On a Problem of Moser

by

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0 Introduction

This paper studies the analytic structure of the local hull of holomorphy of a 2-dimensional, real analytic manifold that is embedded in $\mathbb{C}^2$. Our specific purpose is to solve a problem of Jurgen Moser (see [MOS], [MOW]). [In the statement of this problem we shall use certain standard terminology from the literature that will be defined later.] The result is:

THEOREM 0.1 *Let M be a 2-dimensional, real analytic embedded submanifold of $\mathbb{C}^2$. Suppose that $z_0 \in M$ is a non-degenerate elliptic point of M. Then the local hull of holomorphy $\widetilde{M}$ of M near z_0 is a Levi flat hypersurface which is real analytic across the boundary manifold M.*

Recall that for a general closed subset $E \subseteq \mathbb{C}^n$, we define here the *hull of holomorphy* $\widetilde{E}$ of E to be the intersection of all Stein neighborhoods of E.

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The determination of the hull of holomorphy of a subset $E \subseteq \mathbb{C}^n$ is a fundamental problem in complex analysis. In the 1960's, when it was studied by E. Bishop [BIS], one of the motivations was the study of analytic structure in the maximal ideal space of a Banach algebra. Today, now that the function theory of several complex variables is more developed, there are more basic reasons for studying this problem. The papers, [BEKL], [GR], and [ELI] give several instances of the important role of this circle of ideas in the literature.

In full generality, the aforementioned problem is very difficult. Bishop [BIS] first proposed that the hull of holomorphy be determined by using analytic discs attached to M, in the case when M is a smooth, regularly embedded submanifold of $\mathbb{C}^n$ (it is known in general, however—see [STOLZ]—that a set may have a large hull of holomorphy that contains no analytic discs). He classified the local study of the hull in terms of the local geometry of the base point $z_0 \in M$. Namely, it is now understood that it is important to distinguish the case when the tangent space $T_{z_0}M$ has a complex structure from the case when $T_{z_0}M$ is totally real (that is, $T_{z_0}M \cap \sqrt{-1}\,T_{z_0}M = \{0\}$). Points of the second type are of no interest for us because, by the work of Hörmander-Wermer [HOW], the local hull of holomorphy near such a point contains no new points. The situation in the first case is quite different.

When M is a two dimensional real submanifold in $\mathbb{C}^2$, Bishop [BIS] showed that, in the case that $z_0 \in M$ has an isolated complex tangent and satisfies a certain non-degeneracy condition, then a holomorphic change of variables may be effected so that $z_0 = 0$ and the manifold M may be described in complex coordinates (z, w) by

$$w = h(z) = z\overline{z} + \lambda(z^2 + \overline{z}^2) + o(|z|^3)$$

with $0 \leq \lambda < +\infty$ and $\lambda \neq \frac{1}{2}$. Here the constant λ is a biholomorphic invariant of the manifold M. Now it is standard terminology to say that z_0 is an elliptic or hyperbolic point of M according to whether $\lambda \in [0, 1/2)$ or $\lambda > 1/2$ respectively.

In the elliptic case, Bishop obtained a family of analytic discs attached to M by using a Picard-type iteration theory. In the later work of Bedford-Gaveau [BG] and Kenig-Webster [KW], it was shown that the local hull $\widetilde{M}$ of M is foliated by a family of embedded, pairwise disjoint analytic discs. Bedford-Gaveau proved that $\widetilde{M}$ is Lipschitz 1 continuous near z_0 provided that M itself is of class C^5 near z_0. Kenig-Webster proved that $\widetilde{M}$ is C^∞ near z_0 when M is of class C^∞ near z_0.

The analytic structure of $\widetilde{M}$ was studied in the papers of Moser-Webster [MOW] and Moser [MOS]. In [MOW], it was shown that $\widetilde{M}$ is real analytic up to z_0 when the original manifold M itself is real analytic at z_0, provided that $0 < \lambda < 1/2$. The case $\lambda = 0$ is not treated in [MOW]—indeed, the techniques presented there seem not apply to this case. Instead, in [MOS], Moser showed that a formal power series change of variables could be found in the case $\lambda = 0$ so that the manifold M is defined by an equation of the form

$$w = z\overline{z} + z^s + \overline{z}^s + \phi(z) + \overline{\phi(z)}.$$

Here $z_0 \leftrightarrow 0$ and $s,\ 3 \leq s \leq \infty$, is a biholomorphic invariant of the surface M at z_0. Note also that ϕ is a formal power series in z beginning with terms of order at least $s + 1$.

By using a "rapidly convergent" iteration scheme, Moser was able to prove that, when $s = +\infty$, this formal coordinate change is also a convergent analytic coordinate change. However he was unable to prove this statement when $s < \infty$. Equivalently, he left open the question of whether $\widetilde{M}$ is real analytic near z_0 when $s < \infty$.

The purpose of the present work is to answer the above question of Moser. The ideas that motivate our proof can be described as follows: Instead of considering a normal form for the analytic surface M near the distinguished point z_0, we blow up the point z_0 twice. This process makes M into a totally real cylinder in $\mathbb{C}^2$ with a twisted real analytic boundary. We then use a suitable infinite dimensional implicit function theorem to obtain a real analytically parametrized family of (meromorphic) analytic discs that are attached to this cylinder. Before blowing back down, we verify the real analyticity of our foliation in the normal direction at z_0; we finally obtain the full statement of real analyticity of $\widetilde{M}$ near z_0 by using the uniqueness of analytic continuation.

