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Volume-preserving mappings between Hermitian symmetric spaces of compact type

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A B S T R A C T

In this paper, we establish the rigidity result for local holomorphic volume preserving maps from an irreducible Hermitian symmetric space of compact type into its Cartesian products.

Contents

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1. Introduction

Let M be an irreducible $n-$dimensional Hermitian symmetric space of compact type, equipped with a canonical Kähler-Einstein metric ω. Write ω^n for the associated volume form (up to a positive constant depending only on n). The purpose of this paper is to prove the following rigidity theorem:

Theorem 1.1. *Let (M, ω) be an irreducible $n-$dimensional Hermitian symmetric space of compact type as above. Let $F = (F_1, ..., F_m)$ be a holomorphic mapping from a connected open subset $U \subset M$ into the m-Cartesian product $M \times ... \times M$ of M. Assume that each F_j is generically non-degenerate in the sense that $F_j^*(\omega^n) \not\equiv 0$ over U. Assume that F satisfies the following volume-preserving (or measure-preserving) equation:*

$$\omega^n = \sum_{i=1}^{m} \lambda_i F_i^*(\omega^n), \tag{1}$$

for certain constants $\lambda_j > 0$. Then for each j with $1 \le j \le m$, F_j extends to a holomorphic isometry of (M, ω). In particular, the conformal factors satisfy the identity: $\sum_{j=1}^m \lambda_j = 1$.

Rigidity properties are among the fundamental phenomena in Complex Analysis and Geometry of several variables, that study the global extension and uniqueness for various holomorphic objects up to certain group actions. The rigidity problem that we consider in this paper was initiated by a celebrated paper of Calabi [4]. In [4], Calabi studied the global holomorphic extension and uniqueness (up to the action of the holomorphic isometric group of the target space) for a local holomorphic isometric embedding from a Kähler manifold into a complex space form. He established the global extension and the Bonnet type rigidity theorem for a local holomorphic isometric embedding from a complex manifold with a real analytic Kähler metric into a standard complex space form. The phenomenon discovered by Calabi [4] has been further explored in the past several decades due to its extensive connection with problems in Analysis and Geometry. (See [43], [9], [10], for instance.)

In 2004, motivated by the modularity problem of the algebraic correspondences in algebraic number theory, Clozel and Ullmo [7] were led to study the rigidity problems for local holomorphic isometric maps and even much more general volume-preserving maps between bounded symmetric domains equipped with their Bergman metrics. By reducing the modularity problem to the rigidity problem for local holomorphic isometries, Clozel-Ullmo proved that an algebraic correspondence in the quotient of a bounded symmetric domain preserving the Bergman metric has to be a modular correspondence in the case of the unit disc in the complex plane and in the case of bounded symmetric domains of rank ≥ 2. Notice that in the one dimensional setting, volume preserving maps are identical to the metric preserving maps. Thus the Clozel-Ullmo result also applies to the volume preserving algebraic correspondences in the lowest dimensional case. Motivated by the work in [7], Mok carried out a systematic study of the rigidity problem for local isometric embeddings in a very general setting. Mok in [31–33] proved the total geodesy for a local holomorphic isometric embedding between bounded symmetric domains D and Ω when either (i) the rank of each irreducible component of D is at least two or (ii) $D = \mathbb{B}^n$ and $\Omega = (\mathbb{B}^n)^p$ for $n \geq 2$. In a paper of Yuan-Zhang [48], the total geodesy is obtained in the case of $D = \mathbb{B}^n$ and $\Omega = \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_p}$ with $n \geq 2$ and N_l arbitrary for $1 \leq l \leq p$. Earlier, Ng in [39] had established a similar result when $p = 2$ and $2 \leq n \leq N_1, N_2 \leq 2n-1$. In a paper of Yuan and the second author of this paper [20], we established the rigidity result for local holomorphic isometric embeddings from a Hermitian symmetric space of compact type into the product of Hermitian symmetric spaces of compact type with even negative conformal factors where certain non-cancellation property for the conformal factors holds. (This cancellation condition turns out be the necessary and sufficient condition for the rigidity to hold due to the presence of negative conformal factors.) In a recent paper of Ebenfelt [11], a certain classification, as well as its connection with problems in CR geometry, has been studied for local isometric maps when the cancellation property fails to hold. The recent paper of Yuan [47] studied the rigidity problem for local holomorphic maps preserving the (p, p)-forms between Hermitian symmetric spaces of non-compact type. At this point, we should also mention other related studies for the rigidity of holomorphic mappings. Here, we quote the papers by Chan-Xiao-Yuan [5], Dinh-Sibony [8], Huang [18,19], Huang-Yuan [21], Ji [25], Kim-Zaitsev [26], Mok [30,34], Mok-Ng [35], Ng [37–39], Xiao-Yuan [45,46] and many references therein, to name a few.

The work of Clozel and Ullmo has left open an important question of understanding the modularity problem for volume-preserving correspondences in the quotient of Hermitian symmetric spaces of higher dimension equipped with their Bergman metrics. In 2012, Mok and Ng answered, in the affirmative, the question of Clozel and Ullmo in [36] by establishing the rigidity property for local holomorphic volume preserving maps from an irreducible Hermitian manifold of non-compact type into its Cartesian products.

The present paper continues the above mentioned investigations, especially those in [7], [36] and [20]. Our main purpose is to establish the Clozel-Ullmo and Mok-Ng results for local measure preserving maps between Hermitian symmetric spaces of compact type. Notice that in the Riemann sphere setting, Theorem 1.1 also follows from the isometric rigidity result obtained in an earlier paper of the second author with Yuan [20]. However, the basic approach in this paper fundamentally differs from that in [20]. The method used in [20] is to first obtain the result in the simplest projective space setting and then use the minimal rational curves to reduce the general case to the much simpler projective space case. On the other hand, restrictions of volume preserving maps are no longer volume preserving and thus the reduction method in [20] can not be applied here. The approach we use in this paper is first to establish general results under certain geometric and analytic assumptions (i.e., Propositions (I)–(III)) and then verify that these assumptions are automatically satisfied based on a case by case argument in terms of the type of the Hermitian space.

We now briefly describe the organization of the paper and the basic ideas for the proof of Theorem 1.1. The major part of the paper is devoted to showing the algebraicity for a certain component F_j in Theorem 1.1 with total degree depending only on the geometry of (M, ω). For this, we introduce the concept of Segre family for an embedded projective subvariety. Notice that in the previous work, Segre varieties were only defined for a real submanifold in a complex space through complexification. Our Segre family is defined by slicing the minimal embedding with a hyperplane in the ambient projective space, associated with points in its conjugate space. The Segre family thus defined is invariant under holomorphic isometric transformations, whose defining function is closely related to the complexification of the potential function of the canonical metric. The first step in our proof is to show that a certain component F_j preserves at least locally the Segre family. The next difficult step is then to show that preservation of the Segre foliation gives the algebraicity of F_j. To obtain the algebraicity of F_j, we need to study the size that the space of the jets of the map F_j along the Segre variety directions. Indeed, an important part of the paper is to show that the space of the jets of an associated embedding map r_F along the Segre direction up to a certain order depending only on M and its minimal embedding spans the whole target tangent space. This is a main reason we need to describe precisely what the minimal embedding is for each M. Once this is done, we can then show that the map, when restricted to each Segre variety, stays in the field generated by rational functions and the differentiations of their defining functions as well as their inverse, and thus must be algebraic by a modified version of the Hurwitz theorem. The uniform bound of the total degree of F_j is obtained by the fact that we need only a fixed number of steps to perform algebraic and differential operations to reproduce the map from the minimal embedding functions. After obtaining the algebraicity, we further show that F_j extends to a birational self-map of the space by a monodromy argument, the geometry of the Segre foliation, an iteration argument and the classical Bezout theorem. Finally, a simple argument shows that a birational map which preserves the Segre foliation is the restriction of a holomorphic self-isometry of the space. Once F_j is proved to be an isometry, we can delete F_j from the original equation and then apply an induction argument to conclude the rigidity for other components.

The organization of the paper is as follows: In §2, we first introduce the Segre family for a polarized projective variety. We then describe the canonical and minimal embedding of the space into a complex projective space in terms of the type of the space. In §3, we derive a general theorem for partially degenerate holomorphic embeddings which will play a fundamental role in the later development. In §4, we provide the algebraicity for one of the components of the holomorphic mapping F under additional assumptions which include the partial non-degeneracy condition introduced in §3, the generic transversality of the Segre varieties and the irreducibility of the Segre family. In §5, we show that the partial non-degeneracy holds for local biholomorphisms between any irreducible Hermitian space of compact type. §6 is devoted to proving the generic transversality for the intersection of the Segre varieties. We prove in §7 the irreducibility of the potential functions pulled back to a complex Euclidean space, which has consequences on the irreducibility of the Segre varieties and the Segre families. The argument in §5-§7 varies as the type of the space varies and thus has to be done case by case.

We include several Appendices for convenience of the reader. In Appendix A, we give the concrete functions for a minimal holomorphic embedding of a Hermitian symmetric space of exceptional type into a projective space. In Appendix B, we continue to establish Proposition (I) for the rest cases. In Appendix C, we provide the verification on the transversality for the Segre varieties for the remaining cases not covered in §6.

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2. Irreducible Hermitian symmetric spaces and their Segre varieties

2.1. Segre varieties of projective subvarieties

Write $z = (z_1, \cdots, z_n, z_{n+1})$ for the coordinates of $\mathbb{C}^{n+1}$ and $[z] = [z_1, \cdots, z_n, z_{n+1}]$ for the homogeneous coordinates of $\mathbb{CP}^n$. For a polynomial $p(z)$, we define $\overline{p}(z) := \overline{p(\overline{z})}$. For a connected projective variety $V \subset \mathbb{CP}^n$, write $\mathcal{I}_V$ for the ideal consisting of homogeneous polynomials in z that vanish on V. We define the conjugate variety V^* of V to be the projective variety defined by $\mathcal{I}_V^* := \{\bar{f} : f \in \mathcal{I}_V\}$. Apparently the map $z \mapsto \overline{z}$ defines a diffeomorphism from V to V^*. When $\mathcal{I}_V$ has a basis consisting of polynomials with real coefficients, $V^* = V$. Also if V is irreducible and has a smooth piece parametrized by a neighborhood of the origin of a complex Euclidean space through polynomials with real coefficients, then $V^* = V$.

Next for $[\xi] \in V^*$, we define the Segre variety Q_ξ of V associated with ξ by $Q_\xi = \{[z] \in V : \sum_{j=1}^{n+1} z_j \xi_j = 0\}$ which is a subvariety of codimension one in V. Similarly, for

$[z] \in V$, we define the Segre variety Q_z^* of V^* associated with z by $Q_z^* = \{[\xi] \in V^* : \sum_{j=1}^{n+1} z_j \xi_j = 0\}$. It is clear that $[z] \in Q_\xi$ if and only if $[\xi] \in Q_z^*$. The Segre family of V is defined to be the projective variety $\mathcal{M} := \{([z], [\xi]) \in V \times V^*, [z] \in Q_\xi\}$.

Now, we let (M, ω) be an irreducible Hermitian symmetric space of compact type canonically embedded in a certain minimal projective space $\mathbb{CP}^N$, that we will describe in detail later in this section. Then under this embedding, its conjugate space M^* is just M itself. Taking ω to be the natural restriction of the Fubini-Study metric to M, the holomorphic isometric group of M is then the restriction of a certain subgroup of the unitary actions of the ambient space. Now, for two points $p_1, p_2 \in M$, let U be an $(N+1) \times (N+1)$ unitary matrix such that $\sigma([z]) = [z] \cdot U$ is an isometry sending p_1 to p_2. Then $\sigma^*([\xi]) = [\xi]\overline{U}$ is an isometry of M^*. By a straightforward verification, we see that σ^* biholomorphically sends $Q_{p_1}^*$ to $Q_{p_2}^*$. Similarly, for any $q_1, q_2 \in M^*$, Q_{q_1} is unitary equivalent to Q_{q_2}. In the canonical embeddings which we will describe later, the hyperplane section at infinity of the manifold is a Segre variety. Since the one at infinity is built up from Schubert cells and all Segre varieties are holomorphically equivalent to each other, one deduces that each Segre variety of M is irreducible. This fact will play a role in the proof of our main theorem.

2.2. Canonical embeddings and explicit coordinate functions

We now describe a special type of canonical embedding of the Hermitian symmetric space M of compact type into $\mathbb{CP}^N$. This embedding will play a crucial role in our computation leading to the proof of Theorem 1.1. See [16] for the classification of the irreducible Hermitian symmetric spaces of compact type. See also [28], [29] on the typical canonical embeddings of the Heritian symmetric spaces of compact type and the related theory of Hermitian positive Jordan triple system.

