Here $\{g_j\}_{j=1}^{\rho_0}$ and $\{\widetilde{g}_j\}$ are as in the Theorem. M is apparently a smooth manifold of codimension ρ near $(p, \widetilde{p})$, for $dg_1 \wedge ... \wedge dg_{\eta_0}(p) \neq 0$, $d\widetilde{g}_1 \wedge .... \wedge d\widetilde{g}_{\rho_0}(\widetilde{p}) \neq 0$.

Consider the differential ideals Δ: Δ is generated by $\{\pi^*(\Omega) - \widetilde{\pi}^*(\widetilde{\Omega})\}$, where π is the projection from $V \times \widetilde{V}$ to V and $\widetilde{\pi}$ is the projection from $V \times \widetilde{V}$ to $\widetilde{V}$. We first claim that Δ, when restricted to M, is an integral differential system.

Indeed, on $V \times \widetilde{V}$,

$$d(\pi^*\omega^a - \widetilde{\pi}^*\widetilde{\omega}^a) = \sum (C^a_{jk} \circ \pi)\pi^*(\omega^j) \wedge \pi^*(\omega^k) - \sum (\widetilde{C}^a_{jk} \circ \widetilde{\pi})\widetilde{\pi}^*(\widetilde{\omega}^j) \times \widetilde{\pi}^*(\widetilde{\omega}^k).$$

Since $C^a_{jk} = A^a_{jk}(g_1, ..., g_\rho)$ and $\widetilde{C}^a_{jk} = A^a_{jk}(\widetilde{g}_1, ..., \widetilde{g}_\rho)$, when restricted to M, we see that $C^a_{jk} \circ \pi \equiv \widetilde{C}^a_{jk} \circ \widetilde{\pi}$. Hence on M, we have

$$\begin{aligned} d(\pi^*\omega^a - \widetilde{\pi}^*\widetilde{\omega}^a) &= \sum C^a_{jk} \circ \pi\{\pi^*(\omega^j) \wedge \pi^*(\omega^k) - \widetilde{\pi}^*(\widetilde{\omega}^j) \wedge \pi^*(\widetilde{\omega}^k)\} \\ &= \sum C^a_{jk} \circ \pi\{\pi^*(\omega^j) \wedge (\pi^*\omega^k - \widetilde{\pi}^*\widetilde{\omega}^k) - \widetilde{\pi}^*(\widetilde{\omega}^k) \wedge (\pi^*(\omega^j) - \widetilde{\pi}^*\widetilde{\omega}^j)\}. \end{aligned}$$

Next, we claim that the rank of $\Delta|_M$ is $n - \rho_0$.

Let us write $y_j = g_j$ for $j = 1, ..., \rho_0$ and extend $(y_1, ..., y_{\rho_0})$ to a coordinate system

$$(y_1, ..., y_{\rho_0}, ..., y_n)$$

near $(g_j(p_0), 0, ..., 0)$. Then the regularity assumption at p gives that

$$dy^j = \sum y^j_{|l}(y_1, ..., y_{\rho_0})\omega^l_V \quad for\ j \leq \rho_0.$$

Also the matrix $(y^a_{|j})$ must have rank ρ_0 by the rank assumption of Ω.

By relabelling $\{\omega_j\}$ if necessary, we can assume that $det(y^\alpha_{|\beta})_{1\leq\alpha,\beta\leq\rho_0} \neq 0$. Let $(y^\alpha_{|\beta})^{-1} = g_{\alpha\beta}$. Then

$$\sum g_{\alpha\beta}dy^\beta = \omega^\alpha + \sum b_{a=\rho_0+1,\cdots,n}\omega^a, \text{ or}$$

$$\omega^\alpha = \sum_{\beta=1}^{\rho_0} g_{\alpha\beta}dy^\beta - \sum_{a=\rho_0+1}^{n} b_{\alpha a}\omega^a.$$

Similarly, we have $\widetilde{g}_{\alpha\beta} = (\widetilde{y}^\alpha_{|\beta})^{-1}$,

$$\widetilde{\omega}^\alpha = \sum_{\alpha,\beta=1}^{\rho_0} \widetilde{g}_{\alpha\beta}\widetilde{dy^\beta} - \sum_{a=\rho_0+1,\cdots,n} \widetilde{b}_{\alpha a}\widetilde{w}^a.$$

By the assumption in (ii), $g_{\alpha\beta} = \widetilde{g}_{\alpha\beta}$, $b_{\alpha a} = \widetilde{b}_{\alpha a}$ for $y_j = \widetilde{y}_j$ $(j \leq \rho_0)$. Hence, when restricted to M,

$$\pi^*(\omega^\alpha) = \widetilde{\pi}^*(\omega^\alpha) \mod_{\rho_0+1\leq a\leq n}\{\pi^*(\omega^\alpha) - \widetilde{\pi}^*(\widetilde{\omega}^a)\}.$$

This proves that the rank of Δ, when restricted to M, is bounded by $n - \rho_0$.

We next show that $\{\pi^*(\omega^a) - \widetilde{\pi}^*(\widetilde{\omega}^a)\}_{\rho_0+1\leq a\leq n}$, when restricted to M, is linearly independent.

Indeed, since $dy_1 \wedge ... \wedge dy_{\rho_0} \wedge d\omega^{\rho_0+1} \wedge ... \wedge d\omega^n = det(y^\alpha_\beta)\omega_1 \wedge ... \wedge \omega_n \neq 0$,

$$d\widetilde{y}_1 \wedge ... \wedge d\widetilde{y}_{\rho_0} \wedge d\omega^{\rho_0+1} \wedge ... \wedge \widetilde{\omega}^n \neq 0.$$

We see that $\{dy_1, ..., dy_{\rho_0}, d\omega^{\rho_0+1}, ..., d\omega^n\}$ $\{d\widetilde{y}_1, ..., d\widetilde{y}_{\rho_0}, d\widetilde{\omega}^{\rho_0+1}, ..., d\widetilde{\omega}^n\}$ are co-frames.

Now, in the $(y, \widetilde{y})$-coordinates, M is defined by $y_j = \widetilde{y}_j$ for $j \leq \rho_0$. Hence, it is easy to see that $\{\pi^*(\omega^a)\}_{a\geq\rho_0+1}$ is of rank $n - \rho_0$ when restricted to M near $(p, \widetilde{p})$. Hence the rank of $\pi^*(\omega^a) - \widetilde{\pi}^*(\omega^a)$ is of rank $n - \rho_0$.