As an application of Theorem 0.1, we use a classical result of É. Cartan [CA] (which essentially states that an analytic Levi flat hypersurface in $\mathbb{C}^2$ is locally biholomorphic to an open subset of $\mathbb{R}^3 = \{\mathrm{Im}\, \mathrm{w} = 0\}$) to derive the following consequence:

THEOREM 0.2 *Let M be a real analytic surface in $\mathbb{C}^2$ and let $z_0 \in M$ be an elliptic point with $\lambda = 0$. Then there is a holomorphic change of coordinates near z_0 so that $z_0 \leftrightarrow 0$ and, in the new coordinates, M is given by*

$$w = z\overline{z} + z^s + \overline{z}^s + \sum_{i+j>s} a_{ij} z^i \overline{z}^j$$

with $\overline{a_{ij}} = a_{ji}$ and $s \geq 3$.

Observe that the content of the corollary is that the non-linear functional equation (1.5) on Page 398 of [MOS] possesses a holomorphic solution. (This assertion has also been conjectured by Moser).

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1 Proof of the Main Result

We now present the proof of Theorem 0.1. For clarity, we divide our discussion into two subsections. We first study the analytic dependence of analytic discs on a real parameter. Then we investigate the local hull of M near the exceptional point.

1.1 Analytic Dependence on a Real Parameter

Let $M \subseteq \mathbb{C}^2$ be an embedded, real, two dimensional manifold. Let z_0 be an isolated complex tangent point of M. Moreover we assume that z_0 is non-degenerate and elliptic (see [BIS] for as discussion of this terminology). By the work of Bishop [BIS] and Kenig-Webster [KW] we know that, after a holomorphic change of coordinates, we may assume that $z_0 = 0$ and the manifold M is given, in the new coordinates (z, w), by

$$w = h(z) = z\overline{z} + \lambda(z^2 + \overline{z}^2) + p(z) + \sqrt{-1}\, k(z), \tag{1.1.1}$$

where p, k are real functions with

$$p(z) = \sum_{i+j\geq 3} a_{ij} z^i \overline{z}^j \qquad \text{and} \qquad k(z) = \sum_{i+j>m} b_{ij} z^i \overline{z}^j .$$

Here $m(\geq 6)$ is an *a priori* given positive integer and $\lambda \in [0, \frac{1}{2})$. Of course, we can make $a_{ji} = \overline{a_{ij}}$ and $b_{ji} = \overline{b_{ij}}$.

We mention that, although our argument works equally well for $\lambda \in (0, 1/2)$ as for $\lambda = 0$, we will assume that $\lambda = 0$ in this paper to simplify our notation. Another major reason we do this is that our main theorem has been settled in the case $\lambda \in (0, \frac{1}{2})$ by the work of Moser-Webster [MW].

In what follows, an analytic disc is a continuous function ψ from the closed unit disc $\overline{\Delta}$ in $\mathbb{C}$ to $\mathbb{C}^2$ that is holomorphic on the interior Δ. We say that an analytic disc ψ is *attached* to M if $\psi(\partial\Delta) \subseteq M$.

Next we set up the equation that will describe our analytic discs. Set $I_\epsilon = (-\epsilon, \epsilon) \subseteq \mathbb{R}$, with $\epsilon > 0$ and $\epsilon << 1$. Let S^1 denote the unit circle in $\mathbb{C}$. We consider a function $\Phi : I_\epsilon \times S^1 \to \mathbb{C}^2$. The function Φ_ϵ acts on variables (r, ξ) with $r \approx 0$ and $\xi \in S^1$. We would like to arrange for $\Phi(r, \cdot\,)$ to have a holomorphic extension to Δ for each fixed r and also that $\Phi(r, \xi) \in M$ when $\xi \in S^1$.

Write $\Phi(r, \xi) = (\phi_1(r, \xi), \phi_2(r, \xi))$. For $r \in I_\epsilon$, we let D_r denote the domain

$$D_r \equiv \{z \in \mathbb{C}^1 : |z|^2 + \sum_{i+j\geq 3} a_{ij} r^{i+j-2} z^i \overline{z}^j < 1\}.$$

Here the numbers a_{ij} are just the same as those coefficients in the formula for $p(z)$. For each r, let $\sigma_r(\xi) = \sigma(r, \xi)$ be a conformal mapping of Δ to D_r. Assume in advance that $\sigma(r, 0) = 0$ and $\sigma'(r, 0) > 0$. These last two conditions can always be arranged and make our choice of $\sigma(r, \xi)$ unique.

A useful observation from classical function theory is that the mapping $\sigma(r, \xi)$ will be real analytic in (r, ξ) and holomorphic in the ξ variable in a neighborhood of $\overline{\Delta}$ (see, for example, [Hu, Lemma 4.1]). In particular, we have

$$\sigma(r, \xi) = \sum_{i \geq 0} \sigma_i(\xi) r^i \qquad \xi \in \overline{\Delta}(0,1), \; r \in I_\epsilon.$$

Here $\sigma_i \in \mathrm{Hol}(\Delta) \cap \mathrm{C}^{\mathrm{m},1/2}(\overline{\Delta})$ and $\|\sigma_i\| < R^i$ for some $R >> 1$. Here and in what follows, we use $\| \; \|$ to stand for the standard norm in the Banach space $C^{m,1/2}(S^1)$ and R for a large positive constant which may be different in different contexts. After a suitable renormalization, the mapping

$$\Phi_0(r, \xi) = (r\sigma(r, \xi), r^2)$$

gives rise to all of the analytic discs attached to the model surface M_0 of M. Here the "model" surface is given by

$$M_0 = \{(z, w) : w = z\overline{z} + p(z)\}.$$

Typically, the strategy for constructing discs attached to M is that (i) it is easy to attach discs to the model surface, (ii) the surface M osculates the model surface to high order at z_0, and (iii) we may then obtain discs attached to M itself from those attached to M_0 by a deformation process.