♣1. Grassmannians (spaces of type I): Write $G(p, q)$ for the Grassmannian space consisting of p planes in $\mathbb{C}^{p+q}$. (Since $G(p, q)$ is biholomorphically equivalent to $G(q, p)$, we will assume $p \leq q$ in what follows.) There is a matrix representation of $G(p, q)$ as the equivalence classes of $p \times (p+q)$ non-degenerate matrices under the matrix multiplication from the left by elements of $GL(p, \mathbb{C})$. A Zariski open affine chart $\mathcal{A}$ for $G(p, q)$ is identified with $\mathbb{C}^{pq}$ with coordinates Z for elements of the form:

$$(\, I_{p\times p} \quad Z\,) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & z_{11} & z_{12} & \cdots & z_{1q} \\ 0 & 1 & 0 & \cdots & 0 & z_{21} & z_{22} & \cdots & z_{2q} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & z_{p1} & z_{p2} & \cdots & z_{pq} \end{pmatrix}, \text{ where } \; Z \text{ is a } p \times q \text{ matrix.}$$

The Plücker embedding $G(p, q) \to \mathbb{CP}(\Lambda^p \mathbb{C}^{p+q})$ is given by mapping the $p-$plane Λ spanned by vectors $v_1, ..., v_p \in \mathbb{C}^{p+q}$ into the wedge product $v_1 \wedge v_2 \wedge ... \wedge v_p \in \wedge^p \mathbb{C}^{p+q}$. The action induced by the multiplication through elements of $SU(p+q)$ from the right induces a unitary action in the embedded ambient projective space. In homogeneous coordinates, the embedding is given by the $p \times p$ minors of the $p \times (p+q)$ matrices (up to a sign). More specifically, in the above local affine chart, we have the following (up to a sign in front of the components):

$$Z \to [1, Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}, ...] \tag{2}$$

which is denoted for simplicity of notation, in what follows, by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$. Here and in what follows, $Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ is the determinant of the submatrix of Z formed by its $i_1^{\text{th}}, ..., i_k^{\text{th}}$ rows and $j_1^{\text{th}}, ..., j_k^{\text{th}}$ columns, where the indices run through

$$k = 1, 2, ..., p, 1 \le i_1 < i_2 < ... < i_k \le p, 1 \le j_1 < j_2 < ... < j_k \le q.$$

In particular when $k = 1$, $Z\begin{pmatrix} i_1 \\ j_1 \end{pmatrix} = z_{i_1 j_1}$. Notice that under such an embedding into the projective space, $(G(p, q))^* = G(p, q)$. We thus have the same affine coordinates for $(G(p, q))^*$:

$$(I_{p\times p} \quad \Xi) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \xi_{11} & \xi_{12} & \cdots & \xi_{1q} \\ 0 & 1 & 0 & \cdots & 0 & \xi_{21} & \xi_{22} & \cdots & \xi_{2q} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & \xi_{p1} & \xi_{p2} & \cdots & \xi_{pq} \end{pmatrix}, \quad \Xi \text{ is a } p \times q \text{ matrix.}$$

By the definition in §2.1, it follows that the restriction of the Segre family to the product of these Zariski open affine subsets has the following canonical defining function:

$$\rho(z, \xi) = 1 + \sum_{\substack{1\le i_1<i_2<...<i_k\le p, \\ 1\le j_1<j_2<...<j_k\le q \\ k=1,...,p}} Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \Xi\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \tag{3}$$

Here $z = (z_{11}, z_{12}, ..., z_{pq})$, $\xi = (\xi_{11}, \xi_{12}, ..., \xi_{pq})$. For simplicity of notation and terminology, we call this quasi-projective algebraic variety embedded in $\mathbb{C}^{pq} \times \mathbb{C}^{pq}$, which is defined by (3), the Segre family of $G(p, q)$. Our defining function $\rho(z, \xi)$ of the Segre family is closely related to the generic norm of the corresponding Hermitian positive Jordan triple system (cf. [28], [29]).

♣2. Orthogonal Grassmannians (type II): Write $G_{II}(n, n)$ for the submanifold of the Grassmannian $G(n, n)$ consisting of isotropic n-dimensional subspaces of $\mathbb{C}^{2n}$. Then $\tilde{S} \in G_{II}(n, n)$ if and only if

$$\tilde{S} \begin{pmatrix} 0 & I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix} \tilde{S}^T = 0. \tag{4}$$

In the aforementioned open affine piece of the Grassmannian $G(n, n)$ with $\tilde{S} = (I, S)$, $\tilde{S} \in G_{II}(n, n)$ if and only if S is an $n \times n$ antisymmetric matrix. We identify this open affine chart $\mathcal{A}$ of $G_{II}(n, n)$ with $\mathbb{C}^{\frac{n(n-1)}{2}}$ through the holomorphic coordinate map:

$$(I_{n\times n} \quad Z) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -z_{12} & 0 & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -z_{1n} & -z_{2n} & \cdots & 0 \end{pmatrix} \to (z_{12}, \cdots z_{(n-1)n}). \tag{5}$$

Later in the paper we will sometimes use the notation $z_{ji} := -z_{ij}$ if $j > i$ for this type II case. The Plücker embedding of $G(n,n)$ gives a 2-canonical embedding of $G_{II}(n,n)$. Unfortunately this embedding is not good enough for our purposes later. Therefore, we will use a different embedding in this paper, which is given by the spin representation of O_{2n}. This embedding is what is called a one-canonical embedding of $G_{II}(n,n)$. We briefly describe this embedding as following. More details can be found in [Chapter 12; 41].

Let V be a real vector space of dimension $2n$ with a given inner product, and let $\mathcal{K}(V)$ be the space consisting of all orthogonal complex structures on V preserving this inner product. An element of $\mathcal{K}(V)$ is a linear orthogonal transformation $J : V \to V$ such that $J^2 = -1$. Any two choices of J are conjugate in the orthogonal group $O(V) = O_{2n}$, and thus $\mathcal{K}(V)$ can be identified with the homogeneous space O_{2n}/U_n. On the other hand, there is a one-to-one correspondence assigning the complex J to a complex n-dimensional isotropic subspace W of $V_{\mathbb{C}}(= V \bigotimes \mathbb{C})$. $\mathcal{K}(V)$ has two connected components $\mathcal{K}_{\pm}(V)$: Noticing that any complex structure defines an orientation on V, these two components correspond to the two possible orientations on V. Write one for $\mathcal{K}_+(V)$, which is actually our $G_{II}(n,n)$.

Now fix an isotropic n-dimensional subspace $W \subset V_{\mathbb{C}}$ with the associated complex structure J of $V_{\mathbb{C}}$ and pick a basis for V: $\{x_1, ..., x_n, y_1, ..., y_n\}$ with $J(x_i) = y_i, J(y_i) = -x_i$. Then W is spanned by $\{x_i - \sqrt{-1}y_i\}_{i=1}^n$. Define $\overline{W}$ to be the space spanned by $\{x_i + \sqrt{-1}y_i\}_{i=1}^n$. As shown in [41], there is a holomorphic embedding $\mathcal{K}(V) \hookrightarrow \mathbb{CP}(\Lambda(W))$, where $\Lambda(W)$ is the exterior algebra of W. This embedding is equivariant under the action of $O(V)$. Thus $\mathcal{K}_+(V) \hookrightarrow \mathbb{CP}(\Lambda(W))$ is equivariant under $SO(V)$. Choose the open affine cell of $\mathcal{K}_+(V)$ such that $\{Y \in \mathcal{K}_+(V) | Y \cap \overline{W} = \varnothing\}$. Then it can be identified with (5).

We next describe the 1-canonical embedding by Pfaffians as following: Let Π be the set of all partitions of $\{1, 2, ..., 2n\}$ into pairs without regard to order. An element $\alpha \in \Pi$ can be written as $\alpha = \{(i_1, j_1), (i_2, j_2), ..., (i_n, j_n)\}$ with $i_k < j_k$ and $i_1 < i_2 < ... < i_n$. Let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & ... & 2n \\ i_1 & j_1 & i_2 & j_2 & ... & j_n \end{bmatrix}$$

be the corresponding permutation. Given a partition α as above and a $(2n) \times (2n)$ matrix $A = (a_{jk})$, define

$$A_\alpha = \operatorname{sgn}(\pi) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}.$$

The Pfaffian of A is then given by

$$\mathrm{pf}(A) = \sum_{\alpha \in \Pi} A_\alpha.$$

The Pfaffian of an $m \times m$ skew-symmetric matrix for m odd is defined to be zero.

Therefore in the coordinate system (5), the embedding of $\mathcal{A}$ is given by

$$[1, ..., \mathrm{pf}(Z_\sigma), ...]. \tag{6}$$

Write S_k for the collection of all subsets of $\{1, ..., n\}$ with k elements. The σ in (6) runs through all elements of S_k with $2 \le k \le n$ and k even. For $\sigma = \{i_1 < \cdots < i_k\}$, Z_σ is defined as the submatrix $Z(\begin{smallmatrix} i_1 & \dots & i_k \\ i_1 & \dots & i_k \end{smallmatrix})$. For instance, $(\mathrm{pf}(Z_\sigma))_{\sigma \in S_2} = (z_{12}, ..., z_{(n-1)n})$. We also write (6) as $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$ for simplicity of notation. We choose the local coordinates for $(G_{II}(n,n))^*$ in a similar way

$$\begin{pmatrix} I_{n\times n} & \Xi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \xi_{12} & \cdots & \xi_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -\xi_{12} & 0 & \cdots & \xi_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -\xi_{1n} & -\xi_{2n} & \cdots & 0 \end{pmatrix}. \tag{7}$$

The defining function for the Segre family (in the product of such affine pieces) is given by

$$\rho(z, \xi) = 1 + \sum_{\substack{\sigma \in S_k, \\ 2 \le k \le n, 2|k}} \mathrm{Pf}(Z_\sigma)\mathrm{Pf}(\Xi_\sigma). \tag{8}$$

♣3. Symplectic Grassmannians (type III): Write $G_{III}(n,n)$ for the submanifold of the Grassmannian space $G(n,n)$ defined as follows: Take the matrix representation of each element of the Grassmannian $G(n,n)$ as an $n \times 2n$ non-degenerate matrix. Then $\widetilde{A} \in G_{III}(n,n)$, if and only if,

$$\widetilde{A} \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix} \widetilde{A}^T = 0. \tag{9}$$

In the Zariski open affine piece of the Grassmannian $G(n,n)$ defined before, we can take a representative matrix of the form: $\widetilde{A} = (I, Z)$. Then we conclude that $\widetilde{A} \in G_{III}(n,n)$ if and only if Z is an $n \times n$ symmetric matrix. We identify this Zariski open affine chart $\mathcal{A}$ of $G_{III}(n,n)$ with $\mathbb{C}^{\frac{n(n+1)}{2}}$ through the holomorphic coordinate map:

$$\tilde{A} = \begin{pmatrix} I_{n\times n} & Z \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & z_{11} & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & z_{12} & z_{22} & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & z_{1n} & z_{2n} & \cdots & z_{nn} \end{pmatrix} \to \quad (z_{11}, \cdots, z_{nn}).$$

Later in the paper we sometimes use the notation $z_{ji} := z_{ij}$ if $j > i$ for this type III case. Through the Plücker embedding of the Grassmannian, $G_{III}(n,n)$ is embedded into

$\mathbb{CP}(\Lambda^n\mathbb{C}^{2n})(\cong \mathbb{CP}^{N^*})$. In the above local coordinates, we write down the embedding as (up to a sign)

$$Z \to [1, \cdots, Z(\begin{smallmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{smallmatrix}), \dots] := [1, \psi_1, \cdots, \psi_{N^*}]. \tag{10}$$

Choose the local affine open piece of $(G_{III}(n,n))^*$ consisting of elements in the following form:

$$(\, I_{n\times n} \quad \Xi\,) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ 0 & 1 & 0 & \cdots & 0 & \xi_{12} & \xi_{22} & \cdots & \xi_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & \xi_{1n} & \xi_{2n} & \cdots & \xi_{nn}. \end{pmatrix}.$$

The defining function of Segre family in the product of such affine open pieces is given by

$$\rho(z,\xi) = 1 + \sum_{\substack{1\le i_1<i_2<\dots<i_k\le n,\\ 1\le j_1<j_2<\dots<j_k\le n\\ k=1,\dots,n}} Z(\begin{smallmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{smallmatrix})\Xi(\begin{smallmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{smallmatrix}) \tag{11}$$

However the Plücker embedding is not a useful canonical embedding to us for $G_{III}(n,n)$, due to the fact that $\{\psi_j\}$ is not a linearly independent system. For instance,

$$Z\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + Z\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = Z\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

This embedding can not serve our purposes here. We therefore derive from this embedding a minimal embedding into a certain projective subspace in $\mathbb{CP}(\Lambda^n\mathbb{C}^{2n})(\cong \mathbb{CP}^{N^*})$. We denote this minimal projective subspace by $\mathcal{H} \cong \mathbb{CP}^N$, which is discussed in detail below. We notice that the embedding $G_{III}(n,n) \hookrightarrow \mathbb{CP}^N$ is equivariant under the transitive action of $Sp(n)$.