Now, Δ induces a foliation in M with each leaf of real dimension $2n - \rho_0 - (n - \rho_0) = n$. Letting $\mathcal{L}$ be a leaf in M, that passes through $(p, \widetilde{p})$. We claim that $\pi_* : T_{(p,\widetilde{p})}\mathcal{L} \to T_pV$ is an isomorphism. For this, we need only to show that π_* is injective.

Suppose $X \in T_{(p,\widetilde{p})}\mathcal{L}$ be such that $\pi_*(X) = 0$. Then

$$0 =< \pi^*(\omega^j) - \widetilde{\pi}^*(\widetilde{\omega}^j), X >= -(\widetilde{\omega}^j, \widetilde{\pi}_*(X))$$

for all j. Since $\{\widetilde{\omega}^j\}$ forms a co-frame in $\widetilde{V}$, we get $\widetilde{\pi}_*(X) = 0$. Thus $X = 0$.

Finally, let Φ be such that $\mathcal{L} = \{(x, \Phi(x)) : x \approx p\}$. Then one sees that Φ is precisely the map that we are looking for. ■

Next we give the proof of Theorem 5.3:

Proof of Theorem 5.3: Let $(X_1, ..., X_n)$ be the dual frame of $(\omega_1, ..., \omega_n)$. Namely, $\langle \omega_j, X_l \rangle = \delta_j^l$. Let ϵ_0 be sufficient small such that for any constant vector $(a_1, ..., a_n)$ with $\sum_j |a_j|^2 < \epsilon_0^2$. The integral curve $\gamma_a(t)$ with $\gamma_a(0) = p$ of $\sum_j a_j X_j$ is defined for $|t| < 2$. Namely,

$$\frac{\mathrm{d}\gamma_a(t)}{\mathrm{d}t} = \sum_{j=1}^{n} a_j X_j(\gamma_a(t)), \ \gamma_a(0) = p$$

has a unique solution for $|t| < 2$.

We can similarly define $(\widetilde{X}_1, ..., \widetilde{X}_n)$ and $\widetilde{\gamma_a}(t)$. We then claim that

$$\Gamma_j(\Omega, V)(\gamma_a(t)) \equiv \Gamma_j(\widetilde{\Omega}, \widetilde{V})(\widetilde{\gamma}_a(t)) \quad for \ |t| < 2.$$

To this aim, for $g_j \in \Gamma(\Omega, V)$ and $\widetilde{g}_j \in \Gamma_j(\widetilde{\Omega}, \widetilde{V})$ with the same lexicographic order, we first notice that $g_j(\gamma_a(t)) - \widetilde{g}_j(\widetilde{\gamma}_a(t))$ is real analytic for $|t| < 2$. To prove that $g_j(\gamma_a(t)) \equiv \widetilde{g}_j(\widetilde{\gamma}_a(t))$, we need only to verify that $G_j(t) = g_j(\gamma_a(t)) - \widetilde{g}_j(\widetilde{\gamma}_a(t)))$ vanishes to infinite order at 0. In fact,

$$\begin{aligned}\frac{dG_j(t))}{dt} &= \sum_l g_{j|l}(\gamma_a(t))\langle \omega_l, \sum_k a_k X_k \rangle(\gamma_a(t)) - \widetilde{g}_{j|l}(\widetilde{\gamma}_a(t))\langle \widetilde{\omega}_l, \sum_k a_k \widetilde{X}_k \rangle \\ &= \sum_l \{g_{j|l}(\gamma_a(t))a_l - \widetilde{g_{j|l}}(\widetilde{\gamma}_a(t))a_l\}\end{aligned}$$

Hence, it follows that $G_j(0) = G_j'(0)$. By induction and the given hypothesis, we can conclude that $G_j^{(k)}(t) \equiv 0$ for all k.

Now, we can define $M \subset V \times V' := \{(x, \widetilde{x}) : \Gamma(\Omega, V)(x) = \Gamma(\Omega, \widetilde{V})(\widetilde{x})\}$. Define Δ the same way as in the proof of Theorem 5.2. Then we can similarly construct the required map Φ. (In fact, one can choose Φ that sends $\gamma_a(1)$ to $\widetilde{\gamma}_a(1)$, when $|a| < \epsilon_0$ varies.) ■

For the proof of Theorem 5.4, we refer the reader to [HJ2].

5.2. Segre family of real analytic hypersurfaces: We now explore how Cartan's method can be adapted to the study of the equivalence problem of real hypersurfaces in $\mathbf{C^n}$. We mainly focus on the real analytic category.

Let M be a real analytic hypersurface in $D \subset \mathbf{C^n}$ with real analytic defining function $r \in C^\omega(D)$. Apparently, for any other local defining function r^* of M, $r^* = s^* r$ with $s^*|_M \neq 0$. Hence we can well define its complexification as the complex submanifold: $\mathcal{M} = \{(z,\xi) \in D \times Conj(D) \ : r(z,\xi) = 0\}$. $\mathcal{M}$ is a complex submanifold of complex codimension 1 in $\mathbf{C}^n \times \mathbf{C}^n$ near $M \times conj(M)$. Here for a set $E \subset \mathbf{C}^n$, $Conj(E) := \{\overline{z} \mid z \in E\}$. For each $\xi \sim Conj(M))$, we can define a complex analytic variety $Q_\xi := \{z \in \mathbf{C}^n : r(z,\xi) = 0\}$. We call Q_ξ the *Segre variety* of M with respect to ξ. Notice that $\mathcal{M}$ is foliated by $\{Q_\xi\}$ (In some references, say, in [We1] [Hu4], one defines $Q_\xi := \{z \in D : \ r(z,\overline{\xi}) = 0\}$ for $\xi \in D$). A fundamental fact for Segre family is its invariant property for holomorphic maps. More precisely, if f is a local holomorphic map from (M,p) to $(\widetilde{M},\widetilde{p})$, then $f(Q_{\overline{\xi}}) \subset \widetilde{Q_{\overline{f(\overline{\xi})}}}$ for any ξ near $\overline{p}$. Here $\widetilde{Q_{\overline{f(\overline{\xi})}}}$ is the Segre variety of $\widetilde{M}$ with respect to $\overline{f}(\overline{\xi})$. In particular, when f is a holomorphic map from (M,p) to $(\widetilde{M},\widetilde{p})$, f induces a holomorphic map $(f(z), \overline{f}(\xi))$ from $(\mathcal{M},(p,\overline{p}))$ to $(\widetilde{\mathcal{M}},(\widetilde{p},\overline{\widetilde{p}}))$.