The ideas in the last paragraphs motivate us to consider a map $\Phi(r, \xi) = (\phi_1(r, \xi), \phi_2(r, \xi))$ such that

$$\phi_1(r, \xi) = r\sigma(r, \xi)(1 + \mathcal{F}(r, \xi)), \tag{1.1.2}$$

where $\mathcal{F}(r, \xi) \approx 0$ when $r \approx 0$ and holomorphic in ξ for each fixed r. This will be our approach, and we shall construct such a function $\mathcal{F}$ later.

Given (1.1.1) and (1.1.2), we find that

$$\phi_2(r,\xi) = |\phi_1(r,\xi)|^2 + p(\phi_1(r,\xi)) + \sqrt{-1}\, k(\phi_1(r,\xi)),$$

where $\xi \in S^1$. In particular, we see that

$$\frac{1}{r^2}\phi_2(r,\xi) = |\sigma(r,\xi)(1+\mathcal{F}(r,\xi))|^2 + \frac{1}{r^2}p(\phi_1(r,\xi)) + \sqrt{-1}\ \frac{1}{r^2}k(\phi_1(r,\xi)).$$

Denote by $\mathcal{H}$ the standard Hilbert transform on the unit circle (see [KAT]). Then we see that

$$|\sigma(r,\xi)(1+\mathcal{F}(r,\xi))|^2 + \frac{1}{r^2}p(\phi_1(r,\xi)) = -\frac{1}{r^2}\mathcal{H}(k(\phi_1)) + C$$

for some real constant C. We seek a function $\mathcal{F}$ such that $C = 1$.

Let

$$\Omega(r,\mathcal{F}) = |\sigma(r,\xi)(1+\mathcal{F})|^2 + \sum_{i+j\geq 3} a_{ij} r^{i+j-2}\sigma^i\overline{\sigma}^j(1+\mathcal{F})^i(1+\overline{\mathcal{F}})^j.$$

Linearizing near $\mathcal{F} = 0$, we find that

$$\Omega(r,\mathcal{F}) = \Omega(r,0) + \Omega'(r,\mathcal{F}) + \Omega^*(r,\mathcal{F}).$$

These terms are defined as follows:

$$\begin{aligned}
\Omega(r,0) &= 1; \\
\Omega'(r,\mathcal{F}) &= \lim_{t\to 0}\frac{\Omega(r,t\mathcal{F}) - \Omega(r,0)}{t} \\
&= 2\mathrm{Re}\,\{[|\sigma|^2 + \sum_{ij} a_{ij} r^{i+j-2}\sigma^i\overline{\sigma}^j i]\mathcal{F}\}; \\
\Omega^*(r,\mathcal{F}) &= \sum_{i+j\geq 2} c_{ijl}(\xi) r^l \mathcal{F}^i\overline{\mathcal{F}}^j.
\end{aligned}$$

Here $\overline{c_{ijl}} = c_{jil}$. By standard Taylor expansion estimates we have that $\|c_{ijl}\| \lesssim R^{i+j+l}$.

Let us write

$$c(r,\xi) = 2\Big[|\sigma|^2 + \sum a_{ij} r^{i+j-2} \sigma^i \overline{\sigma}^j i\Big].$$

We see that

$$\mathrm{Re}\,\{c(r,\xi)\mathcal{F}\} + \Omega^*(r,\mathcal{F}) = -\mathcal{H}\left(\frac{1}{r^2}k(r\sigma(1+\mathcal{F}))\right).$$

Now notice that $\mathrm{Ind}_{S^1} c(r,\xi) = 0$ for $|r| << 1$. Therefore, if we write $c(r,\xi) = a(r,\xi)b(r,\xi)$ with $a = |c|$, $b = c/|c|$, then $\ln b$ is a well-defined real analytic function. Let

$$c^*(r,\xi) = \frac{1}{a(r,\xi)} e^{\sqrt{-1}\,\mathcal{H}(\ln b(r,\xi))}.$$

Then, by the real analytic hypoellipticity of the Hilbert transform (which can be easily proved, for instance, by using the classical Schwartz reflection principle), we can see that $c^*(r,\xi)$ is also real analytic in $(r,\xi) \in I_\epsilon \times S^1$. (More precisely, what we need in our argument is the fact that: $c^*(r,\xi) = \sum c_j^*(\xi) r^j$ with $\|c_j^*\| < R^j$). We observe that c^* is a positive real function.

Write $d(r,\xi) = c^* c$, which is real analytic in $(r,\xi) \in I_\epsilon \times S^1$ and holomorphic in ξ. In fact, let $d(r,\xi) = \sum_{j\geq 0} d_j(\xi) r^j$. Then we need, in what follows, that $d_j(\xi) \in \mathrm{Hol}(\Delta) \cap \mathrm{C}^{\mathrm{m}+1/2}(\overline{\Delta})$ and $\|d_j\| < R^j$ for $j \geq 0$. Therefore

$$\mathrm{Re}\,(d(r,\xi)\mathcal{F}) = -c^*(r,\xi)\Omega^*(r,\mathcal{F}) - c^*(r,\xi)\mathcal{H}\left(\frac{1}{r^2}k(r\sigma(r,\xi)(1+\mathcal{F}))\right).$$