Following the notations we set up in the Grassmannian case, we write $[1, \psi_1, \cdots, \psi_{N^*}]$ for the map of the Plücker embedding into $\mathbb{CP}^{N^*}$. Write $(\psi_{i_1}, \dots, \psi_{i_{m_k}})$ for those components of degree k in z among $\{\psi_j\}_{j=1}^{N^*}$. Here $1 \le k \le n$, and $\{i_1, \dots, i_{m_k}\}$ depends on k. For instance, if $k = 1$, then

$$(\psi_{i_1}, \dots, \psi_{i_{m_1}}) = (z_{11}, \dots, z_{nn}),$$

where z_{ij} is repeated twice if $i \neq j$. Let $\{\psi_1^{(k)}, \cdots, \psi_{m_k^*}^{(k)}\}$ be a maximally linearly independent subset of $\{\psi_{i_1}, \dots, \psi_{i_{m_k}}\}$ over $\mathbb{R}$ (and thus also over $\mathbb{C}$). For instance,

$$\{\psi_1^{(1)}, \cdots, \psi_{m_1^*}^{(1)}\} = \{z_{ij}\}_{i\le j}.$$

Let A_k be the $m_k^* \times m_k$ matrix such that $(\psi_{i_1}, \cdots, \psi_{i_{m_k}}) = (\psi_1^{(k)}, \cdots, \psi_{m_k^*}^{(k)}) \cdot A_k$. Apparently A_k has real entries and is of full rank. Hence $A_k \cdot A_k^t$ is positive definite.

Then $\{\psi_1^*, \cdots, \psi_N^*\} := \{\psi_1^{(k)}, \cdots, \psi_{m_k^*}^{(k)}\}_{1 \le k \le n}$ forms a basis of $\{\psi_1, \cdots, \psi_{N^*}\}$, where $N = m_1^* + ... + m_n^*$. Moreover, if we write A as the $(m_1^* + ... + m_n^*) \times (m_1 + ... + m_n)$ matrix:

$$A = \begin{pmatrix} A_1 & & \\ & \cdots & \\ & & A_n \end{pmatrix},$$

then A has full rank and we have a real orthogonal matrix U such that

$$U = \begin{pmatrix} U_1 & & \\ & \cdots & \\ & & U_n \end{pmatrix}, \quad U^t(A \cdot A^t)U = \begin{pmatrix} \mu_1 & & \\ & \cdots & \\ & & \mu_N \end{pmatrix} \text{ with each } \mu_j > 0.$$

Here $U_k, 1 \le k \le n$, is an $m_k^* \times m_k^*$ orthogonal matrix. Now we define

$$\begin{aligned} &(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n) \\ &:= (\psi_1^*, \cdots, \psi_N^*) \cdot U \cdot \begin{pmatrix} \sqrt{\mu_1} & & & \\ & \sqrt{\mu_2} & & \\ & & \cdots & \\ & & & \sqrt{\mu_N} \end{pmatrix}. \end{aligned}$$

Here $N_1 + ... + N_{n-1} + N_n = N^*$, where we set $N_n = 1$. We will also sometimes write $\psi_{N_n}^n = \psi^n$. As a direct consequence,

$$\begin{aligned} &(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n) \\ &\quad \cdot (\overline{\psi_1^1}, ..., \overline{\psi_{N_1}^1}, \overline{\psi_1^2}, ..., \overline{\psi_{N_2}^2}, ..., \overline{\psi_1^{n-1}}, ..., \overline{\psi_{N_{n-1}}^{n-1}}, \overline{\psi^n}) \\ &= (\psi_1, \cdots, \psi_{N^*}) \cdot (\overline{\psi_1}, \cdots, \overline{\psi_{N^*}}) = \det(I + Z\bar{Z}^t) = \rho(z, \overline{z}). \end{aligned} \tag{12}$$

Moreover $\{\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n\}$ forms a linearly independent system; and $\{\psi_1^k, ..., \psi_{N_k}^k\}$ are polynomials in z of degree k for $k = 1, ..., n$. Now our canonical embedding of the aforementioned affine piece $\mathcal{A}$ of $G_{III}(n, n)$ is taken as

$$z \in \mathbb{C}^{\frac{n(n+1)}{2}} \to [1, \psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n].$$

For simplicity, we will still denote $(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n)$ by

$$r_z = (\psi_1, \psi_2, ..., \psi_N) = \left(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n\right). \tag{13}$$

Here, for instance, $(\psi_1, ..., \psi_{\frac{n(n+1)}{2}}) = (\psi_1^1, ..., \psi_{N_1}^1) = (a_{ij} z_{ij})_{1 \le i \le j \le n}$, where a_{ij} equals to 1 if $i = j$, equals to $\sqrt{2}$ if $i < j$. Hence the defining function of the Segre family, which is the same as (11), is given by $\rho(z, \xi) = 1 + \sum_{i=1}^N \psi_i(z)\psi_i(\xi)$.

♣4. Hyperquadrics (type IV): Let Q^n be the hypersurface in $\mathbb{CP}^{n+1}$ defined by

$$\left\{[x_0, ..., x_{n+1}] \in \mathbb{CP}^{n+1} : \sum_{i=1}^{n} x_i^2 - 2x_0x_{n+1} = 0\right\},$$

where $[x_1, ..., x_{n+2}]$ are the homogeneous coordinates for $\mathbb{CP}^{n+1}$. It is invariant under the action of the group $SO(n+2)$. We mention that under the present embedding, the action is not the standard $SO(n+2)$ in $GL(n+2)$. However it is conjugate to the standard $SO(n+2)$ action by a certain element $g \in U(n+2)$. A Zariski open affine piece $\mathcal{A} \subset Q^n$ identified with $\mathbb{C}^n$ is given by $(z_1, ..., z_n) \mapsto [1, \psi_1, ..., \psi_{n+1}] = [1, z_1, ..., z_n, \frac{1}{2}\sum_{i=1}^{n} z_i^2]$, which will be denoted by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_{n+1}]$. Choose the same local chart for $(Q^n)^*$: $(\xi_1, ..., \xi_n) \to [1, \xi_1, ..., \xi_n, \frac{1}{2}\sum_{i=1}^{n} \xi_i^2]$. Then the defining function of the Segre family restricted to $\mathbb{C}^n \times \mathbb{C}^n \hookrightarrow Q^n \times (Q^n)^*$ is given by

$$\rho(z, \xi) = 1 + \sum_{i=1}^{n} z_i\xi_i + \frac{1}{4}(\sum_{i=1}^{n} z_i^2)(\sum_{i=1}^{n} \xi_i^2) \tag{14}$$

♣5. The exceptional manifold $M_{16} := E_6/SO(10) \times SO(2)$: As shown in [23], [24], this exceptional Hermitian symmetric space can be realized as the Cayley plane. Take the exceptional 3×3 complex Jordan algebra

$$\mathcal{J}_3(\mathbb{O}) = \left\{ \begin{pmatrix} c_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & c_2 & x_1 \\ x_2 & \bar{x}_1 & c_3 \end{pmatrix} : c_i \in \mathbb{C}, x_i \in \mathbb{O} \right\} \cong \mathbb{C}^{27}. \tag{15}$$

Here $\mathbb{O}$ is the complexified algebra of octonions, which is a complex vector space of dimension 8. Denote a standard basis of $\mathbb{O}$ by $\{e_0, e_1, ..., e_7\}$. The multiplication rule in terms of this basis is given in Appendix A. The conjugation operator appeared in (15) is for octonions, which is defined as follows: $\bar{x} = x_0e_1 - x_1e_1 - ... - x_7e_7$, if $x = x_0e_0 + x_1e_1 + x_2e_2 + ... + x_7e_7, x_i \in \mathbb{C}$. Moreover under this basis, $\mathcal{J}_3(\mathbb{O}) \cong \mathbb{C}^{27}$ is realized by identifying each matrix

$$X = \begin{pmatrix} \xi_1 & \eta & \bar{\kappa} \\ \bar{\eta}_3 & \xi_2 & \tau \\ \kappa & \bar{\tau} & \xi_3 \end{pmatrix} \in \mathcal{J}_3(\mathbb{O})$$

with the point $(\xi_1, \xi_2, \xi_3, \eta_0, \eta_1, \ldots, \eta_7, \kappa_0, \kappa_2, \ldots, \kappa_7, \tau_0, \tau_1, \ldots, \tau_7) \in \mathbb{C}^{27}$, where $\eta = \sum_{i=0}^{7} \eta_i e_i, \kappa = \sum_{i=0}^{7} \kappa_i e_i$ and $\tau = \sum_{i=0}^{7} \tau_i e_i$.

The Jordan multiplication is defined as $A \circ B = \frac{1}{2}(AB + BA)$ for $A, B \in \mathcal{J}_3(\mathbb{O})$. The subgroup $SL(\mathbb{O})$ of $GL(\mathcal{J}_3(\mathbb{O}))$ consisting of automorphisms preserving the determinant is the adjoint group of type E_6. The action of E_6 on the projectivization $\mathbb{CP}\mathcal{J}_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface, the regular part of this hypersurface, and its singular part which is the closed E_6-orbit. The closed orbit is the Cayley plane or the hermitian symmetric space of compact type corresponding to E_6. It can be defined by the quadratic equation

$$X^2 = \mathrm{trace}(X)X, \qquad X \in \mathcal{J}_3(\mathbb{O}),$$

or as the closure of the affine cell $\mathcal{A}$

$$\mathbb{OP}_1^2 = \left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & x\bar{x} & y\bar{x} \\ \bar{y} & x\bar{y} & y\bar{y} \end{pmatrix} : x, y \in \mathbb{O} \right\} \cong \mathbb{C}^{16}$$

in the local coordinates $(x_0, x_1, ..., x_7, y_0, ..., y_7)$. The precise formula for the canonical embedding map is given in Appendix B. We denote this embedding by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$.

To find the defining function for its Segre family over the product of such standard affine sets, we choose local coordinates for the conjugate Cayley plane to be $(\kappa_0, \kappa_1, ..., \kappa_7, \eta_0, \eta_1, ..., \eta_7)$. Then

$$\rho(z,\xi) = 1 + \sum_{i=0}^{7} x_i\kappa_i + \sum_{i=0}^{7} y_i\eta_i + \sum_{i=0}^{7} A_i(x,y)A_i(\kappa,\eta) + B_0(x,y)B_0(\kappa,\eta) + B_1(x,y)B_1(\kappa,\eta), \tag{16}$$

where A_j, B_j are defined as in Appendix A, $z = (x_0, ..., x_7, y_0, ..., y_7)$ and $\xi = (\kappa_0, ..., \kappa_7, \eta_0, ..., \eta_7)$.

♣6. The other exceptional manifold $M_{27} = E_7/E_6 \times SO(2)$: As shown in [6], it can be realized as the Freudenthal variety. Consider the Zorn algebra

$$\mathcal{Z}_2(\mathbb{O}) = \mathbb{C} \bigoplus \mathcal{J}_3(\mathbb{O}) \bigoplus \mathcal{J}_3(\mathbb{O}) \bigoplus \mathbb{C}$$

One can prove that there exists an action of E_7 on that 56−dimensional vector space (see [13]). The closed E_7−orbit inside $\mathbb{CP}\mathcal{Z}_2(\mathbb{O})$ is the Freudenthal variety $E_7/E_6 \times SO(2)$. An affine cell $\mathcal{A}$ of Freudenthal variety is $[1, X, \mathrm{Com}(X), \det(X)] \in \mathbb{CP}\mathcal{Z}_2(\mathbb{O})$. Here X belongs to $\mathcal{J}_3(\mathbb{O})$; $\mathrm{Com}(X)$ is the comatrix of X such that $X\mathrm{Com}(X) = \det(X)I$ under the usual matrix multiplication rule. Notice that $\mathrm{Com}(X) = X \times X$, where $X \times X$ is the Freudenthal multiplication defined as follows (see [40]):

$$X \times X := X^2 - \mathrm{tr}(X)X + \frac{1}{2}(\mathrm{tr}(X)^2 - \mathrm{tr}(X^2))I.$$

For explicit expressions for $X \times X$ and $\det(X)$ in terms of the entries of X, see [40] or Appendix A in this paper.