We mention that the above simple property for Segre family has been a basic tool to study the analyticity problem for CR mappings between real analytic hypersurfaces, based on ideas from the original paper of Webster. (See [Hu1] [Hu4] for historic discussions and many related references.) Here, we will use it for a different purpose.

In what follows, we assume that $0 \in M$ and we use (z,ξ) for the coordinates of $\mathbf{C}^n \times \mathbf{C}^n$. Also, we can assume, without loss of generality, that M is defined by an equation of the form $r = 2\mathrm{Im}(z_n) + O(|z| + |\mathrm{Re}(z_n)|)$. In what follows, the indices α, β will have range from 1 to $n-1$. Occasionally, we will write w, η for z_n, ξ_n, respectively. We also use the summation convention: repeated indices imply summation.

On $\mathcal{M}$, there are $(n-1)$ independent holomorphic one forms

$$\theta^\alpha = dz^\alpha|_{\mathcal{M}},\ \theta_\alpha = d\xi_\alpha|_{\mathcal{M}},\ \theta = i d_z r|_{\mathcal{M}} = i r_\alpha dz^\alpha|_{\mathcal{M}} + i r_n dz^n|_{\mathcal{M}}. \tag{5.6}$$

$\{\theta, \theta^\alpha, \theta_\alpha\}$ is a co-frame for $\mathcal{M}$, which depends on the choice of the defining functions.

Next, let $(\widetilde{M},\widetilde{p})$ be another real analytic hypersurface near $\widetilde{p} = 0$ in $\mathbf{C}^n$ with a defining function $\widetilde{r} = 2Im(\widetilde{z}_n) + O(|\widetilde{z}|)$. Define similarly the co-frame $\{\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}_\alpha\}$ on $\widetilde{\mathcal{M}}$ near $(\widetilde{p},\widetilde{p})$.

If there is a biholomorphic map f from $(M,0)$ to $(\widetilde{M},0)$, then we have a holomorphic map $(f(z), \overline{f}(\xi))$ from $(\mathcal{M},0)$ to $(\widetilde{\mathcal{M}},0)$. We say that $(\mathcal{M},0)$ is Segre equivalent to $(\widetilde{\mathcal{M}},0)$ if there is a holomorphic map $\Phi = (\Phi_1(z), \Phi_2(\xi))$ from $(\mathbf{C}^{2n},0)$ to $(\mathbf{C}^{2n},0)$ such that Φ sends each Segre variety Q_ξ of $\mathcal{M}$ near 0 to the Segre variety $Q_{\Phi_2(\xi)}$ of $\widetilde{\mathcal{M}}$. (Apparently, such a map sends

$(\mathcal{M},0)$ to $(\widetilde{\mathcal{M}},0)$). In particular, we see that when $(M,0)$ is equivalent to $(\widetilde{M},0)$, then $(\mathcal{M},0)$ is Segre equivalent to $(\widetilde{\mathcal{M}},0)$. We mention that even if M, $\widetilde{M}$ are strongly pseudoconvex, Faran constructed in [Fa] examples showing that the converse of the above statement fails. However, see Remark 5.7.

Lemma 5.5: $(\mathcal{M},0)$ is equivalent to $(\widetilde{\mathcal{M}},0)$ if and only if there is a holomorphic map $\Phi=(\Phi_1,\Phi_2)=(\phi_j,\psi_k)$ from $(\mathcal{M},0)$ to $(\widetilde{\mathcal{M}},0)$ such that

$$\begin{cases} \Phi^*(\widetilde{\theta})=u\theta, \\ \Phi^*(\widetilde{\theta}^\alpha)=u^\alpha\theta+u^\alpha_\beta\theta^\beta, \\ \Phi^*(\widetilde{\theta}_\alpha)=u_\alpha\theta+v_\alpha\theta_\beta. \end{cases} \tag{5.7}$$

where $u,u^\alpha,u^\alpha_\beta,v_\beta$ are holomorphic near 0 and the holomorphic 1-forms are defined as in (5.6).

Proof of Lemma 5.5: Suppose the existence of the Segre isomorphism $\Phi=(\Phi_1(z),\Phi_2(\xi))$. Notice that $\widetilde{r}(\Phi)$ is also a defining function for $\mathcal{M}$ near 0. Hence $\widetilde{r}(\Phi)=r(z,\xi)S(z,\xi)$ near $\mathcal{M}$ with $S(z,\xi)\neq 0$ near $\mathcal{S}$.

Since $\widetilde{\theta}=i\partial\widetilde{r}$ on $\mathcal{M}$, we have

$$\Phi^*(\widetilde{\theta})=i\partial\widetilde{r}(\Phi_1(z),\Phi_2(\xi))=iS(z,\xi)\partial r(z,\xi)=iS(z,\xi)\theta.$$

$$\begin{aligned}\Phi^*(\widetilde{\theta^\alpha})=\Phi^*(d\widetilde{z}_\alpha)=d\phi_\alpha(z)&=\frac{\partial\phi_\alpha}{\partial z_\beta}dz_\beta+\frac{\partial\phi_\alpha}{\partial z_n}dz_n\\ &=\frac{\partial\phi_\alpha}{\partial z_\beta}dz_\beta+\frac{\partial\phi_\alpha}{\partial z_n}\Big(-i\frac{\theta}{r^n}-\frac{r^\beta}{r_n}dz_\beta\Big)\\ &=\Big(\frac{\partial\phi_\alpha}{\partial z_\beta}-\frac{\partial\phi_\alpha}{\partial z_n}\frac{r_\beta}{r_n}\Big)\theta^\beta-i\frac{\partial\phi_\alpha}{\partial z_n}\frac{1}{r_n}\theta.\end{aligned}$$

Similarly, we can verify the last equality in (5.7). This proves the first part of the lemma. Similarly, if $\Phi=(\Phi_1,\Phi_2)$ is a holomorphic map satisfying (5.7). Then $\frac{\partial}{\partial\xi_\alpha}\Phi_1=0$ and $\frac{\partial\Phi_2}{\partial z_\alpha}=0$. Hence

$$\Phi_1=\Phi_1(z_1,...,z_{n-1},z_n,\xi^n)\ \ and\ \ \Phi_2=\Phi_2(\xi_1,...,\xi_{n-1},z_n,\xi^n).$$