Let $\widetilde{\mathcal{F}} = d(r,\xi)\mathcal{F} \equiv U(r,\xi) + \sqrt{-1}\,\mathcal{H}(U(r,\xi))$. We observe that $d(r,\xi) \neq 0$. Then we obtain the equation

$$\begin{aligned} U(r,\xi) \quad &= \quad -c^*(r,\xi)\Omega^*\left(r, \frac{U + \sqrt{-1}\,\mathcal{H}(U)}{d(r,\xi)}\right) \\ &\quad -c^*(r,\xi)\mathcal{H}\left(\frac{1}{r^2}k\left(r\sigma(r,\xi)\left(1 + \frac{U+\sqrt{-1}\,\mathcal{H}U}{d}\right)\right)\right). \end{aligned}$$

Let

$$\begin{aligned}
\Lambda_1(r,U) &= -c^*(r,\xi)\Omega^*\left(r, \frac{U+\sqrt{-1}\,\mathcal{H}U}{d}\right) \\
&= -c^*(r,\xi)\sum_{i+j\geq 2} c_{ijl}(\xi) r^l \frac{(U+\sqrt{-1}\,\mathcal{H}(U))^i}{d^i}\cdot\frac{(U-\sqrt{-1}\,\mathcal{H}(U))^j}{\overline{d^j}} \\
&= \sum_{i+j\geq 2} \widetilde{c}_{ijl}(\xi) r^l U^i (H(U))^j.
\end{aligned}$$

Observe that the real function $\widetilde{c}_{ijl}(\xi)$ satisfies the same sort of Cauchy-type estimates as does c_{ijl}.

Write

$$\begin{aligned}
k^*(r,U) &= \frac{1}{r^2}k\left(r\sigma\left(1+\frac{U+\sqrt{-1}\,\mathcal{H}U}{d}\right)\right) \\
&= \sum_{i+j\geq m} b_{ij} r^{i+j-2}\sigma^i\overline{\sigma}^j \frac{1}{d^i}\frac{1}{\overline{d}^j}(d+U+\sqrt{-1}\,\mathcal{H}U)^i(\overline{d}+U-\sqrt{-1}\,\mathcal{H}U)^j \\
&\equiv \sum_{i,j;l\geq m-2} \widetilde{b}_{ijl}(\xi) r^l U^i (\mathcal{H}U)^j.
\end{aligned}$$

Then the real function $\widetilde{b}_{ijl}$ satisfies the Cauchy estimates $\|\widetilde{b}_{ijl}\| \lesssim R^{i+j+l}$.

Denote

$$\Lambda_2(r,U) = -c^*(r,\xi)\mathcal{H}\left(\sum_{i,j\geq 0;l\geq m-2} \widetilde{b}_{ijl}(\xi) r^l U^i (\mathcal{H}U)^j\right).$$

We then need to solve the equation

$$U = \Lambda_1(r,U) + \Lambda_2(r,U) \tag{1.1.3}$$

for U. We note here that for the model surface, i.e. $k \equiv 0$, the only solution for each $r \sim 0$ is $U \equiv 0$.

Now we are going to apply a suitable version of the (real analytic) implicit function theorem to (1) to obtain a solution U that is real analytic in the variable r. To this end, we write

$$B_\epsilon^{m,1/2} = \{\phi \in C^{m,1/2}(S^1) : \phi \text{ is real valued and } \|\phi\| \equiv \|\phi\|_{m,1/2} < \epsilon\}.$$

The index m is the same as before.

Consider now the operator

$$\begin{aligned} \Lambda : B_\epsilon^{m,1/2} &\rightarrow C^{m,1/2}(S^1) \\ (r,U) &\mapsto \Lambda_1(r,U) + \Lambda_2(r,U). \end{aligned}$$

By the boundedness of the Hilbert transform acting on the Banach space $C^{m,\frac{1}{2}}(S^1)$, we easily see that Λ is a well-defined operator when $\epsilon << 1$. Moreover, by a tedious but routine verification, one sees that Λ is smooth and real analytic in (r,U) near the origin in a sense which we will describe below. Detailed arguments of this nature can be found in Claim 1 and Lemma 4.3 of Chapter 4 of [Hu] (see also Lemma 5.1 of [HT] for the proof of the smoothness part):

$$\Lambda(r,U) = \sum_{i,j} \frac{r^i D^{(i,j)}(\Lambda(0,0))(U^j)}{i!j!},$$

for any $(r,U) \sim 0$. Here $D^{(i,j)}(\Lambda(r_0,U_0))(U^j)$ is the (i,j)-mixed derivative of Λ with respect to (r,U), evaluated at (r_0,U_0), and then applied to U^j. Note that $D^{i,j}(\Lambda(r,U))$ is an $i+j$- multilinear operator; however the action in the first i components factors out as r^i. We also require the Cauchy estimates for the operator norm: $\|D^{(i,j)}\Lambda(r,U)\| <$

$i!j! \cdot R^{i+j}$ when $(r,U) \sim 0$. For completeness and clarity, we shall verify in what follows the analyticity of Λ_2 (the analyticity of Λ_1 can be checked similarly).