The embedding of $E_7/E_6 \times SO(2) \hookrightarrow \mathbb{CP}^N$ in local coordinates z is given in Appendix A. Choose the local affine open piece for $(E_7/E_6 \times SO(2))^*$ with coordinates

$$\xi = (\xi_1, \xi_2, \xi_3, \eta_0, ..., \eta_7, \kappa_0, ..., \kappa_7, \tau_0, ..., \tau_7).$$

We denote this embedding by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$. The defining function for the Segre family is then $\rho(z, \xi) = 1 + r_z \cdot r_\xi$, where

$$\begin{aligned} r_z &= (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7, A(z), B(z), C(z), D_0(z), ..., D_7(z), \\ &\quad E_0(z), ..., E_7(z), F_0(z), ..., F_7(z), G(z)) \\ r_\xi &= (\psi_1(\xi), \psi_2(\xi), ..., \psi_N(\xi)) = (\xi_1, \xi_2, \xi_3, \eta_0, ..., \eta_7, \kappa_0, ..., \kappa_7, \tau_0, ..., \tau_7, \\ &\quad A(\xi), B(\xi), C(\xi), D_0(\xi), ..., D_7(\xi), E_0(\xi), ..., E_7(\xi), F_0(\xi), ..., F_7(\xi), G(\xi)) \end{aligned} \tag{17}$$

Here see Appendix A for the definition of the functions appeared in the formula.

Summarizing the above, for each irreducible Hermitian symmetric space of compact type M of dimension n, we now have described a canonical embedding from M into a projective space $\mathbb{P}^N$, which restricted to a certain Zariski open affine piece $\mathcal{A}$ holomorphically equivalent to $\mathbb{C}^n$ takes the form: $z(\in \mathbb{C}^n) \mapsto [1, \kappa_1 z_1, \cdots, \kappa_i z_i, \cdots, \kappa_n z_n, O(z^2)]$. Here $\kappa_i = 1$ for all i except in the case of type III where κ_i can be 1 or $\sqrt{2}$. This is the embedding we will use in later discussions. Notice in our embedding, the conjugate space M^* is the same as M. For simplicity of notation, we will also write $\mathcal{M}$ for the restriction of the Segre family of M restricted to $\mathcal{A} \times \mathcal{A}^* = \mathbb{C}^n \times \mathbb{C}^n$. From this embedding and the invariant property of Segre varieties, we immediately conclude the following:

Lemma 2.1. *Assume A and B are two distinct points of M. Then their associated Segre varieties are different, namely, $Q_A \neq Q_B$.*

Proof of Lemma 2.1: Since the holomorphic isometric group acts transitively on M, we can assume $A = (0, 0, ..., 0) \in \mathbb{C}^n \cong \mathcal{A} \subset M$. Therefore Q_A is the hyperplane section of $M \hookrightarrow \mathbb{P}^N$ at infinity, namely, $Q_A = M \backslash \mathcal{A}$. Now if $B \in \mathcal{A}$, because $B \neq (0, 0, ..., 0)$, there are non-trivial linear terms in the defining function of Q_B. This leads to the fact that the defining function of Q_B has to be a non-constant polynomial in $\mathbb{C}[\xi_1, ..., \xi_n]$. Therefore $Q_B \cap \mathbb{C}^n \neq \emptyset$ and thus does not coincide with Q_A. If $B \in \mathcal{M} \backslash \mathcal{A}$, by the symmetric property of Segre varieties, we have $(0, ..., 0) \in Q_B$. Therefore $Q_B \neq Q_A$. We then arrive at the conclusion. □

Finally, since in our setting, $M^* = M$ and the Segre family on M and M^* are the same. For simplicity of notation, we do not distinguish, in what follows, Q^* and $\mathcal{M}^*$ from Q and $\mathcal{M}$, respectively.

2.3. Explicit expression of the volume forms

From now on, we assume that M is an irreducible Hermitian symmetric space of compact type and we choose the canonical embedding $M \hookrightarrow \mathbb{CP}^N$ as described in §2.2 according to its type. We denote the metric on M induced from Fubini-Study of $\mathbb{CP}^N$ by ω, and the volume form by $d\mu = \omega^n$ (up to a positive constant). Notice that the metric we obtained is always invariant under the action of a certain transitive subgroup $G \subset Aut(M)$ (which comes from the restriction of a subgroup of the unitary group of the ambient projective space). Hence by a theorem of Wolf [44], ω is the unique G invariant metric on M up to a scale. We claim ω must be Kähler-Einstein. Indeed, since the Ricci form $\mathrm{Ric}(\omega)$ of ω is invariant under G, for a small ϵ, $\omega + \epsilon \mathrm{Ric}(\omega)$ is thus also a G invariant metric on M. By [44], it is a multiple of ω, and thus $\mathrm{Ric}(\omega) = \lambda\omega$. Write $d\mu$ as the product of V and the standard Euclidean volume form over the affine subspace $\mathcal{A}$, where V is a positive function in z. Since $\mathrm{Ric}(\omega) = -i\partial\bar{\partial}\log V$, $-i\partial\bar{\partial}\log V = \lambda\omega$. Notice that $\lambda > 0$. In the local affine open piece $\mathcal{A}$ defined before, $\omega = i\partial\bar{\partial}\log\rho(z,\bar{z})$, where $\rho(z,\xi)$ is the defining function for the associated Segre family. As we will see later (§7), $\rho(z,\xi)$ is an irreducible polynomial in (z,ξ). Then we have

$$\partial\bar{\partial}\log(V\rho(z,\bar{z})^{\lambda}) = 0.$$

Hence, $\log(V\rho(z,\bar{z})^{\lambda}) = \phi(z) + \overline{\psi(z)}$, where both ϕ and ψ are holomorphic functions. Therefore $V = \frac{e^{\phi(z)+\overline{\psi(z)}}}{\rho(z,\bar{z})^{\lambda}}$. Because $\rho(z,\xi)$ is an irreducible polynomial, from the way V is defined, V must be a rational function of the form $\frac{p(z,\bar{z})}{\rho(z,\bar{z})^m}$ with p,ρ relatively prime to each other. Since ϕ,ψ are globally defined, by a monodromy argument, it is clear that λ has to be an integer. Also both $e^{\phi(z)}$ and $e^{\overline{\psi(\xi)}}$ must be rational functions. Again, since ϕ,ψ are also globally defined, this forces ϕ,ψ to be constant functions. Therefore, we conclude that $V = c\rho(z,\bar{z})^{-\lambda}$. Here λ is a certain positive integer and c is a positive constant. Next by a well-known result (see [1]), two Kähler-Einstein metrics of M are different by an automorphism of M (up to a positive scalar multiple). Therefore, to prove Theorem 1.1, we can assume, without loss of generality, that the Kähler-Einstein metric in Theorem 1.1 is the metric obtained by restricting the Fubini-Study metric to M through the embedding described in this section.

3. A basic property for partially degenerate holomorphic maps

In this section, we introduce a notion of degeneracy for holomorphic maps and derive an important consequence, which will be fundamentally applied in the proof of our main theorem.

Let $\psi(z) := (\psi_1(z),...,\psi_N(z))$ be a vector-valued holomorphic function from a neighborhood U of 0 in $\mathbb{C}^m, m \geq 2$, into $\mathbb{C}^N, N > m$, with $\psi(0) = 0$. Here we write $z = (z_1,...,z_m)$ for the coordinates of $\mathbb{C}^m$. In the following, we will write $\widetilde{z} = (z_1,...,z_{m-1})$, i.e., the vector z with the last component z_m being dropped out. Write $\frac{\partial^{|\alpha|}}{\partial\widetilde{z}^{\alpha}} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}...\partial z_{m-1}^{\alpha_{m-1}}}$ for an $(m-1)-$multiindex α, where $\alpha = (\alpha_1,...,\alpha_{m-1})$. Write

$$\frac{\partial^{|\alpha|}}{\partial\widetilde{z}^{\alpha}}\psi(z) = \left(\frac{\partial^{|\alpha|}}{\partial\widetilde{z}^{\alpha}}\psi_1(z),...,\frac{\partial^{|\alpha|}}{\partial\widetilde{z}^{\alpha}}\psi_N(z)\right).$$

We introduce the following definition.

Definition 3.1. Let $k \geq 0$. For a point $p \in U$, write $E_k(p) = \mathrm{Span}_{\mathbb{C}}\{\frac{\partial^{|\alpha|}}{\partial \widetilde{z}^\alpha}\psi(z)|_{z=p} : 0 \leq |\alpha| \leq k\}$. We write r for the greatest number such that for any neighborhood O of 0, there exists $p \in O$ with $\dim_{\mathbb{C}} E_k(p) = r$. r is called the $k-$th $\widetilde{z}-$rank of ψ at 0, which is written as $\mathrm{rank}_k(\psi, \widetilde{z})$. F is called $\widetilde{z}-$nondegenerate if $\mathrm{rank}_{k_0}(\psi, \widetilde{z}) = N$ for some $k_0 \geq 1$.

Remark 3.2. It is easy to see that $\mathrm{rank}_k(\psi, \widetilde{z}) = r$ if and only if the following matrix

$$\begin{pmatrix} \frac{\partial^{|\alpha^0|}}{\partial \widetilde{z}^{\alpha^0}}\psi(z) \\ \cdots \\ \cdots \\ \frac{\partial^{|\alpha^s|}}{\partial \widetilde{z}^{\alpha^s}}\psi(z) \end{pmatrix}$$

has an $r \times r$ submatrix with determinant not identically zero for $z \in U$ for some multiindices $\{\alpha^0, ..., \alpha^s\}$ with all $0 \leq |\alpha^j| \leq k$. Moreover, any $l \times l$ $(l > r)$ submatrix of the matrix has identically zero determinant for any choice of $\{\alpha^0, ..., \alpha^s\}$ with $0 \leq |\alpha^j| \leq k$.

In particular, ψ is $\widetilde{z}-$nondegenerate if and only if there exist multiindices $\beta^1, ..., \beta^N$ such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}}{\partial \widetilde{z}^{\beta^1}}\psi_1(z) & \cdots & \frac{\partial^{|\beta^1|}}{\partial \widetilde{z}^{\beta^1}}\psi_N(z) \\ \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^N|}}{\partial \widetilde{z}^{\beta^N}}\psi_1(z) & \cdots & \frac{\partial^{|\beta^N|}}{\partial \widetilde{z}^{\beta^N}}\psi_N(z) \end{vmatrix}$$

is not identically zero. Moreover, $\mathrm{rank}_{i+1}(\psi, \widetilde{z}) \geq \mathrm{rank}_i(\psi, \widetilde{z})$ for any $i \geq 0$.

For the rest of this section, we further assume that the first m components of ψ, i.e., $(\psi_1, ..., \psi_m) : \mathbb{C}^m \to \mathbb{C}^m$ is a biholomorphic map in a neighborhood of $0 \in \mathbb{C}^m$. Then we have,

Lemma 3.3. *It holds that* $\mathrm{rank}_0(\psi, \widetilde{z}) = 1, \mathrm{rank}_1(\psi, \widetilde{z}) = m$, *and for* $k \geq 1, \mathrm{rank}_k(\psi, \widetilde{z}) \geq m$.

Proof of Lemma 3.3: We first notice that it holds trivially that $\mathrm{rank}_0(\psi, \widetilde{z}) = 1$, for F is not identically zero. We now prove $\mathrm{rank}_1(\psi, \widetilde{z}) = m$. First notice that $\mathrm{rank}_1(\psi, \widetilde{z}) \leq m$ as there are only m distinct multiindices β such that $|\beta| \leq 1$. On the other hand, since ψ has full rank at 0, we have,

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial z_1} & \cdots & \frac{\partial \psi_m}{\partial z_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial \psi_1}{\partial z_m} & \cdots & \frac{\partial \psi_m}{\partial z_m} \end{vmatrix}(0) \neq 0.$$

This together with the fact $\psi(0) = 0$ implies that the z_m derivative of

$$\begin{vmatrix} \psi_1 & \cdots & \psi_m \\ \frac{\partial \psi_1}{\partial z_1} & \cdots & \frac{\partial \psi_m}{\partial z_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial \psi_1}{\partial z_{m-1}} & \cdots & \frac{\partial \psi_m}{\partial z_{m-1}} \end{vmatrix} \tag{18}$$

is nonzero at $p = 0$. Consequently, the quantity in (18) is not identically zero in U. By the definition of the $\widetilde{z}$-rank, we then arrive at the conclusion. □

We now prove the following degeneracy theorem in terms of its $\widetilde{z}$-rank, which will be used to derive Theorem 3.10.