Since $\mathcal{M}$ can be parameterized either by (z_α,ξ) or (z,ξ_α), Φ_1 and Φ_2 can be completely expressed as holomorphic functions in z or ξ, respectively. Also, it is obvious that Φ preserves the Segre varieties. ■

Now let $(\mathcal{M},P)$ be as before with holomorphic co-frame $\{\theta,\theta^\alpha,\theta_\alpha\}$. Then we can form a G-structure co-frame bundle $\mathcal{Y}$ over $\mathcal{M}$, where G consists of invertible matrices of the form

$$\begin{pmatrix} u & 0 & 0\\ u^\alpha & u^\beta_\alpha & 0\\ v_\alpha & 0 & v^\beta_\alpha\end{pmatrix}$$

To solve the Segre equivalence problem by using Cartan's method, the key step is to find the co-frame on $\mathcal{Y}$, through which the G-equivalence can be reduced to the $\{e\}$-equivalence problem.

Assume that M is strongly pseudoconvex at 0 and $\mathcal{M}$ is defined by

$$r = z_n - \rho(z^\alpha, \xi_\alpha, \xi^n) = z_n - \xi^n + o(|z^\alpha| + |\rho^\alpha|).$$

It is easy to see that $(\rho_\alpha^\beta)|_0$ is precisely the Levi-form of M at 0. Hence, $\det(\rho_\alpha^\beta) \neq 0$ near 0.

Following Chern [Ch] and Chern-Ji [CJ], we choose a co-frame over $\mathcal{M}$ of the following form:

$$\begin{cases} \theta = i(dz^n + r_\alpha dz^\alpha) \\ \theta^\alpha = dz^\alpha, \\ \theta_\alpha = i\frac{r_\alpha^n}{r^n}\theta - \left(r_\alpha^\beta - \frac{r_\alpha^n r^\beta}{r_n}\right)d\xi_\beta, \end{cases} \tag{5.8}$$

where and in what follows, we write $r_\alpha = \frac{\partial r}{\partial z_\alpha}$, $r^\beta = \frac{\partial r}{\partial \xi^\beta}, r_\alpha^\beta = \frac{\partial^2 r}{\partial z_\alpha \partial \xi_\beta}$, etc..

For forms in (5.8), we have:

$$d\theta = i\theta^\alpha \wedge \theta_\alpha. \tag{5.9}$$

Indeed, notice that

$$\begin{aligned} dr_\alpha &= r_{\alpha\beta}dz^\beta + r_\alpha^n d\xi_n + r_\alpha^\beta d\xi^\beta \\ &= r_{\alpha\beta}dz^\beta + \frac{r_\alpha^n}{r^n}(-i\theta - r^\beta d\xi^\beta) + r_\alpha^\beta d\xi_\beta \\ &= r_{\alpha\beta}dz^\beta - \theta_\alpha. \end{aligned}$$

Hence $\theta_\alpha = -dr_\alpha + r_{\alpha\beta}dz^\beta$ and $d\theta = idr_\alpha \wedge dz^\alpha = i\theta^\alpha \wedge \theta_\alpha$.

Now, by the Levi non-degeneracy of M at 0,

$$\begin{cases} z^\alpha = z^\alpha \\ z_n = z_n \\ \rho_\alpha = \rho_\alpha(z^\alpha, \xi_\alpha, \xi_n) \end{cases}$$

can be used to uniquely solve for $(z^\alpha, \xi_\alpha, z_n)$ by the data $(z^\alpha, \rho_\alpha, z^n)$. Hence, we can use $(z^\alpha, \rho_\alpha, z^n)$ for the coordinates of $\mathcal{M}$. In the $(z^\alpha, z_n, \rho_\beta)$ coordinates, we have the following formula:

$$\begin{cases} \theta = i(dz^n - \rho_\alpha dz^\alpha), \\ \theta^\alpha = dz^\alpha, \\ \theta_\alpha = d\rho_\alpha - \rho_{\alpha\beta}dz^\beta, \quad d\theta = i\theta^\alpha \wedge \theta_\alpha. \end{cases} \tag{5.10}$$

Here $\rho_{\alpha\beta}$ are holomorphic functions in (z, ρ_α).

Next, let $(\widetilde{\mathcal{M}},0)$ be the complexification of another real analytic hypersurface $(\widetilde{M},0)$. We also choose the same type of the co-frame $(\widetilde{\theta},\widetilde{\theta}^{\alpha},\widetilde{\theta}_{\alpha})$ on $(\widetilde{\mathcal{M}},0)$ as in (5.10). Now, suppose that Φ is a Segre isomorphism from $(\mathcal{M},0)$ to $(\widetilde{\mathcal{M}},0)$, then

$$\begin{cases} \Phi^*(\widetilde{\theta}) = u\theta, \\ \Phi^*(\widetilde{\theta}^{\alpha}) = u^{\alpha}_{\beta}\theta^{\beta} + u^{\alpha}\theta, \\ \Phi^*(\widetilde{\theta}_{\alpha}) = v^{\beta}_{\alpha}\theta_{\beta} + v_{\alpha}\theta. \end{cases}$$

with $d\widetilde{\theta} = i\widetilde{\theta}^{\alpha}\wedge\widetilde{\theta}_{\alpha}$ and $u, u^{\alpha}_{\beta}, v^{\alpha}_{\beta}, u^{\alpha}, v_{\beta}$ holomorphic near the origin.