Let $\Lambda = c^*(r,\xi)\Lambda^*$. By noting that c^* is real analytic, we only have to worry about the analyticity of Λ^*. Since

$$\|\widetilde{b}_{ijs}\| \lesssim R^{i+j+s}$$

for some $R >> 1$ and since when $|r|, \|U\| << 1$ then $\|\mathcal{H}U\| << 1$, we see that

$$D^{(\alpha,\beta)}\Lambda_2^*(r,U) = \mathcal{H}\left(\sum \widetilde{b}_{ijs} D^{(\alpha,\beta)}(r^s U^i [\mathcal{H}(U)]^j)\right).$$

We claim that

$$\begin{aligned} &\|D^{(\alpha,\beta)}(r^i U^j [\mathcal{H}(U)]^s\| \\ &\lesssim \quad i \cdot (i-1) \cdots (i-\alpha+1) \cdot (j+s) \cdot (j+s-1) \\ &\qquad\qquad \cdots (j+s-\beta+1) R^{i+j+s} \epsilon^{i+j+s-\alpha-\beta}, \end{aligned}$$

where $\epsilon R << 1$. This assertion can be verified using an induction argument. For example, we present here the case when $\alpha = \beta = 1$. Then

$$D^{(1,1)}(r^i U^j [\mathcal{H}(U)]^s) = i r^{i-1} D^{(0,1)}(U^j [\mathcal{H}(U)]^s).$$

For any $U_1 \in C^{m,1/2}$, we see that

$$\begin{aligned} \|D^{(1,1)}(r^i U^j [\mathcal{H}(U)]^s)(U_1)\| \quad &\leq \quad i|r|^{i-1} c^{j+s} (j\|U_1\| \cdot \|U\|^{j-1} \|\mathcal{H}U\|^s \\ &\qquad + s\|\mathcal{H}U_1\| \cdot \|U\|^j \cdot \|\mathcal{H}U\|^{s-1}). \end{aligned}$$

Notice here that $\|\mathcal{H}U\| \leq c\|U\|$ and $\|UV\| \leq c\|U\| \cdot \|V\|$ for some constant c. Next,

$$\|D^{(1,1)}(r^i U^j [\mathcal{H}(U)]^s\| \quad \leq \quad i|r|^{i-1} c^{j+s} (jc^s \|U\|^{j+s-1} + sc^s \|U\|^{j+s-1})$$

$$\lesssim \quad i(j+s)c^{j+2s}|r|^{i-1}\|U\|^{j+s-1}$$

$$\lesssim \quad i(j+s)R^{i+j+s}\epsilon^{i+j+s-2}\,,$$

where we have selected $R >> c$ and $|r|, \|U\| < \epsilon$. In conclusion,

$$\begin{aligned}
&\|D^{(\alpha,\beta)}(\Lambda_2^*(r,U)\| \\
&\quad \leq \quad C\sum \|\widetilde{b_{ijs}}\| s(s-1)\cdots(s-\alpha+1) \\
&\qquad\qquad \cdot(i+j)(j+i-1)\cdots(j+i-\beta+1)R^{i+j+l}\epsilon^{i+j+s-\alpha-\beta} \\
&\quad < \quad \alpha!\beta!R_*^{\alpha+\beta}
\end{aligned}$$

as long as $R_* >> 1$ and $|r|, \|U\| << 1$. As a result, from the Taylor formula, we can conclude that $\Lambda_2^*(r,U)$ is real analytic near $(0,0)$. This concludes the proof of the claim.

Now we notice that $\Lambda(0,0) = 0$ and $D\Lambda(0,0) = 0$. Thus, from the implicit function theorem in the analytic case (Proposition 1.1 below), we conclude that for (1.1.3) there is a unique analytic solution $\mathcal{R}(r) = \sum_{j=1}^{\infty} \phi_j(\xi) r^j$ with each $\phi_j \in C^{m,1/2}$ and $\|\phi_j\| \lesssim R^j$ for some $R >> 1$.

Now we formulate the precise version of the implicit function theorem that we have just used.

Proposition 1.1 *Let $\mathcal{O}$ be an open subset of the (real) Banach space E and let $o \in \mathcal{O}$. Suppose that $f : I_\epsilon \times \mathcal{O} \to E$ is analytic near $(0, o)$ and that $f(0, o) = 0, D^{(0,1)} f(0, o) = 0$. Then there is a unique analytic solution X near 0 with $X(0) = o$ for the equation $X = f(r, X)$.*

Remark: Variants of this proposition have been commonly used in the literature. See, for example, [LEM]. Its proof is analogous to the finite dimensional case (for which see, for instance, [KRP]); however, for lack of a reference with a cogent statement and a proof, we include the following details.

Proof of Proposition 1.1:

First we note that

$$f(r,X) = \sum_{j+k\geq 1} r^j \mathcal{L}_k(X^k),$$

where $\mathcal{L}_k$ is a symmetric k-multilinear form and $\|\mathcal{L}_k\| \lesssim R^k$. Let $E_c \equiv E \otimes \mathbb{C}$ be the complexification of E. We let Δ_ϵ be the disc with center 0 and radius $\epsilon > 0$ in $\mathbb{C}$. Let us define

$$F(r,Z) = \sum r^k \mathcal{L}_k^c(Z^k),$$

where $r \in \Delta_\epsilon$, $Z \in E_c$, $Z = X + iY$, and $\mathcal{L}_k^c$ is the natural complexification of $\mathcal{L}_k$. Thus we see that $F(r,Z)$ is also analytic near $(0,o) \in \Delta_\epsilon \times E_c$.