Theorem 3.4. *Let $\psi = (\psi_1, ..., \psi_m, \psi_{m+1}, ..., \psi_N)$ be a holomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into $\mathbb{C}^N$ with $\psi(0) = 0$. Recall that $\widetilde{z} = (z_1, ..., z_{m-1})$, i.e., the vector z with the last component z_m being dropped out. Assume that $(\psi_1, ..., \psi_m)$ is a biholomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into a neighborhood of $0 \in \mathbb{C}^m$. Suppose*

$$\mathrm{rank}_{N-m+1}(\psi, \widetilde{z}) < N. \tag{19}$$

Then there exist N holomorphic functions $g_1(z_m), ..., g_N(z_m)$ near 0 in the z_m-Gauss plane with $\{g_1(0), ..., g_N(0)\}$ not all zero such that the following holds for any $(z_1, ..., z_m)$ near 0.

$$\sum_{i=1}^{N} g_i(z_m)\psi_i(z_1, ..., z_m) \equiv 0. \tag{20}$$

In particular, one can make one of the $\{g_i\}_{i=1}^N$ to be identically one.

The geometric intuition for the theorem is as follows: The space of 1-jets has dimension m by Lemma 3.3. We expect that at least one more dimension is increased when we go from the space of k-jets to the space of $(k+1)$-jets until we reach the maximum possible value N. The theorem says that if this process fails, namely, the assumption in (19) holds, we then end up with a function relationship as in (20).

Proof of Theorem 3.4: We consider the following set,

$$\mathcal{S} = \{l \geq 1 : \mathrm{rank}_l(\psi, \widetilde{z}) \leq l + m - 2\}.$$

Note that $1 \notin \mathcal{S}$, for $\mathrm{rank}_1(F) = m$. We claim that $\mathcal{S}$ is not empty. Indeed, we have $1 + N - m \in \mathcal{S}$ by (19). Now write t' for the minimum number in $\mathcal{S}$. Then $2 \leq t' \leq 1 + N - m$. Moreover, by the choice of t',

$$\mathrm{rank}_{t'}(\psi, \widetilde{z}) \leq t' + m - 2, \ \mathrm{rank}_{t'-1}(\psi, \widetilde{z}) \ \geq t' + m - 2. \tag{21}$$

This yields that

$$\operatorname{rank}_{t'}(\psi, \widetilde{z}) = \operatorname{rank}_{t'-1}(\psi, \widetilde{z}) = t' + m - 2. \tag{22}$$

We write $t := t' - 1,\ n := t' + m - 2$. Here we note $t \geq 1, m \leq n \leq N - 1$. Then there exist multiindices $\{\gamma^1, ..., \gamma^n\}$ with each $|\gamma^i| \leq t$ and $j_1, ..., j_n$ such that

$$\Delta(\gamma^1, ..., \gamma^n | j_1, ..., j_n) := \begin{vmatrix} \frac{\partial^{|\gamma^1|}\psi_{j_1}}{\partial \bar{z}^{\gamma^1}} & ... & \frac{\partial^{|\gamma^1|}\psi_{j_n}}{\partial \bar{z}^{\gamma^1}} \\ ... & ... & ... \\ \frac{\partial^{|\gamma^n|}\psi_{j_1}}{\partial \bar{z}^{\gamma^n}} & ... & \frac{\partial^{|\gamma^n|}\psi_{j_n}}{\partial \bar{z}^{\gamma^n}} \end{vmatrix} \text{ is not identically zero in } U. \tag{23}$$

Since $\operatorname{rank}_1(\psi, \widetilde{z}) = m$, we can choose $(\gamma^1, ..., \gamma^n | j_1, ..., j_n)$ such that

$$\gamma^1 = (0, .., 0), \gamma^2 = (1, 0, ..., 0), ..., \gamma^m = (0, ..., 0, 1).$$

For any $\alpha^1, ..., \alpha^{n+1}$ with $|\alpha^i| \leq t + 1$, and $l_1, ..., l_{n+1}$, we have

$$\Delta(\alpha^1, ..., \alpha^{n+1} | l_1, ..., l_{n+1}) = \begin{vmatrix} \frac{\partial^{|\alpha^1|}\psi_{l_1}}{\partial \bar{z}^{\alpha^1}} & ... & \frac{\partial^{|\alpha^1|}\psi_{l_n}}{\partial \bar{z}^{\alpha^1}} & \frac{\partial^{|\alpha^1|}\psi_{l_{n+1}}}{\partial \bar{z}^{\alpha^1}} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ \frac{\partial^{|\alpha^{n+1}|}\psi_{l_1}}{\partial \bar{z}^{\alpha^{n+1}}} & ... & \frac{\partial^{|\alpha^{n+1}|}\psi_{l_n}}{\partial \bar{z}^{\alpha^{n+1}}} & \frac{\partial^{|\alpha^{n+1}|}\psi_{l_{n+1}}}{\partial \bar{z}^{\alpha^{n+1}}} \end{vmatrix} \equiv 0 \text{ in } U. \tag{24}$$

We write Γ for the collection of $(\gamma^1, ..., \gamma^n | j_1, ..., j_n), j_1 < ... < j_n$, with $\gamma^1 = (0, .., 0)$ and with (23) being held. We associate each $(\gamma^1, ..., \gamma^n | j_1, ..., j_n)$ with an integer $s(\gamma^1, ..., \gamma^n | j_1, ..., j_n) := s_0$ where s_0 is the least number $s \geq 0$ such that

$$\frac{\partial^{s_1 + ... + s_{m-1} + s} \Delta(\gamma^1, ..., \gamma^n | j_1, ..., j_n)}{\partial z_1^{s_1} \partial z_2^{s_2} ... \partial z_{m-1}^{s_{m-1}} \partial z_m^s}(0) \neq 0$$

for some integers $s_1, ..., s_{m-1}$. Then $s(\gamma^1, ..., \gamma^n | j_1, ..., j_n) \geq 0$ for any $(\gamma^1, ..., \gamma^n | j_1, ..., j_n) \in \Gamma$.

Let $(\beta^1, ..., \beta^n | i_1, ..., i_n) \in \Gamma, i_1 < ... < i_n$ be indices with the least $s(\gamma^1, ..., \gamma^n | j_1, ..., j_n)$ among all $(\gamma^1, ..., \gamma^n | j_1, ..., j_n) \in \Gamma$.

We write $\{i_{n+1}, ..., i_N\} = \{1, ..., N\} \setminus \{i_1, .., i_n\}$, where $i_{n+1} < ... < i_N$. Write $\tilde{U} = \{z \in U : \Delta(\beta^1, ..., \beta^n | i_1, ..., i_n) \neq 0\}$. We then have the following:

Lemma 3.5. *Fix $j \in \{i_{n+1}, ..., i_N\}$. Let $i \in \{i_1, .., i_n\}$. Write $\{i'_1, ..., i'_{n-1}\} = \{i_1, ..., i_n\} \setminus \{i\}$. There exists a holomorphic function $g_i^j(z_m)$ in $\tilde{U}$ which only depends on z_m such that the following holds for $z \in \tilde{U}$:*

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial\tilde{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}\psi_j}{\partial\tilde{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial\tilde{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}\psi_j}{\partial\tilde{z}^{\beta^n}} \end{vmatrix}(z) = g_i^j(z_m) \begin{vmatrix} \frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial\tilde{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}\psi_i}{\partial\tilde{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial\tilde{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}\psi_i}{\partial\tilde{z}^{\beta^n}} \end{vmatrix}(z), \tag{25}$$

or equivalently,

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial\tilde{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}(\psi_j - g_i^j(z_m)\psi_i)}{\partial\tilde{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial\tilde{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}(\psi_j - g_i^j(z_m)\psi_i)}{\partial\tilde{z}^{\beta^n}} \end{vmatrix} \equiv 0. \tag{26}$$

Proof of Lemma 3.5: For simplicity of notation, we write $\frac{\partial}{\partial\tilde{z}^{\beta^i}}$ for $\frac{\partial^{|\beta^i|}}{\partial\tilde{z}^{\beta^i}}$, and for $\mu = i$ or j, write the matrix

$$V_\mu := \begin{pmatrix} \frac{\partial\psi_{i'_1}}{\partial\tilde{z}^{\beta^1}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^1}} & \frac{\partial\psi_\mu}{\partial\tilde{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\psi_{i'_1}}{\partial\tilde{z}^{\beta^n}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\tilde{z}^{\beta^n}} & \frac{\partial\psi_\mu}{\partial\tilde{z}^{\beta^n}} \end{pmatrix} = \begin{bmatrix} \mathbf{v}_\mu^1 \\ \vdots \\ \mathbf{v}_\mu^n \end{bmatrix},$$

where $\mathbf{v}_\mu^1, \cdots, \mathbf{v}_\mu^n$ are the row vectors of V_μ. To prove (25), one just needs to show that, for each $1 \le \nu \le m-1$,

$$\frac{\partial}{\partial z_\nu}\frac{\det(V_j)}{\det(V_i)} \equiv 0 \text{ in } \tilde{U}. \tag{27}$$

Indeed, by the quotient rule, the numerator of the left-hand side of (27) equals to

$$\det\begin{pmatrix} \det(V_i) & \det(V_j) \\ \frac{\partial}{\partial z_\nu}\det(V_i) & \frac{\partial}{\partial z_\nu}\det(V_j) \end{pmatrix}$$

$$= \det\begin{pmatrix} \det(V_i) & \det(V_j) \\ \det\begin{bmatrix} \frac{\partial}{\partial z_\nu}\mathbf{v}_i^1 \\ \mathbf{v}_i^2 \\ \vdots \\ \mathbf{v}_i^n \end{bmatrix} & \det\begin{bmatrix} \frac{\partial}{\partial z_\nu}\mathbf{v}_j^1 \\ \mathbf{v}_j^2 \\ \vdots \\ \mathbf{v}_j^n \end{bmatrix} \end{pmatrix} + \cdots + \det\begin{pmatrix} \det(V_i) & \det(V_j) \\ \det\begin{bmatrix} \mathbf{v}_i^1 \\ \vdots \\ \mathbf{v}_i^{n-1} \\ \frac{\partial}{\partial z_\nu}\mathbf{v}_i^n \end{bmatrix} & \det\begin{bmatrix} \mathbf{v}_j^1 \\ \vdots \\ \mathbf{v}_j^{n-1} \\ \frac{\partial}{\partial z_\nu}\mathbf{v}_j^n \end{bmatrix} \end{pmatrix}.$$

By (24) and Lemma 4.4 in [2], each term on the right-hand side of the equation above equals 0. For instance, the last term above equals to

$$
\begin{vmatrix}
\begin{vmatrix}
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^1}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^1}} & \frac{\partial\psi_{i}}{\partial\bar z^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^n}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^n}} & \frac{\partial\psi_{i}}{\partial\bar z^{\beta^n}}
\end{vmatrix}
&
\begin{vmatrix}
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^1}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^1}} & \frac{\partial\psi_{j}}{\partial\bar z^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^n}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^n}} & \frac{\partial\psi_{j}}{\partial\bar z^{\beta^n}}
\end{vmatrix}
\\
\begin{vmatrix}
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^1}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^1}} & \frac{\partial\psi_{i}}{\partial\bar z^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^{n-1}}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^{n-1}}} & \frac{\partial\psi_{i}}{\partial\bar z^{\beta^{n-1}}} \\
\frac{\partial}{\partial z_\nu}(\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^n}}) & \cdots & \frac{\partial}{\partial z_\nu}(\frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^n}}) & \frac{\partial}{\partial z_\nu}(\frac{\partial\psi_{i}}{\partial\bar z^{\beta^n}})
\end{vmatrix}
&
\begin{vmatrix}
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^1}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^1}} & \frac{\partial\psi_{j}}{\partial\bar z^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^{n-1}}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^{n-1}}} & \frac{\partial\psi_{j}}{\partial\bar z^{\beta^{n-1}}} \\
\frac{\partial}{\partial z_\nu}(\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^n}}) & \cdots & \frac{\partial}{\partial z_\nu}(\frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^n}}) & \frac{\partial}{\partial z_\nu}(\frac{\partial\psi_{j}}{\partial\bar z^{\beta^n}})
\end{vmatrix}
\end{vmatrix}. \tag{28}
$$

It is a multiple of the following determinant (by Lemma 4.4 in [2]):

$$
\begin{vmatrix}
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^1}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^1}} & \frac{\partial\psi_{i}}{\partial\bar z^{\beta^1}} & \frac{\partial\psi_{j}}{\partial\bar z^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^n}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^n}} & \frac{\partial\psi_{i}}{\partial\tilde Z^{\beta^n}} & \frac{\partial\psi_{j}}{\partial\bar z^{\beta^n}} \\
\frac{\partial\psi_{i'_1}}{\partial\bar z^{\beta^{n+1}}} & \cdots & \frac{\partial\psi_{i'_{n-1}}}{\partial\bar z^{\beta^{n+1}}} & \frac{\partial\psi_{i}}{\partial\bar z^{\beta^{n+1}}} & \frac{\partial\psi_{j}}{\partial\bar z^{\beta^{n+1}}}
\end{vmatrix}, \tag{29}
$$

where $\frac{\partial}{\partial\bar z^{\beta^{n+1}}} = \frac{\partial}{\partial z_\nu}(\frac{\partial}{\partial\bar z^{\beta^n}})$, which is identically zero by (24). This establishes Lemma 3.5. $\square$