Hence $du\wedge\theta + ud\theta = i(u^{\alpha}_{\beta}\theta^{\beta} + u^{\alpha}\theta)\wedge(v^{\beta}_{\alpha}\theta_{\beta} + v_{\alpha}\theta)$, from which we get the following

$$\begin{cases} \delta^l_k u = u^{\alpha}_l v^k_{\alpha}, \\ du = iu^{\alpha}_{\beta}v_{\alpha}\theta^{\beta} - iu^{\alpha}v^{\beta}_{\alpha}\theta_{\beta} + t\theta. \end{cases}$$

Next, we consider the $\mathbf{C}^* := \mathbf{C}\setminus\{0\}$ structure bundle $\mathcal{E}_0 = \mathcal{M}\times\mathbf{C}^*$ on $\mathcal{M}$, which can be identified with the $\mathbf{C}^*$-fiber bundle whose fiber $\pi^{-1}(P)$ over $P\in\mathcal{M}$ is precisely $\{u\theta\}$ with $u\in\mathbf{C}^*$. Then $\omega = u\theta$ is a tautological global holomorphic 1-form on $\mathcal{E}_0$. Notice that

$$d\omega = ud\theta + du\wedge\theta = iu\theta^{\alpha}\wedge\theta_{\alpha} + \omega\wedge(-\frac{du}{u}).$$

Define co-frame

$$\begin{cases} \omega^{\alpha} = u^{\alpha}\theta + u^{\alpha}_{\beta}\theta^{\beta}, \\ \omega_{\alpha} = v_{\alpha}\theta + v^{\beta}_{\alpha}\theta_{\beta}, \end{cases}$$

where $u^{\gamma}_{\beta}v^{\beta}_{\kappa} = \delta^{\gamma}_{\kappa}u$. Then

$$d\omega = i\omega^{\alpha}\wedge\omega_{\alpha} + \omega\wedge\left(-\frac{du}{u} - i\frac{u^{\alpha}}{u}v^{\beta}_{\alpha}\theta_{\beta} + iu^{\alpha}_{\beta}v_{\alpha}\theta^{\beta}\right).$$

Let $\phi = -\frac{du}{u} - i\frac{u^{\alpha}}{u}v^{\beta}_{\alpha}\theta_{\beta} + iu^{\alpha}_{\beta}v_{\alpha}\theta^{\beta} + t\omega$. Then, the above motivates us to consider co-frames of the following form:

$$\begin{cases} \omega = ud\theta, \\ \omega^{\alpha} = u^{\alpha}\theta + u^{\alpha}_{\beta}\theta^{\beta}, \\ \omega_{\alpha} = v_{\alpha}\theta + v^{\beta}_{\alpha}\theta_{\beta}, \\ \phi = -\frac{du}{u} - i\frac{u^{\alpha}}{u}v^{\beta}_{\alpha}\theta_{\beta} + iu^{\alpha}_{\beta}v_{\alpha}\theta^{\beta} + t\theta \\ \delta^l_k u = u^l_{\alpha}v^{\alpha}_k. \end{cases} \tag{5.11}$$

A basic property for the above co-frames is the relation:

$$d\omega = i\omega^{\alpha}\wedge\omega_{\alpha} + \omega\wedge\phi. \tag{5.12}$$

Choose a special co-frame:

$$\begin{cases} \omega^0 = ud\theta, \\ \omega^{0\alpha} = u\theta^\alpha, \\ \omega^0_\alpha = \theta_\alpha, \\ \phi^0 = -\frac{du}{u}. \end{cases}$$

Then, we have

$$\begin{cases} \omega = \omega^0, \\ \omega^\alpha = \frac{u^\alpha}{u}\omega^0 + \frac{u^\alpha_\beta}{u}\omega^{0\beta} \\ \omega_\alpha = \frac{v_\alpha}{u}\omega^0 + v^\beta_\alpha\omega^0_\alpha, \\ \phi = \phi^0 - i\frac{u^\alpha}{u}v^\beta_\alpha\omega^0_\beta + iu^\alpha_\beta\frac{v_\alpha}{u^2}\omega^{0\alpha} + t\omega^0. \end{cases}$$

Hence the space of the co-frames in (5.11) form a G_1-structure bundle $\mathcal{Y}$ over $\mathcal{M}$, where G_1 consists of matrices of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{u^\alpha}{u} & \frac{u^\alpha_\beta}{u} & 0 & 0 \\ \frac{v_\alpha}{u} & 0 & v^\beta_\alpha & 0 \\ t & i\frac{v^\alpha}{u^2}u^\beta_\alpha & -i\frac{u_\alpha}{u}v^\alpha_\beta & 1 \end{pmatrix}$$

with $u^l_\alpha v^\alpha_k = \delta^l_k u$. Or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 & 0 \\ v_\alpha & 0 & v^\beta_\alpha & 0 \\ t & iv^\alpha u^\beta_\alpha & -iu_\alpha v^\alpha_\beta & 1 \end{pmatrix}$$

with $u^l_\alpha v^\alpha_k = \delta^l_k$.

Now, the Segre family $(\mathcal{M}, 0)$ and $(\widetilde{\mathcal{M}}, 0)$ are equivalent if and only if there is a holomorphic map F from $\mathcal{E}_0$ to $\widetilde{\mathcal{E}}_0$, sending a certain point in the fiber over 0 to a certain point in the fiber of 0, such that

$$F^* \begin{pmatrix} \widetilde{\omega} \\ \widetilde{\omega}^\alpha \\ \widetilde{\omega}_\alpha \\ \widetilde{\phi} \end{pmatrix} = \gamma_F \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega_\alpha \\ \phi \end{pmatrix}$$

with γ_F valued in G_1. Indeed, this assertion follows directly from the holomorphic version of Lemma 5.1.

Now, we consider the G_1-structure bundle $\mathcal{Y}$ over $\mathcal{E}_0$ and lift the above co-frames to globally defined forms over $\mathcal{Y}$. To be able to use the Cartan theorem, one needs to further complete these forms into a certain co-frame over $\mathcal{Y}$ so that the G_1-equivalence problem is to be reduced to an e-equivalence problem over $\mathcal{Y}$. This completion is done in the paper of Chern-Moser and Chern, which we state as follows:

Theorem 5.6 (Chern [Ch], Chern-Moser [CH]): Let $(M,0)$ be a strongly pseudoconvex real analytic hypersurface at 0 with $(\mathcal{M},0)$ its Segre family. From the holomorphic forms $\omega, \omega^\alpha, \omega_\alpha, \phi$ in (5.11)-(5,12), after lifting them up to $\mathcal{Y}$ (which we still denote by the same letters), one can construct holomorphic 1-forms $\phi^\alpha_\beta, \phi^\alpha, \phi_\beta, \psi$ on $\mathcal{Y}$ such that

$$\Omega := \{\Omega^j,\ 1 \le j \le (n+2)^2-1\} := \{\omega, \omega^\alpha, \omega_\beta, \phi, \phi^\alpha_\beta, \phi^\alpha, \phi_\beta, \psi\} \text{ forms an e-structure on } \mathcal{Y}$$

and these 1-forms are uniquely determined by the following structure equations

$$\begin{aligned}
&d\omega = i\omega^\alpha \wedge \omega_\alpha + \omega \wedge \phi\\
&d\omega^\alpha = \omega^\beta \wedge \phi^\alpha_\beta + \omega \wedge \phi^\alpha\\
&d\omega_\alpha = \phi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \phi + \omega \wedge \phi_\alpha\\
&d\phi = i\omega^\alpha \wedge \phi_\alpha + i\phi^\alpha \wedge \omega_\alpha + \omega \wedge \psi\\
&d\phi^\beta_\alpha = \phi^\gamma_\alpha \wedge \phi^\beta_\gamma + i\omega_\alpha \wedge \phi^\beta - i\phi_\alpha \wedge \omega^\beta - i\delta^\beta_\alpha(\phi_\sigma \wedge \omega^\sigma) - \tfrac{1}{2}\delta^\beta_\alpha \psi \wedge \omega + \Phi^\beta_\alpha\\
&d\phi^\alpha = \phi \wedge \phi^\alpha + \phi^\beta \wedge \phi^\alpha_\beta - \tfrac{1}{2}\psi \wedge \omega^\alpha + \Phi^\alpha\\
&d\phi_\alpha = \phi^\beta_\alpha \wedge \phi_\beta - \tfrac{1}{2}\psi \wedge \omega_\alpha + \Phi_\alpha\\
&d\psi = \phi \wedge \psi + 2i\phi^\alpha \wedge \phi_\alpha + \Psi
\end{aligned}$$

where $\Phi^\beta_\alpha = S^{\beta\sigma}_{\alpha\rho}\omega^\rho \wedge \omega_\sigma + R^\beta_{\alpha\gamma}\omega \wedge \omega^\gamma + T^{\beta\gamma}_\alpha \omega \wedge \omega_\gamma$

$$\begin{aligned}
&\Phi^\alpha = T^{\alpha\gamma}_\beta \omega^\beta \wedge \omega_\gamma - \tfrac{i}{2}Q^\alpha_\beta \omega \wedge \omega^\beta + L^{\alpha\beta}\omega \wedge \omega_\beta\\
&\Phi_\alpha = R^\beta_{\alpha\gamma}\omega^\gamma \wedge \omega_\beta + P_{\alpha\beta}\omega \wedge \omega^\beta - \tfrac{i}{2}Q^\beta_\alpha \omega \wedge \omega_\beta\\
&\Psi = Q^\beta_\alpha \omega^\alpha \wedge \omega_\beta + H_\alpha \omega \wedge \omega^\alpha + K^\alpha \omega \wedge \omega_\alpha
\end{aligned}$$

and $S^{\beta\sigma}_{\alpha\rho} = S^{\beta\sigma}_{\rho\alpha} = S^{\alpha\beta}_{\alpha\rho}$, $R^\beta_{\alpha\gamma} = R^\beta_{\gamma\alpha}$, $T^{\alpha\gamma}_\beta = T^{\gamma\alpha}_\beta$, $L^{\alpha\beta} = L^{\beta\alpha}$, $P_{\alpha\beta} = P_{\beta\alpha}$, $S^{\beta\sigma}_{\alpha\sigma} = R^\alpha_{\alpha\beta} = T^{\alpha\beta}_\alpha = Q^\alpha_\alpha = 0$.

Remark 5.7 Since the Segre isomorphism does not induce the equivalence of the underlying hypersurfaces as demonstrated by Faran in [Fa], the existence of the e-equivalence map Ψ from $(\mathcal{Y},0)$ to $(\widetilde{\mathcal{Y}},0)$ does not induce automatically the biholomorphic equivalence of $(M,0)$ with $(\widetilde{M},0)$. However,if an element $P \in \mathcal{Y}$ with a certain reality condition is mapped to $\widetilde{P}$ with a certain reality property, then we do have the holomorphic equivalence of $(M,0)$ with $(\widetilde{M},0)$. We will briefly discuss this in the following subsection.

5.3: Cartan-Chern-Moser theory for germs of strongly pseudoconvex hypersurfaces: The materials in §5.2 can be directly used to study the equivalence problem for strongly pseudoconvex (or Levi non-degenerate) hypersurfaces. Here we give a quick account on this matter. The reader is referred to [CM] for more details.

Let $(M,0)$ be the germ of a smooth strongly pseudoconvex hypersurface, defined by $r=0$. Here, we assume that $\frac{\partial r}{\partial z_n}(0) \neq 0$. As before, let $\theta = i\partial r$ and $\theta^\alpha = \mathrm{d}z_\alpha$. We have a co-frame

$\{\theta, \theta^\alpha, \overline{\theta^\alpha}\}$ on M. Let

$$G := \left\{ \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 \\ \overline{v^\alpha} & 0 & \overline{u^\alpha_\beta} \end{pmatrix} \,\middle|\, u \in \mathbf{R},\ u^\alpha_\beta, u^\alpha \in \mathbf{C},\ u > 0,\ det(u^\alpha_\beta) \neq 0, \right\} \tag{5.13}$$

be the connected linear subgroup of $G(2n-1, \mathbf{C})$. $M \times G$ is a G-space. Similarly, let $(\widetilde{M}, 0)$ be another strongly pseudoconvex real hypersurface with a similar co-frame $\{\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\overline{\theta^\alpha}}\}$.

It can be verified that there exists a smooth CR mapping $\Phi(z)$ such that $\Phi(M) \subset \widetilde{M}$ if and only if there is a C^∞ diffeomorphism $\Phi : M \to \widetilde{M}$ satisfying

$$\Phi^* \begin{pmatrix} \widetilde{\theta} \\ \widetilde{\theta}^\alpha \\ \widetilde{\theta}_\alpha \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 \\ \overline{u^\alpha} & 0 & \overline{u^\alpha_\beta} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \end{pmatrix} = (\gamma^\alpha_\beta) \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \end{pmatrix} \tag{5.14}$$

where the $(2n-1)\times(2n-1)$ matrix (γ^α_β) defines a smooth mapping from M into G. By Lemma 5.1, there exists a CR isomorphism $\Phi : M \to \widetilde{M}$ if and only if there exists a smooth diffeomorphism $\Phi^1 : M \times G \to \widetilde{M} \times G$ such that

$$\Phi^{1*}\widetilde{\omega} = \omega,\ \Phi^{1*}\widetilde{\omega}^\alpha = \omega^\alpha,\ \Phi^{1*}\overline{\widetilde{\omega}_\alpha} = \overline{\omega_\alpha}, \tag{5.15}$$

where ω, ω^α are similarly defined as in Lemma 5.1.