Moreover, it can be seen that

$$\frac{\partial F}{\partial \overline{r}} = \frac{1}{\sqrt{-1}} \left(\frac{\partial F}{\partial \mu} + \sqrt{-1}\, \frac{\partial F}{\partial \eta} \right) = 0$$

and

$$\frac{\partial F}{\partial \overline{Z}} = \frac{1}{\sqrt{-1}} \left(\frac{\partial F}{\partial X} + \sqrt{-1}\, \frac{\partial F}{\partial Y} \right) = 0, \tag{1.1.4}$$

where $r = \mu + \sqrt{-1}\,\eta$ and $Z = X + \sqrt{-1}\,Y$. Now we apply the standard implicit function theorem to the equation

$$Z = F(r,Z).$$

We obtain a C^∞ solution

$$Z = \mathcal{R}^c(r)$$

with $\mathcal{R}^c(0) = o$ such that $\mathcal{R}^c(r) = F(r, \mathcal{R}^c(r))$. By (1.1.4),

$$\frac{\partial \mathcal{R}^c(r)}{\partial \overline{r}} = \frac{\partial F}{\partial \overline{r}} + \frac{\partial F}{\partial Z}\frac{\partial \mathcal{R}^c}{\partial \overline{r}}.$$

As a result,

$$(\mathrm{id} - \frac{\partial F}{\partial Z})\frac{\partial \mathcal{R}^c}{\partial \overline{r}} = 0$$

hence

$$\frac{\partial \mathcal{R}^c}{\partial \overline{r}} = 0;$$

for $\left[\mathrm{id} - \frac{\partial F}{\partial Z}\right]$ is invertible when $Z \sim 0$.

In conclusion, $\mathcal{R}^c$ is holomorphic in r (see [Rud]). Therefore $\mathcal{R}^c(r) = \sum_j \phi_j r^j$ with $\|\phi_i\| \lesssim R^i$. Notice that when r is real (and small) then we also have a real C^∞ solution. Thus, by the uniqueness part of the implicit function theorem, the proof is complete. ■

1.2 The Hull of Holomorphy

We continue our discussion from the last subsection and retain all notation introduced there.

Let

$$U(r,\xi) = \mathcal{R}(r)(\xi) = \sum_{j=1}^{\infty} \phi_j(\xi) r^j \in C^{m,1/2}(S^1 \times I_\epsilon).$$

A useful observation is that $\|U(r,\xi)\| \lesssim r^{m-2}$. In fact,

$$\begin{aligned} \|U(r,\xi)\| &\leq \|\Lambda_1(r,U)\| + \|\Lambda_2(r,U)\| \\ &\leq \epsilon\|U\| + O(r^{m-2}), \end{aligned}$$

that is,

$$\|U(r,\xi)\| \leq C \cdot \frac{1}{1-\epsilon} r^{m-2}.$$

Now, returning to the function

$$\mathcal{F} = \frac{U + \sqrt{-1}\,\mathcal{H}(U)}{d(r,\xi)},$$

we have that

$$\mathcal{F} = \sum_{j \geq m-2} f_j r^j \quad \text{with} \quad \|f_j\| \lesssim R^j.$$

Here $f_j \in \mathrm{Hol}(\Delta)$. Notice that

$$\Phi(r,\xi) = (\phi_1, \phi_2) = (r\sigma(1+\mathcal{F}), \mathcal{C}(h(r\sigma(r,\xi)(1+\mathcal{F}))),$$

where $\mathcal{C}$ is the Cauchy integral operator. We have the following properties:

(i) $\phi_1(r,\xi) = r\sigma(r,\xi)(1+\mathcal{F}) = r\sigma(r,\xi) + \ell_1(r,\xi)$

with $\ell_1(r,\xi) = O(r^{m-2})$;

(ii) Let $h(r\sigma(r,\xi)(1+\mathcal{F})) = \sum_j h_j r^j = r^2 + \sum_{j=m}^{\infty} h_j r^j$ with $\|h_j\| \lesssim R^j$ for each j. Then

$$\phi_2 = r^2 + \sum_{j=m}^{\infty} \mathcal{C}(h_j) r^j = r^2 + \ell_2(r,\xi)$$

and

$$\ell_2(r,\xi) = \sum_{j=m}^{\infty} \widetilde{h}_j r^j$$

with $\widetilde{h}_j \in C^{m,1/2}(\overline{\Delta}) \cap \mathrm{Hol}(\Delta)$ and $\|\widetilde{h}_j\| \underset{\sim}{<} R^j$.

Now we study the hull of M near 0. Let

$$M_0 = \{(z,w) \in \mathbb{C}^2 : w = q(z) \equiv z\overline{z} + p(z)\},$$

as before. Then $\widetilde{M}_0$, the hull of M_0, is the set $\{(z,w) \in \mathbb{C}^2 : u \geq q(z) \equiv |z|^2 + p(z)\} \cap \{v = 0\}$ and is foliated in a trivial and natural fashion by a family of imbedded discs $\Phi_0(r,\xi) = (r\sigma(r,\xi), r^2)$ $(r > 0)$.

Let $0 < u << 1$. Define

$$\Psi : \widetilde{M}_0 - 0 \to \bigcup_{0<r<<1} \Phi(r, \overline{\Delta})$$

by

$$\Psi(z,u) = \Phi(\sqrt{u}, \xi(u,z)),$$

where $\xi(u,z)$ is determined by the equations $z = r\sigma(r,\xi)$, $r = \sqrt{u}$.

Claim: The function Ψ has a C^1 extension across $0 \in \widetilde{M}_0$ and is real analytic on $\widetilde{M}_0 \setminus M_0$ (in fact, Ψ is of class $C^{(m-2)/2}$ near 0, but we do not need this result here).

Proof of the Claim: Now

$$\Psi(z,u) = (z,u) + (\widetilde{\phi}_1, \widetilde{\phi}_2),$$

where

$$\widetilde{\phi}_1(z,u) = \ell_1(\sqrt{u},\xi(u,z)) = u^2\ell_1^*(\sqrt{u},\xi(u,z))$$
$$\widetilde{\phi}_2(z,u) = \ell_2(\sqrt{u},\xi(u,z)) = u^2\ell_2^*(\sqrt{u},\xi(u,z)).$$

We note that

$$\xi(u,z) = [(\sigma(r,\ \cdot\ )]^{-1}\left(\frac{z}{\sqrt{u}}\right) = [\sigma(\sqrt{u},\ \cdot\ )]^{-1}\left(\frac{z}{\sqrt{u}}\right) \equiv \sigma^{-1}(\sqrt{u}, z/\sqrt{u}),$$

and $\sigma^{-1}(r,z^*)$ is jointly real analytic in its variables for $(r,z^*)\in I_\epsilon\times D_r$ with $\epsilon << 1$.