The extendability of $g_i^j(z_m)$ will be needed for our later argument:

Lemma 3.6. *For any i, j as above, the holomorphic function $g_i^j(z_m)$ can be extended holomorphically to a neighborhood of 0 in the z_m-plane.*

Proof of Lemma 3.6: First, g_i^j is defined on the projection $\pi_m(\tilde U)$ of $\tilde U$, where π_m is the natural projection of $(z_1, ..., z_m)$ to its last component z_m. If $0 \in \pi_m(\tilde U)$, the claim follows trivially. Now assume that $0 \notin \pi_m(\tilde U)$. If we write $s = s(\beta_1, ..., \beta_n|i_1, ..., i_n)$, by its definition, then there exists $(a_1, ..., a_{m-1}) \in \mathbb{C}^{m-1}$ close to 0, such that

$$
\begin{vmatrix}
\frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial\bar z^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial\bar z^{\beta^1}} & \frac{\partial^{|\beta^1|}\psi_{i}}{\partial\bar z^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial\bar z^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial\bar z^{\beta^n}} & \frac{\partial^{|\beta^n|}\psi_{i}}{\partial\bar z^{\beta^n}}
\end{vmatrix}(a_1, ..., a_{m-1}, z_m) = cz_m^s + o(|z_m|^s),\ c \neq 0. \tag{30}
$$

Then there exists $r > 0$ small enough such that for any $0 < |z_m| < r, (a_1, ..., a_{m-1}, z_m) \in \tilde U$. That is, at any of such points, equation (30) is not zero.

We now substitute $(a_1, ..., a_{m-1}, z_m), 0 < |z_m| < r$, into the equation (25), and compare the vanishing order as $z_m \to 0$:

$$
c_1 z_m^{s'} + o(|z_m|^{s'}) = g_i^j(z_m)(cz_m^s + o(|z_m|^s)),\ c \neq 0, \tag{31}
$$

for some $s' \geq 0$. Note that $0 \leq s \leq s'$ by the definition of s and the choice of $(\beta_1, ..., \beta_n | i_1, ..., i_n)$. The holomorphic extendability across 0 of $g_i^j(z_m)$ then follows easily. $\square$

We next make the following observation:

Claim 3.7. *For each fixed $j \in \{i_{n+1}, ..., i_N\}$ and any $i'_1 < ... < i'_{n-1}$ with $\{i'_1, ..., i'_{n-1}\} \subset \{i_1, ..., i_n\}$, we have:*

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \dots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}(\psi_j - \sum_{k=1}^n g_{i_k}^j \psi_{i_k})}{\partial \bar{z}^{\beta^1}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \dots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}(\psi_j - \sum_{k=1}^n g_{i_k}^j \psi_{i_k})}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z) \equiv 0, \forall z \in \tilde{U}. \tag{32}$$

Proof of Claim 3.7: Note that for each $i'_l, 1 \leq l \leq n-1$, the following trivially holds:

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \dots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}(g_{i'_l}^j \psi_{i'_l})}{\partial \bar{z}^{\beta^1}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \dots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}(g_{i'_l}^j \psi_{i'_l})}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z) \equiv 0, \tag{33}$$

for the last column in the matrix is a multiple of one of the first $(n-1)$ columns. Then (32) is an immediate consequence of (26) and (33). $\square$

Lemma 3.8. *For each fixed $j \in \{i_{n+1}, ..., i_N\}$, we have $\psi_j(z) - \sum_{k=1}^n g_{i_k}^j(z_m)\psi_{i_k}(z) \equiv 0$ for any $z \in \tilde{U}$, and thus it holds also for all $z \in U$.*

Proof of Lemma 3.8: This can be concluded easily from the following Lemma 3.9 and Claim 3.7. Here one needs to use the fact that $\beta^1 = (0, ..., 0)$. $\square$

Lemma 3.9. *([2], Lemma 4.7) Let $\mathbf{b}_1, \cdots, \mathbf{b}_n$ and $\mathbf{a}$ be n-dimensional column vectors with elements in $\mathbb{C}$, and let $B = (\mathbf{b}_1, \cdots, \mathbf{b}_n)$ denote the $n \times n$ matrix. Assume that* $\det B \neq 0$ *and* $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \cdots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0$ *for any $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n$. Then* $\mathbf{a} = 0$.

Theorem 3.4 now follows easily from Lemma 3.8. $\square$

If we further assume that $\psi_i(z), m+1 \leq i \leq N$, vanishes at least to the second order, then we have the following, which plays a crucial role in our proof of Theorem 1.1.

Theorem 3.10. *Let $\psi = (\psi_1, ..., \psi_m, \psi_{m+1}, ..., \psi_N)$ be a holomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into $\mathbb{C}^N$ with $\psi(0) = 0$. Assume that $(\psi_1, ..., \psi_m)$ is a biholomorphic* *map from a neighborhood of* $0 \in \mathbb{C}^m$ *into a neighborhood of* $0 \in \mathbb{C}^N$. *Assume that* $\psi_j(z) = O(|z|^2)$ *for* $m+1 \le j \le N$. *Suppose that* $\mathrm{rank}_{N-m+1}(\psi) < N$. *Then there exist* $a_{m+1}, ..., a_N \in \mathbb{C}$ *that are not all zero such that*

$$\sum_{i=m+1}^{N} a_j \psi_j(z_1, ..., z_{m-1}, 0) \equiv 0, \tag{34}$$

for all $(z_1, ..., z_{m-1})$ *near* 0.

Proof of Theorem 3.10: We first have the following:

Claim 3.11. *For each* $1 \le i \le m$, $g_i(0) = 0$.

Proof of Claim 3.11: Suppose not. Write $\mathbf{c} := (g_1(0), ..., g_m(0)) \neq 0$. Then $(g_1(z_m), ..., g_m(z_m)) = \mathbf{c} + O(|z_m|)$. The fact that $\psi_i(z) = O(|z|^2), i \ge m+1$, implies

$$\sum_{i=1}^{m} g_i(z_m)\psi_i(z) = O(|z|^2). \tag{35}$$

Notice that (the Jacobian of) $(\psi_1, ..., \psi_m)$ is of full rank at 0. Hence

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial z_1}(0) & \cdots & \frac{\partial \psi_m}{\partial z_1}(0) \\ \cdots & \cdots & \cdots \\ \frac{\partial \psi_1}{\partial z_m}(0) & \cdots & \frac{\partial \psi_m}{\partial z_m}(0) \end{pmatrix} \mathbf{c}^t \neq 0. \tag{36}$$

This is a contradiction to (35). □

Finally, letting $z_m = 0$ in equation (20), we obtain (34). By Claim 3.11, $(g_{m+1}(0), ..., g_N(0)) \neq 0$. This establishes Theorem 3.10. □

4. Proof of the main theorem assuming three extra propositions

In this section, we give a proof of our main theorem under several extra assumptions (i.e., Propositions (I)–(III)), which will be verified one by one in the later sections.

Let $M \subset \mathbb{CP}^N$ be an irreducible Hermitian symmetric space of compact type, which has been canonically (and isometrically) embedded in the complex projective space through the way described in §2. In this section, we write n as the complex dimension of M. We also have on M an affine open piece $\mathcal{A}$ that is biholomorphically equivalent to the complex Euclidean space of the same dimension, such that $M \setminus \mathcal{A}$ is a codimension one complex subvariety of M. We identify the coordinates of $\mathcal{A}$ by the parametrization map with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ through what is described in §2, which we wrote as $[1, \psi_1, ..., \psi_N]$, where $\psi_1, ..., \psi_N$ are polynomial maps in $(z_1, ..., z_n)$ with $\psi_j = \kappa_j z_j$, where

$\kappa_j = 1$ or $\sqrt{2}$, for $j = 1, \cdots, n$. We also write $\overline{F}(\xi)$ for $\overline{F(\overline{\xi})}$ for $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$. We still use $\rho(z, \xi)$ for the defining function of the Segre family of M restricted to $\mathcal{A} \times \mathcal{A}^*$, which will be canonically identified with $\mathbb{C}^n \times \mathbb{C}^n$. Since the coefficients of $\psi_1, ..., \psi_N$ are all real, $\overline{\psi} = \psi$ and $\mathcal{A}^* = \mathcal{A}$. Hence, we have

$$\rho(z, \xi) = 1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi). \tag{37}$$

Recall the standard metric ω of M on $\mathcal{A}$ is given by

$$\omega = i\partial\overline{\partial}\log(\rho(z, \bar{z})). \tag{38}$$

The volume form $d\mu = c_n\omega^n$ associated to ω, by §2, is now given in $\mathcal{A}$ by the multiplication of V with the standard Euclidean volume form, where

$$V = \frac{c}{(\rho(z, \bar{z}))^{\lambda}} \tag{39}$$

with $c > 0$ and λ a certain positive integer depending on M. For instance, $\lambda = p + q$ when $X = G(p, q)$ [15]. Here c_n is a certain positive constant depending only on n.

Theorem 4.1. *Let $\mathcal{A} \subset M$ be as above equipped with the standard metric ω. Let $F_j, j = 1, ..., m$, be a holomorphic map from $U \subset \mathcal{A}$ into M, where U is a connected open neighborhood of $\mathcal{A}$. Assume that $F_j^*(d\mu) \not\equiv 0$ for each j and assume that*

$$d\mu = \sum_{j=1}^{m} \lambda_j F_j^*(d\mu), \tag{40}$$

for certain positive constants $\lambda_j > 0$ with $j = 1, \cdots, m$. Then for any $j \in \{1, 2, ..., m\}$, F_j extends to a holomorphic isometry of (M, ω).

For convenience of our discussions, we first fix some notations: In what follows, we identify $\mathcal{A}$ with $\mathbb{C}^n$ having $z = (z_1, \cdots, z_n)$ as its coordinates. On $U \subset \mathcal{A} \subset M$ and after shrinking U if needed, we write the holomorphic map F_j, for $j = 1, ..., m$, from $U \to \mathcal{A} = \mathbb{C}^n$, as follows:

$$F_j = (F_{j,1}, F_{j,2}, ..., F_{j,n}), \; j = 1, ..., m. \tag{41}$$

Still write the holomorphic embedding from $\mathcal{A}$ into $\mathbb{CP}^N$ as $[1, \psi_1, \cdots, \psi_N]$. We define $\mathcal{F}_j(z) = (\mathcal{F}_{j,1}, ..., \mathcal{F}_{j,N}) = (\psi_1(F_j), \psi_2(F_j), ..., \psi_N(F_j))$ for $j = 1, ..., m$. Finally, all Segre varieties and Segre families are restricted to $\mathcal{A} = \mathbb{C}^n$.

The main purpose of this section is to give a proof of Theorem 4.1, assuming the following three propositions hold. These propositions will be separately established in terms of the type of M in §5, §6 and §7. This then completes the proof of our main theorem.

Proposition (I). *Write* $\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z,\xi)}{\frac{\partial \rho}{\partial z_n}(z,\xi)} \frac{\partial}{\partial z_n}, 1 \leq i \leq n-1$, *which are holomorphic vector fields (whenever defined) tangent to the Segre family* $\mathcal{M}$ *of* $M \hookrightarrow \mathbb{CP}^N$ *restricted to* $\mathcal{A} \times \mathcal{A}^* = \mathbb{C}^n \times \mathbb{C}^n$ *defined by* $\rho(z,\xi) = 0$. *Under the notations we set up above, for any local biholomorphic map* $F = (f_1, \cdots, f_n) : U \to \mathbb{C}^n$ *with* $F(0) = 0$, *there are* $z^0 \in U, \xi^0 \in Q_{z^0}, \beta^1, ..., \beta^N$, *such that*

$$\frac{\partial \rho}{\partial z_n}(z^0,\xi^0) \neq 0, \quad \Lambda(\beta^1,...,\beta^N)(z^0,\xi^0) := \begin{vmatrix} \mathcal{L}^{\beta^1}\mathcal{F}_1 & ... & \mathcal{L}^{\beta^1}\mathcal{F}_N \\ ... & ... & ... \\ \mathcal{L}^{\beta^N}\mathcal{F}_1 & ... & \mathcal{L}^{\beta^N}\mathcal{F}_N \end{vmatrix} (z^0,\xi^0) \neq 0. \tag{42}$$

Here $\beta^l = (k_1^l, ..., k_{n-1}^l)$, $k_1^l, ..., k_{n-1}^l$ *are non-negative integers, for* $l = 1, 2, ..., N$; $\beta^1 = (0, 0, ..., 0)$; $\mathcal{L}^{\beta^l} = \mathcal{L}_1^{k_1^l}\mathcal{L}_2^{k_2^l}\mathcal{L}_3^{k_3^l}...\mathcal{L}_{n-1}^{k_{n-1}^l}$; $\mathcal{F}(z) = (\mathcal{F}_1, ..., \mathcal{F}_N) = (\psi_1(F), \psi_2(F), ..., \psi_N(F))$. *Moreover,* $s_l := \sum_{i=1}^{n-1} k_i^l$ $(l = 1, ..., N)$ *is a non-negative integer bounded from above by a universal constant depending only on* (M, ω). *Also, in what follows, when we like to emphasize the dependence of* $\Lambda(\beta^1, ..., \beta^N)$ *on* F, *we also write it as* $\Lambda_F(\beta^1, ..., \beta^N)$.