Define

$$E = M \times \{\omega = u\theta : \omega = \overline{\omega},\ u > 0\}$$

Choose $\theta^\alpha := u^\alpha\theta + u^\alpha_\beta dz^\beta$ for some smooth functions u^α, u^α_β so that $d\theta = i\theta^\alpha \wedge \overline{\theta^\alpha},\ mod(\theta)$. We obtain a co-frame $(\omega, \theta^\alpha, \overline{\theta^\alpha}, \phi_0)$ on E, where $d\omega = iu\theta^\alpha \wedge \overline{\theta^\alpha} + \omega \wedge \phi_0$. Let G_1 be as before. Let $\widetilde{E}$ be the associated bundle over $\widetilde{M}$ with the corresponding co-frame $\{\widetilde{\omega}, \widetilde{\theta}^\alpha, \widetilde{\overline{\theta^\alpha}}, \widetilde{\phi}_0\}$.

Let $\Phi : M \to \widetilde{M}$ be a CR isomorphism. It is easy to verify that Φ induces a unique smooth diffeomorphism, still denoted as Φ, from E to $\widetilde{E}$ satisfying

$$\Phi^* \begin{pmatrix} \widetilde{\theta} \\ \widetilde{\theta}^\alpha \\ \widetilde{\theta}_\alpha \\ \widetilde{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 & 0 \\ \overline{u^\alpha} & 0 & \overline{u^\alpha_\beta} & 0 \\ s & i\overline{u^\alpha}u^\alpha_\beta & -iu^\alpha\overline{u^\alpha_\beta} & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \\ \phi \end{pmatrix} = (\gamma^\alpha_\beta) \begin{pmatrix} \theta \\ \theta^\alpha \\ \theta_\alpha \\ \phi \end{pmatrix}$$

where the $(2n+2)\times(2n+2)$ matrix (γ^α_β) defines a smooth mapping from E into G^r_1, $\theta_\alpha = \overline{\theta^\alpha}$,etc. ($G^r_1$ consists of the matrices of the above form). By Lemma 5.1, the existence of a CR isomorphism $\Phi : M \to \widetilde{M}$ is equivalent to the existence of a smooth diffeomorphism $\Phi^1 : Y := E \times G^r_1 \to \widetilde{Y} := \widetilde{E} \times G^r_1$ such that

$$\Phi^{1*}\widetilde{\omega} = \omega,\ \Phi^{1*}\widetilde{\omega}^\alpha = \omega^\alpha,\ \Phi^{1*}\widetilde{\omega}_\alpha = \omega_\alpha, \Phi^{1*}\widetilde{\phi} = \phi.$$

(Y, π, E) is called the *CR-structure bundle over M*.

The fundamental theorem proved by Cartan-Chern-Moser [CM] asserts that from ω, ω^α, ω_α, ϕ, one can construct 1-forms $\phi^\alpha_\beta, \phi^\alpha, \overline{\phi^\alpha}, \psi$ on Y, with $\omega = \overline{\omega}$, $\phi = \overline{\phi}$, $\psi = \overline{\psi}$, such that

$$\Omega := \{\Omega^j,\ 1 \le j \le (n+2)^2 - 1\} := \{\omega, \omega^\alpha, \overline{\omega^\alpha}, \phi, \phi^\alpha_\beta, \phi^\alpha, \overline{\phi^\alpha}, \psi\}$$

forms an e-structure on Y, and they are uniquely determined by certain structure equations. These structure equations are precisely the restriction of those in Theorem 5.6 from $\mathcal{Y}$ to Y, together with several other reality conditions (see [Theorem 5.5, pp 151, Fa] [BS]), if we assume that M is real analytic.

We let $\theta,\ \theta_\alpha$ be again as defined in §5.2. Since we have the embedding $M \to \mathcal{M}$, by mapping $z \to (z, \overline{z})$, we can regard the bundles E, Y as the subbundles of $\mathcal{E}, \mathcal{Y}$, respectively, as follows (cf.[Fa, (5.9)][BS]): Let

$$E^* := \{(z, \overline{z}, u\theta) \mid z \in M,\ , \overline{u\theta} = u\theta, (u\theta)(T) > 0\} \subset \mathcal{E},$$

over M. Here T is a certain real tangent vector field transversal to $\mathrm{T}^{(1,0)}M + \mathrm{T}^{(0,1)}M$. On E^*, we see $\omega^* = \overline{\omega^*} := u\theta$. Let Y^* be the collection of the frames in $\mathcal{Y}$ restricted to E^* such that $\omega^*_\alpha = \overline{\omega^{*\alpha}}$, $\phi^* = \overline{\phi^*}$ over E^*. Since $\omega^* = \overline{\omega^*}$, $\omega^*_\alpha = \overline{\omega^{*\alpha}}$ and $\phi^* = \overline{\phi^*}$ hold on Y^*, one can check that the structure equations defining Ω^* over Y^* are the same ones defining Ω on Y. Hence, Y and Y^* are G_1^τ-isomorphic. Identify E and Y with E^* and Y^*, respectively. Then the restriction of a function $g \in \Gamma(\Omega, \mathcal{Y})$ on Y equals to the lexicographically corresponding function $g|_Y \in \Gamma(\Omega|_Y, Y)$.

Finally, we mention that the reality condition mentioned in Remark 5.7 is precisely the condition that $P, \widetilde{P}$ are in Y^* or $\widetilde{Y}^*$, respectively. This explains the statement in Remark 3.7

[Alx]: H. Alexander, Proper holomorphic maps in $\mathbf{C}^n$, *Ind. Univ. Math. Journal* (26), 137–146, 1977.

[BER1] S. Baouendi, P. Ebenfelt and L. Rothschild, Real submanifolds in complex spaces and their mappings, Princeton Mathematical Series (47), Princeton, New Jersey, 1999

[BER2] M.S. Baouendi, P. Ebenfelt and L.P. Rothschild, Local geometric properties of real submanifolds in complex spaces, Bull. AMS, 37(2000), 309-336.