Thus we easily see that $\Psi(z,u)$ is real analytic jointly in z,u, when $(z,u)\in\widetilde{M}_0\setminus M_0$. Observe that $\|z/\sqrt{u}\| = O(1)$ when $(z,u)\in\widetilde{M}_0$. We have

$$\frac{\partial^{i+j+s}\ell_k}{\partial z^i\partial\overline{z}^j\partial u^s} = O(1) \qquad \text{when} \qquad k=1,2; \qquad i+j+s\le 2; \quad \text{and} \quad r\to 0.$$

Thus Ψ is of class C^1 near 0 and $d\Psi|_0 = \text{id}$. In particular, we see that $\Psi(\widetilde{M}_0)$ is a C^1 smooth, (Levi flat) hypersurface spanned by M near 0, which is certainly real analytic on the interior points.

However, by the Kenig-Webster theorem, we know that $\widetilde{M}$ is of class C^∞ near 0 with boundary M (we may also see this assertion by using our own arguments just presented by first letting $m\to\infty$ and then proving a sort of unique determination theorem of the local hull by analytic discs). Thus we can easily conclude that $\Psi(\widetilde{M}_0) = \widetilde{M}$ near 0 by noting the fact that $\Psi(\widetilde{M}_0)\subset\widetilde{M}$.

As an immediate application of our function Ψ, we see that the tangent space $T_0\widetilde{M} = \mathbb{R}^3$ $(= \mathrm{T}_0(\widetilde{\mathrm{M}}_0))$. Now let us define the projection mapping $\pi:\widetilde{M}\to\mathbb{R}^3$ by $\pi(z,u+iV) = (z,u)$. Then we can conclude that π is a C^∞ diffeomorphism. Moreover, since

$\pi(\partial\widetilde{M}) = \pi(M) = M_0$ and $(\pi(\widetilde{M}\setminus M))\cap(\widetilde{M_0}\setminus M_0)$ is not empty, we see that $\pi(\widetilde{M}) = \widetilde{M}_0$. That is, $\widetilde{M}$ can be viewed as the graph of some function $V(z,u)$ over $\widetilde{M}_0$. That is, $\widetilde{M} = \{(z, u+iV(z,u)) : (z,u) \in \widetilde{M}_0\}$.

We remark now that the graph of a function is a C^α smooth manifold if and only if the function itself is C^α smooth (here α is either a positive integer or ∞ or ω). Thus we have that $V(x,u) \in C^\infty(\widetilde{M}_0)$ and is real analytic on $\widetilde{M}_0 \setminus M_0$.

We are going to show next that

$$V(z,u) = \sum_{i,j,s} \frac{1}{i!j!s!} \left(\frac{\partial^{i+j+s} V(z,u)}{\partial z^i \partial \overline{z}^j \partial u^s} \right)_{(0,0)} z^i \overline{z}^j u^s$$

when $|z|, u \approx 0$. That is, $V(z,u)$ is real analytic near 0. This will complete the proof of our theorem.

To this end, we first note that the point $(z, u+iV(z,u)) \in \widetilde{M} - M$ if and only if there is a unique pair $(r,\xi) \in I_\epsilon^+ \times \Delta$ such that

$$\begin{aligned} z &= \phi_1(r,\xi) \\ u + iV &= \phi_2(r,\xi), \end{aligned}$$

that is,

$$z = r\sigma(r,\xi)(1 + \mathcal{F}(r,\xi)) \tag{1}$$

$$u = \operatorname{Re}\phi_2 = r^2 + \operatorname{Re}\ell_2(r,\xi) \tag{2}$$

$$V = \operatorname{Im}\,\phi_2 = \operatorname{Im}\,\ell_2(r,\xi) \tag{3}$$

Here we note that $\ell_2(r,\xi) = \sum_{i+j\geq m} \ell_{ij} r^i \xi^j$ with $(r,\xi) \in I_\epsilon \times \Delta$ for $\epsilon << 1$. From (2), we obtain $\sqrt{u} = r + r^2 \ell_1^*(r, \xi)$ with $\ell_1^*(r, \xi)$ real analytic jointly in r, ξ when $(r, \xi) \in I_\epsilon \times \Delta$ $(|r| << 1)$.

When $|r| << 1$, applying the implicit function theorem we then see that

$$r = \tilde{g}(\eta_1, \xi) = \eta_1 \cdot (1 + \tilde{g}^*(\eta_1, \xi)),$$

where $\eta_1 = \sqrt{u}$ in case $1 >> u \geq 0$ and $\tilde{g}^* = o(|\eta_1|)$ is jointly real analytic in (η_1, ξ). Thus by (1) we see that

$$z = \sqrt{u}(1 + \tilde{g}^*(\eta_1, \xi))\sigma(\tilde{g}(\eta_1, \xi), \xi)(1 + \mathcal{F}(r, \xi)).$$

Write $\eta_2 = z/\sqrt{u}$. We then have

$$\eta_2 = (1 + \tilde{g}^*(\eta_1, \xi))\sigma(\tilde{g}(\eta_1, \xi), \xi)(1 + \mathcal{F}(\tilde{g}(\eta_1, \xi), \xi)) = \eta_2(\eta_1, \xi). \tag{1.2.1}$$

Notice that when $\eta_1,\ \xi \approx 0$, we have (a) η_2 is real analytic in ξ and η_1; (b) $\eta_2(0, 0) = 0$; (c) $d_\xi(\tilde{g}^*(\eta_1, \xi), d_\xi \mathcal{F}(\tilde{g}(\eta_1, \xi), \xi) \approx 0$; and (d) $d_\xi \sigma(\tilde{g}(\eta_1, \xi), \xi) \neq 0$.