Proposition (II). *Suppose that* $\xi^0 \in \mathbb{C}^n$ *with* $\xi^0 \neq (0, 0, ..., 0)$. *Then for a generic smooth point* z^0 *on the Segre variety* Q_{ξ^0} *and a small neighborhood* $U \ni z^0$, *there is a* $z^1 \in U \cap Q_{\xi^0}$ *such that* Q_{z^0} *and* Q_{z^1} *both are smooth at* ξ^0 *and intersect transversally at* ξ^0, *too. Moreover, we can find a biholomorphic parametrization near* ξ^0: $(\xi_1, \xi_2, ..., \xi_n) = \mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n)$ *with* $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$, *where* U_1 *and* U_2 *are small neighborhoods of* $1 \in \mathbb{C}$, *and* U_j *for* $j \geq 3$ *are small neighborhoods of* $0 \in \mathbb{C}$ *such that (i).* $\mathcal{G}(1, 1, 0, \cdots, 0) = \xi_0$, *(ii).* $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}, \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$, *and (iii).* $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times ... \times U_n)$ *or* $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times ... \times U_n), s \in U_1, t \in U_2$ *is an open piece of a certain Segre variety for each fixed* t *and* s. *Moreover* $\mathcal{G}$ *consists of algebraic functions with total degree bounded by a constant depending only on the manifold* M.

Proposition (III). *For any* $\xi \neq 0 (z \neq 0,$ respectively$) \in \mathbb{C}^n$, $\rho(z,\xi)$ *is an irreducible polynomial in* z *(and in* ξ, *respectively). (In particular,* Q_ξ^* *and* Q_z *are irreducible.) Moreover, if* U *is a connected open set in* $\mathbb{C}^n$, *then the Segre family* $\mathcal{M}$ *restricted to* $U \times \mathbb{C}^n$ *is an irreducible complex subvariety and thus its regular points form a connected complex submanifold. In particular,* $\mathcal{M}$ *is an irreducible complex subvariety of* $\mathbb{C}^n \times \mathbb{C}^n$.

The rest of this section is splitted into several subsections. In the first subsection, we discuss a partial algebraicity for a certain component F_{j_0} in Theorem 4.1. In §4.2, we show F_{j_0} is algebraic. In §4.3, we further prove the rationality of F_{j_0}. §4.4 is devoted to proving that F_{j_0} extends to a birational map from M to itself and extends to a holomorphic isometry, which can be used, through an induction argument, to prove Theorem 4.1 assuming Propositions (I)–(III).

4.1. An algebraicity lemma

We use the notations we have set up so far. We now proceed to the proof Theorem 4.1 under the hypothesis that Propositions (I)–(III) hold.

Denote by $J_f(z)$ the determinant of the complex Jacobian matrix of a holomorphic map $f : B \to \mathbb{C}^n$, where $B \subset \mathbb{C}^n$ is an open subset and $z = (z_1, \cdots, z_n) \in B$. For any holomorphic map $g(\xi)$ from an open subset of $\mathbb{C}^n$ to $\mathbb{C}^m$, where $\xi \in \mathbb{C}^n$, we define $\overline{g}(\xi) := \overline{g(\overline{\xi})}$.

Now from (37)(38)(39)(40), we obtain

$$\sum_{j=1}^{m} \lambda_j \frac{|J_{F_j}(z)|^2}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\overline{F}_j(\bar{z})))^\lambda} = \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\bar{z}))^\lambda}, \quad z = (z_1, ..., z_n) \in U. \tag{43}$$

Recall that $F_j = (F_{j,1}, F_{j,2}, ..., F_{j,n}), j = 1, ..., n$. Complexifying (43), we have

$$\sum_{j=1}^{m} \lambda_j \frac{J_{F_j}(z)\overline{J_{F_j}}(\xi)}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\overline{F}_j(\xi)))^\lambda} = \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda}, \quad (z, \xi) \in U \times \mathrm{conj}(U). \tag{44}$$

Here $\mathrm{conj}(U) =: \{z : \overline{z} \in U\}$. Using the transitive action of the holomorphic isometric group of (M, ω) on M, we assume that $0 \in U$, $F_j(0) = 0 \in \mathcal{A}$ and $J_{F_j}(0) \neq 0$ for each j. Also, letting $U = B_r(0)$ for a sufficiently small $r > 0$, we have $\mathrm{conj}(U) = U$. Hence, we will assume that (44) holds for $(z, \xi) \in U \times U$.

We will need the following algebraicity lemma.

Lemma 4.2. *Let F_j's be as in Theorem 4.1. Then there exist Nash algebraic maps*

$$\widehat{F}_1(z, X_1, ..., X_m), ..., \widehat{F}_m(z, X_1, ..., X_m)$$

holomorphic in $(z, X_1, ..., X_m)$ near $(0, \overline{J_{F_1}}(0), ..., \overline{J_{F_m}}(0)) \in \mathbb{C}^n \times \mathbb{C}^m$ such that

$$\overline{F}_j(z) = \widehat{F}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_m}}(z)), j = 1, ..., m \tag{45}$$

for $z = (z_1, ..., z_n)$ near 0.

Proof of Lemma 4.2: Recall that $\psi_i = \kappa_i z_i$, where $\kappa_i = 1$ or $\sqrt{2}$, for $i = 1, \cdots, n$ and $\psi_i = O(|z|^2)$ is a polynomial of z for each $n + 1 \leq i \leq N$. We obtain from (44) the following:

$$\sum_{j=1}^{m}\lambda_j\big(J_{F_j}(z)\overline{J_{F_j}}(\xi)-\lambda(\sum_{i=1}^{n}(J_{F_j}(z)\kappa_i F_{j,i}(z))(\overline{J_{F_j}}(\xi)\kappa_i\overline{F}_{j,i}(\xi)))+P_j(z,\overline{F_j}(\xi),\overline{J_{F_j}}(\xi))\big)$$
$$=\frac{1}{(1+\sum_{i=1}^{N}\psi_i(z)\psi_i(\xi))^{\lambda}}. \tag{46}$$

Here each $P_j(z,\overline{F_j}(\xi),\overline{J_{F_j}}(\xi))$ is a rational function in $z,\overline{F_j}(\xi)$ and $\overline{J_{F_j}}(\xi)$.

We now set $X_j=J_{F_j}, 1\leq j\leq m$. Set $Y_j, 1\leq j\leq m$, to be the vectors:

$$Y_j=(Y_{j1},...,Y_{jn}):=(\kappa_1 J_{F_j}F_{j,1},...,\kappa_n J_{F_j}F_{j,n}).$$

Then equation (46) can be rewritten as

$$\sum_{j=1}^{m}\lambda_j\left(X_j(z)\overline{X}_j(\xi)-\lambda Y_j(z)\cdot\overline{Y}_j(\xi)+Q_j(z,\overline{X}_j(\xi),\overline{Y}_j(\xi))\right)=\frac{1}{(1+\sum_{i=1}^{N}\psi_i(z)\psi_i(\xi))^{\lambda}} \tag{47}$$

over $U\times U$. Here each Q_j with $1\leq j\leq m$ is rational in $\overline{X}_j,\overline{Y}_j$. Moreover, each $Q_j, 1\leq j\leq m$, has no terms of the form $\overline{X}_j^k\overline{Y}_{js}^l$ with $l\leq 1$ for any $s\geq 1$ in its Taylor expansion at $(\overline{X_j}(0),\overline{Y_j}(0))$.

We write $D^\alpha=\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}...\partial z_n^{\alpha_n}}$ for an $n-$multiindex $\alpha=(\alpha_1,...,\alpha_n)$. Taking differentiation in (47), we obtain, for each multiindex α, the following:

$$\sum_{j=1}^{m}\big((D^\alpha X_j(z))\overline{X}_j(\xi)-\lambda(D^\alpha Y_j(z))\cdot\overline{Y}_j(\xi)+D^\alpha Q_j(z,\overline{X}_j(\xi),\overline{Y}_j(\xi))\big)$$
$$=D^\alpha\Big(\frac{1}{(1+\sum_{i=1}^{N}\psi_i(z)\psi_i(\xi))^{\lambda}}\Big).$$

Again each $D^\alpha Q_j, 1\leq j\leq m$, is rational in $(\overline{X}_j,\overline{Y}_j)$ and has no terms of the form $\overline{X}_j^k\overline{Y}_{js}^l$ with $l\leq 1$ and $s\geq 1$ in its Taylor expansion at $(\overline{X_j}(0),\overline{Y_j}(0))$. Applying a similar argument as in [Proposition 3.1, [20]], we can algebraically solve for $\overline{F_j}$ to complete the proof of the lemma. $\square$

Let $\mathcal{R}$ be the field of rational functions in $z=(z_1,...,z_n)$. Consider the field extension

$$\mathcal{E}=\mathcal{R}(\overline{J_{F_1}}(z),...,\overline{J_{F_m}}(z)).$$

Let K be the transcendental degree of the field extension $\mathcal{E}/\mathcal{R}$. If $K=0$, then each of $\{\overline{J_{F_1}},...,\overline{J_{F_m}}\}$ is Nash algebraic. As a consequence of Lemma 4.2, each F_j with $1\leq j\leq m$ is Nash algebraic. Otherwise, by re-ordering the indices if necessary, we let $\mathcal{G}=\{\overline{J_{F_1}},...,\overline{J_{F_K}}\}$ be the maximal algebraic independent subset of $\{\overline{J_{F_1}},...,\overline{J_{F_m}}\}$. It follows that the transcendental degree of $\mathcal{E}/\mathcal{R}(\mathcal{G})$ is zero. For any $l>K$, there exists a minimal polynomial $P_l(z, X_1, ..., X_K, X)$ such that $P_l(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z), \overline{J_{F_l}}(z)) \equiv 0$. Moreover,

$$\frac{\partial P_l(z, X_1, ..., X_K, X)}{\partial X}(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z), \overline{J_{F_l}}(z)) \not\equiv 0$$

in a small neighborhood V of 0, for otherwise, P_l cannot be a minimal polynomial of $\overline{J_{F_l}}(z)$. Now the union of the vanishing set of the partial derivative with respect to X in the above equation for each l forms a proper local complex analytic variety near 0. Applying the algebraic version of the implicit function theorem, there exists a small connected open subset $U_0 \subset U$, with $0 \in \overline{U}_0$ and a holomorphic algebraic function $\widehat{h}_l, l > K$, in a certain neighborhood $\widehat{U}_0$ of $\{(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)) : z \in U_0\}$ in $\mathbb{C}^n \times \mathbb{C}^K$, such that

$$\overline{J_{F_l}}(z) = \widehat{h}_l(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)),$$

for any $z \in U_0$. (We can assume here U_0 is the projection of $\widehat{U}_0$.) Substitute this into

$$\widehat{F}_i(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_m}}(z)),$$

and still denote it, for simplicity of notation, by $\widehat{F_j}(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z))$ with

$$\widehat{F_j}(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)) = \widehat{F}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_m}}(z)) \text{ for } z \in U_0.$$

In the following, for simplicity of notation, we also write for $j \leq K$,

$$\widehat{h}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)) = \overline{J_{F_j}}(z) \text{ or } \widehat{h}_j(z, X_1, ..., X_K) = X_j.$$

Now we replace $\overline{F_j}(\xi)$ by $\widehat{F_j}(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi))$, and replace $\overline{J_{F_j}}(\xi)$ by $\widehat{h}_j(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi))$, for $1 \leq j \leq m$, in (44). Furthermore, we write $X = (X_1, ..., X_K)$, and replace $\overline{J_{F_j}}(\xi)$ by X_j for $1 \leq j \leq K$ in

$$\widehat{F_j}(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi)), \widehat{h}_j(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi)), 1 \leq j \leq m.$$

We define a new function Φ as follows:

$$\Phi(z, \xi, X) := \sum_{j=1}^{m} \lambda_j \frac{J_{F_j}(z)\widehat{h}_j(\xi, X)}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\widehat{F}_j(\xi, X)))^\lambda} - \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda}. \quad (48)$$

Lemma 4.3. *Shrinking U if necessary, we have $\Phi(z, \xi, X) \equiv 0$, i.e.,*

$$\sum_{j=1}^{m} \lambda_j \frac{J_{F_j}(z)\widehat{h}_j(\xi, X)}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\widehat{F}_j(\xi, X)))^\lambda} = \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda}, \quad (49)$$

or,

$$(1+\sum_{i=1}^{N}\psi_i(z)\psi_i(\xi))^{\lambda}\sum_{j=1}^{m}\left(\lambda_j J_{F_j}(z)\widehat{h}_j(\xi,X)\prod_{1\le k\le m,k\ne j}(1+\sum_{i=1}^{N}\psi_i(F_k(z))\psi_i(\widehat{F}_k(\xi,X)))^{\lambda}\right)$$
$$=\prod_{1\le j\le m}(1+\sum_{i=1}^{N}\psi_i(F_j(z))\psi_i(\widehat{F}_j(\xi,X)))^{\lambda} \tag{50}$$

for $z\in U$ *and* $(\xi,X)\in\widehat{U}_0$.