[BJT] S. Baouendi, H. Jacobowitz and F. Treves, On the analyticity of CR mappings, *Ann. of Math.*(122), 365-400,1985

[BRZ] S. Baouendi, L. Rothschild and D. Zaitsev, Equivalences of real submanifolds in complex space, *Jour. Diff. Geom.* 59 (2001), no. 2, 301–351.

[BG] E. Bedford and B. Gaveau, Envelopes of holomorphy of certain 2-spheres in $\mathbf{C}^2$, *Amer. J. Math.* (105), 975-1009, 1983.

[EK] E. Bedford and W. Klingenberg, On the envelopes of holomorphy of a 2-sphere in $\mathbf{C}^2$, *Journal of AMS*(4), 623-655, 1991.

[Bis] E. Bishop, Differentiable manifolds in complex Euclidean space, *Duke Math. J.* (32), 1-21, 1965.

[BS1] D. Burns and S. Shnider, Projective connections in CR geometry, Manuscripta Math., 33(1980), 1-26.

[BSW] D. Burns, S. Shnider and R. Wells, On deformations of strictly pseudoconvex domains, *Invent. Math.* (46), 237-253, 1978.

[Ch] S.-S. Chern, On the projective structure of a real hypersurface in C_{n+1}, Math. Scand. 36(1975), 74-82.

[CJ] S.-S. Chern and S. Ji, Projective geometry and Riemann's mapping problem, Math Ann. 302(1995), 581-600.

[CM] S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds *Acta Math.* (133), 219-271, 1974

[DF] K. Diederich and E. Fornaess, Pseudoconvex domains with real analytic boundaries, *Ann. Math.* (107), 371-384, 1978.

[Da] J. P. D'Angelo, Several Complex Variables and the Geometry of Real Hypersurfaces, CRC Press, Boca Raton, 1993

[EHZ1] P. Ebenfelt, X. Huang and D. Zaitsev, The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics, *Amer. Jour. of Math.*, 2002, to appear.

[EHZ2] P. Ebenfelt, X. Huang and D. Zaitsev, Rigidity of CR-immersions into spheres, preprint, 2002.

[Ga] R. Gardner, The method of equivalence and its applications, CBMS-NSF, regional conference series in applied mathematics, 1989.

[Gon] X. Gong, On the convergence of normalizations of real analytic surfaces near hyperbolic complex tangents, *Comment. Math. Helv.* 69 (1994), no. 4, 549–574.

[Fa] J. Faran, Segre families and real hypersurfaces, *Invent. Math.* 60, 135-172, 1980.

[Fe] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.* 26, 1-65, 1974.

[GK] R. Greene and S. Krantz, Deformation of complex structures, estimates for $\overline{\partial}$-equation, and stability of the Bergman kernel, *Advanced in Math.* (43), 1-86, 1982.

[HS] Zheng-Xu He and O. Schramm, Fixed points, Koebe uniformization and circle packings, *Ann. of Math.* (137),369-406, 1993.

[Ho] L. Hörmander, Introduction to Complex Analysis in Several Variables, North Holland, Amsterdam, 1973.

[Hu1] X. Huang, On some problems in several complex variables and Cauchy-Riemann Geometry, Proceedings of ICCM (edited by L. Yang and S. T. Yau), AMS/IP Stud. Adv. Math. 20, 383-396, 2001.

[Hu2] X. Huang, On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions, *Jour of Diff. Geom.* (51), 13-33, 1999

[Hu3] X. Huang, On an n-manifold in $\mathbf{C}^n$ near an elliptic complex tangent *J. Amer. Math. Soc.* (11), 669–692, 1998.

[Hu4] X. Huang, Schwarz reflection principle in complex spaces of dimension two, *Comm. in PDE* (21), 1781-1828, 1996.

[HJ1] X. Huang and S. Ji, Global extension of local holomorphic maps and the Riemann mapping theorem for algebraic domains, *Math. Res. Lett.* (5), 247–260, 1998.

[HJ2] X. Huang and S. Ji, Cartan-Chern-Moser theory on real algebraic hypersurfaces and applications, *Annales. de L'Inst. Fourier* (52), 1793-1831, 2002.

[HJY] X. Huang, S. Ji, and S.S.T. Yau, An example of real analytic strongly pseudoconvex hypersurface which is not holomorphically equivalent to any algebraic hypersurfaces, *Ark. Mat.*, 39(2001), 75-93.

[HK] X. Huang and S. Krantz, On a problem of Moser *Duke Math. J.* (78), 213-228, 1995.

[Ja] H. Jacobowitz, An introduction to CR structures, Mathematical Surveys and Monographs (32), American Mathematical Society, Providence, RI, 1990

[KW1] C. Kenig and S. Webster, The local hull of holomorphy of a surface in the space of two complex variables, *Invent. Math.* (67), 1-21, 1982.

[KW2] C. Kenig and S. Webster, On the hull of holomorphy of an n-manifold in $\mathbf{C}^n$, *Annali Scoula Norm. Sup. de Pisa IV* (11), 261-280, 1984

[Kr] S. Krantz, Function Theory of Several Complex Variables, John Wiley & Sons, New York, 1982.

[Le] H. Lewy, On the boundary behavior of holomorphic mappings, *Acad. Zaz. Lincei.* (35), 1-8, 977.

[Mos] J. Moser, Analytic surfaces in $\mathbf{C}^2$ and their local hull of holomorphy, *Annales Aca - demiæFennicae Series A.I. Mathematica* (10), 397-410, 1985.

[MW] J. Moser and S. Webster, Normal forms for real surfaces in $\mathbf{C}^2$ near complex tangents and hyperbolic surface transformations, *Acta Math.* (150), 255-296, 1983.

[Pi] S. Pinchuk, On the proper holomorphic mappings of real analytic pseudoconvex domains, *Siberian Math.* (15), 909-917, 1974.

[SM] C. L. Siegel and J. Moser, Lectures on Celestial Mechanics, Classics in Mathematics, Springer-verlag, 1995

[Vit] A.G. Vitushkin, Holomorphic mappings and geometry of hypersurfaces, Encyclopaedia of Mathematical Sciences, Vol. 7, Several Complex Variables I, Springer-Verlag, Berlin, 1985, 159-214.

[We1] S. Webster, The rigidity of C-R hypersurfaces in a sphere. *Indiana Univ. Math. J.*(28), 405–416, 1979.

[We2] S.M. Webster, On the mapping problem for algebraic real hypersurfaces, *Invent. Math.* (43), 53-68, 1977.