We see from the implicit function theorem that (1.2.1) can be solved as

$$\xi = f(\eta_1, \eta_2)$$

with f real analytic near $(0, 0)$ and $f(0, 0) = 0$. Now

$$r = \tilde{g}(\eta_1, f(\eta_1, \eta_2)) = g(\eta_1, \eta_2),$$

which is also real analytic near 0.

Returning to (3), we see that

$$V(z, u) = \text{Im } \phi_2(g(\eta_1, \eta_2), f(\eta_1, \eta_2)),$$

which is also analytic in η_1, η_2 when $\eta_1, \eta_2 \approx 0$. Let η_1 be real and write

$$\text{Im } \phi_2(g(\eta_1, \eta_2), f(\eta_1, \eta_2)) = \sum_{i,j,s \geq 0} S_{ijs} \eta_1^i \eta_2^j \overline{\eta}_2^s$$

with $|S_{ijs}| \lesssim R^{i+j+s}$ for some $R >> 1$.

Now, when $0 < u < \epsilon^2$ and $|z|/\sqrt{u} < \epsilon$ with $0 < \epsilon << 1$, we have that

$$V(z, u) = \sum_{i,j,s} S_{ijs} u^{(1/2)(i-j-s)} z^j \overline{z}^s.$$

However we note that $V(z, u)$ is C^∞ near 0. In particular,

$$\frac{\partial^{j+s} V(z, u)}{\partial z^j \partial \overline{z}^s}\Big|_{(0,u)}$$

is C^∞ in u, as long as $0 \leq u << 1$.

Meanwhile,

$$V^{(j,s)}(0, u) = \frac{\partial^{j+s} V(z, u)}{\partial z^j \partial \overline{z}^s}\Big|_{(0,u)} = \sum_{i=0}^{\infty} j! s! S_{ijs} u^{(1/2)(i-j-s)} \; .$$

This obviously implies that $S_{ijs} = 0$ when $(1/2)(i - j - s)$ is not a positive integer. Thus

$$V(z, u) = \sum_{i,j,s} S_{ijs} u^{\frac{1}{2}(i-j-s)} z^j \overline{z}^s = \sum_{\tau, j, s \in \mathbb{Z}^+} S_{2\tau+j+s,j,s} u^\tau z^j \overline{z}^s$$

when $0 < u < \epsilon^2$, $|z| < \epsilon\sqrt{u}$.

On the other hand,

$$|S_{2\tau+j+s,j,s}| \lesssim R^{2\tau+j+s+j+s} \lesssim (R^2)^{\tau+j+s}.$$

Thus we conclude that

$$\widetilde{V}(z, u) = \sum_{\tau,j,s} S_{2\tau+j+s,j,s} u^\tau z^j \overline{z}^s$$

is real analytic when $u, |z| \approx 0$. Also $\widetilde{V}(z,u) \equiv V(z,u)$ when $0 < u < \epsilon^2$ and $|z| < \epsilon\sqrt{u}$. Notice that $V(z,u)$ is real analytic on $\widetilde{M}_0 \setminus M_0$ and it is C^∞ on $\widetilde{M}_0$. By the unique continuation property of real analytic functions, it follows that $\widetilde{V}(z,u) \equiv V(z,u)$ for all $z, |u| \approx 0$ and $(z,u) \in \widetilde{M_0}$.

At last this completes the proof of the theorem.

2 Closing Remarks

Our main theorem fails in the case that $0 \in M$ is a degenerate elliptic point. As an example, consider the real analytic submanifold

$$M = \{(z,w) : w = |z\overline{z}|^2 + |z\overline{z}|^3(|z|^2 z + \sqrt{-1}\,)\}$$

in two dimensional complex space. It turns out the $\widetilde{M}$ is only $C^{3/2}$ up to the point $0 \in M$—certainly not real analytic.

To verify this last assertion, we invoke the unique determination of the local hull in terms of the attached analytic discs. If we can write down a foliation by analytic discs that spans $\widetilde{M}$ near 0, then that is a unique representation of the local holomorphic hull.

In detail, consider the model domain $M_0 = \{(z,w) : w = |z|^4\}$. Near the point $(0,0) \in M_0$, we may attach analytic discs to M_0 in this way: $\{(z,u) : \text{ for each fixed u, } |z|^4 < u\}$. Here the parameter is the real variable $u \geq 0$: for each value of u, the associated disc is attached to M_0. Now we may map M_0 to M_1, and $\widetilde{M}_0$ to (what will turn out to be) $\widetilde{M}_1$, by way of the mapping

$$\Phi(z,w) = (\phi_1(z,w), \phi_2(z,w))$$

with

$$\phi_1(z,w) = z$$

$$\phi_2(z,w) = u + u^{3/2}(u^{1/2}z + i).$$

One can easily see that this Φ takes each analytic disc in $\widetilde{M}_0$ to an analytic disc attached to M_1. Thus $\Phi(\widetilde{M}_0)$ becomes $\widetilde{M}_1$. After some calculation (which we omit), it can also be verified that $\Phi(\widetilde{M}_0)$ is only $C^{3/2}$ and no better. We leave the details to the interested reader. Examples of this type, and extensions of some of our results to higher dimensions, will be developed in a future paper.

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