Proof of Lemma 4.3: Suppose not. Notice Φ is Nash algebraic in (ξ,X) for each fixed $z\in U$, by Lemma 4.2. For a generic fixed $z=z_0$ near 0, since $\Phi(z,\xi,X)\not\equiv 0$, there exist polynomials $A_l(\xi,X)$ for $0\le l\le N$ with $A_0(\xi,X)\not\equiv 0$ such that

$$\sum_{0\le l\le N}A_l(\xi,X)\Phi^l(z,\xi,X)\equiv 0.$$

As $\Phi(z_0,\xi,\overline{J_{F_1}}(\xi),...,\overline{J_{F_K}}(\xi))\equiv 0$ for $\xi\in U_0$, then it follows that $A_0(\xi,\overline{J_{F_1}}(\xi),...,\overline{J_{F_K}}(\xi))$ $\equiv 0$ for $\xi\in U_0$. This is a contradiction to the assumption that $\{\overline{J_{F_1}}(\xi),...,\overline{J_{F_K}}(\xi)\}$ is an algebraic independent set. $\square$

Now that $\widehat{F}_j(\xi,X), 1\le j\le m$, is algebraic in its variables, if $\widehat{F}_j, 1\le j\le m$, is independent of X, then F_j is algebraic by Lemma 4.2. This fact motivates the remaining work in this section.

4.2. Algebraicity and rationality with uniformly bounded degree

In this subsection, we prove the algebraicity and rationality for at least one of the $F_j's$. We start with the following:

Lemma 4.4. *Let* $F_j(z), j\in\{1,...,m\}$, *be a local holomorphic map defined on a neighborhood of* $0\in U$ *as in* (44). *Suppose that there exist* $z^0\in U$ *and* $\xi^0\in Q_{z^0}$ *such that* $\Lambda(\beta^1,...,\beta^N)(z^0,\xi^0)$ *is well defined and non-zero with* $\beta^1=(0,0,...,0)$. *Then there is an analytic variety* $W\subsetneq U$ *such that when* $z\in U\backslash W$, $\Lambda(\beta^1,...,\beta^N)(z,\xi)$ *is a rational function in* ξ *over* Q_z *and* $\Lambda(\beta^1,...,\beta^N)(z,\xi)\not\equiv 0$ *on* Q_z.

Proof of Lemma 4.4: By the assumption, $\frac{\partial\rho}{\partial z_n}(z_0,\xi_0)\ne 0$ and

$$\Lambda(\beta^1,...,\beta^N)(z^0,\xi^0)=\begin{vmatrix}\mathcal{L}^{\beta^1}\mathcal{F}_{j,1} & ... & \mathcal{L}^{\beta^1}\mathcal{F}_{j,N}\\ ... & ... & ...\\ \mathcal{L}^{\beta^N}\mathcal{F}_{j,1} & ... & \mathcal{L}^{\beta^N}\mathcal{F}_{j,N}\end{vmatrix}(z^0,\xi^0) \tag{51}$$

is non-zero with $\beta^1=(0,0,...,0)$.

By the definition, $\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z,\xi)}{\frac{\partial \rho}{\partial z_n}(z,\xi)} \frac{\partial}{\partial z_n}$ and $\mathcal{L}^{\beta^l} = \mathcal{L}_1^{k_1^l} \mathcal{L}_2^{k_2^l} \mathcal{L}_3^{k_3^l} ... \mathcal{L}_{n-1}^{k_{n-1}^l}$ for $\beta^l = (k_1^l, ..., k_{n-1}^l)$, $k_1^l, ..., k_{n-1}^l$. Hence $\Lambda(\beta^1, ..., \beta^N)(z,\xi)$ can be written in the form $\Lambda(\beta^1, ..., \beta^N)(z,\xi) = \frac{\mathcal{G}_1(z,\xi)}{\mathcal{G}_2(z,\xi)}$. Here $\mathcal{G}_1(z,\xi) = \sum_{|I|=0}^{M_1} \Phi_I(z)\xi^I$, $\mathcal{G}_2(z,\xi) = \sum_{|J|=0}^{M_2} \Psi_J(z)\xi^J$, with Φ_I and Ψ_J being holomorphic functions defined over $U \subset \mathbb{C}^n$. In fact, $\mathcal{G}_2(z,\xi)$ is simply taken as a certain sufficiently large power of $\rho_{z_n} := \frac{\partial \rho}{\partial z_n}$.

By our assumption, we have $\mathcal{G}_1, \mathcal{G}_2$ not equal to zero at (z^0, ξ^0). Hence, $\mathcal{G}_1, \mathcal{G}_2$ are not zero elements in $\mathcal{O}(U)[\xi_1, ..., \xi_n]$, the polynomial ring of ξ with coefficients from the holomorphic function space over U.

By Proposition (III), the defining function of the Segre family ρ can be written in the form $\rho(z,\xi) = \sum_{|\alpha|=0}^{M_3} \Theta_k(z)\xi^\alpha$, which is an irreducible polynomial in (z,ξ). And for each fixed z, by Proposition (III), we also have $\rho(z,\xi)$ irreducible as a polynomial of ξ only.

Then the set of $z \in U$ where $\Lambda(\beta^1, ..., \beta^N)(z,\xi)$ is undefined over Q_z is a subset of $z \in U$ where $\mathcal{G}_2(z,\xi)$, as a polynomial of ξ, contains the factor $\rho(z,\xi)$ as a polynomial in ξ. We denote the latter set by W_2. Similarly, the set of $z \in U$ with $\Lambda(\beta^1, ..., \beta^N)(z,\xi) \equiv 0$ over Q_z is a subset of $z \in U$ where $\mathcal{G}_1(z,\xi)$, as a polynomial of ξ, contains a factor $\rho(z,\xi)$, which we denote by W_1.

Notice that $\rho(z,\xi) \in \mathcal{O}(U)[\xi_1, ..., \xi_n]$ depends on each ξ_j for $1 \le j \le n$. Also notice that $\mathcal{G}_2(z,\xi)$, as a certain power of $\rho_{z_n}(z,\xi)$, depends on ξ_n.

We next characterize W_2 by the resultant R_2 of $\mathcal{G}_2(z,\xi)$ and $\rho(z,\xi)$ as polynomials in ξ_n. We rewrite $\mathcal{G}_2$ and ρ as polynomials of ξ_n as follows:

$$\mathcal{G}_2 = \sum_{i=0}^{k} a_i(z, \xi_1, ..., \xi_{n-1})\xi_n^i, \;\; \rho = \sum_{j=0}^{l} b_j(z, \xi_1, ..., \xi_{n-1})\xi_n^j.$$

Here the leading terms $a_k, b_l \not\equiv 0$ with $k, l \ge 1$. We write the resultant as $R_2(z, \xi_1, ..., \xi_{n-1}) = \sum_I c_I(z){\xi'}^I$, where $c_I's$ are holomorphic functions of $z \in U$.

For those points $z \in W_2$, $R_2(z, \cdot) \equiv 0$ as a polynomial of $\xi_1, ..., \xi_{n-1}$. Then W_2 is contained in the complex analytic set $\widetilde{W_2} := \{c_I = 0, \forall I\}$. If $\widetilde{W_2} = U$, then we can find non-zero polynomials $f, g \in \mathcal{O}(U)[\xi_1, ..., \xi_{n-1}][\xi_n]$ such that $f\rho + g\mathcal{G}_2 \equiv 0$, where the degree of g in ξ_n is less than the degree of ρ in ξ_n. Hence $\{\mathcal{G}_2 = 0\} \cup \{g = 0\} \supset \{\rho = 0\} \cap (U \times \mathbb{C}^n)$. Again by the irreducibility of $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$, since $\{g = 0\}$ is a thin set in $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$, $\mathcal{G}_2$ vanishes on $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$. This contradicts $\mathcal{G}_2(z^0, \xi^0) \neq 0$. Hence $W_2 \subset \widetilde{W_2}$ and $\widetilde{W_2}$ is a proper complex analytic subset of U.

By a similar argument, we can prove that W_1 is contained in $\widetilde{W_1}$ that is also a proper analytic set of U. Let $W = \widetilde{W_1} \cup \widetilde{W_2}$. Then when $z \in U \backslash W$, $\Lambda(\beta^1, ..., \beta^N)(z,\xi)$ is well-defined over Q_z as a rational function in ξ and $\Lambda(\beta^1, ..., \beta^N)(z,\xi) \not\equiv 0$ on Q_z. □

Lemma 4.5. *Let $\psi(\xi, X)$ be a non-zero Nash-algebraic function in $(\xi, X) = (\xi_1, ..., \xi_n, X_1, ..., X_m) \in \mathbb{C}^n \times \mathbb{C}^m$. Write E for a proper complex analytic variety of $\mathbb{C}^n \times \mathbb{C}^m$ that contains the branch locus of ψ and the zeros of the leading coefficient in the minimal polynomial of ψ. Then there exists a proper analytic set W_1 in $\mathbb{C}^n$ such that*

$$\{\xi|\ \exists X^0, (\xi, X^0) \notin E\} \supset \mathbb{C}^n \backslash W_1.$$

Proof of Lemma 4.5: Since ψ is algebraic, there is an irreducible polynomial $\Phi(\xi, X; Y) = \sum_{i=0}^{k} \phi_i(\xi, X)Y^i$ such that $\Phi(\xi, X, \psi(\xi, X)) \equiv 0$. If $k = 1$ then ψ is a rational function and thus E is just the poles and points of indeterminacy. The proof is then obvious and we hence assume $k \geq 2$.

Define $\Psi(\xi, X, Y) = \frac{\partial \Phi}{\partial Y}$. Since $k \geq 2$, the degree of Ψ in Y is at least one. Consider Φ, Ψ as polynomials in Y, and write $R(\xi, X)$ for their resultant. Then the branch locus is contained in $\{(\xi, X)|R(\xi, X) = 0\}$. Notice that $R \not\equiv 0$, for Φ is irreducible. Write $R = \sum_I r_I(\xi)X^I$ with some $r_I \neq 0$. Write $\phi_k(\xi, X) = \sum \phi_{k,i}(\xi)X^i$ and $W_1 = \{r_I(\xi) = 0\ , \forall I\} \cup \{\phi_{k,i}(\xi) = 0\ , \forall\ i\}$, which is a proper complex analytic set in $\mathbb{C}^n$. Then $\{\xi|\ \exists X^0, (\xi, X^0) \notin E\} \supset \mathbb{C}^n \backslash W_1$. □

Let E be a proper complex analytic variety containing the union of the branch loci of $\widehat{h}_j, \widehat{F}_j$ for $j = 1, \cdots, m$ and the zeros of the leading coefficients in their minimal polynomials. For any point $(z^0, \xi^0, X^0) \in U \times ((\mathbb{C}^n \times \mathbb{C}^K)\backslash E)$, we can find a smooth Jordan curve γ in $U \times ((\mathbb{C}^n \times \mathbb{C}^K)\backslash E)$ connecting (z^0, ξ^0, X^0) with a certain point in $U \times (\widehat{U}_0 \setminus E)$. We can holomorphically continue the following equation along γ:

$$\begin{aligned}(\rho(z,\xi))^\lambda \sum_{j=1}^{m} \left(\lambda_j J_{F_j}(z)\widehat{h}_j(\xi, X) \prod_{1\leq k\leq m, k\neq j} (1 + \sum_{i=1}^{N} \psi_i(F_k(z))\psi_i(\widehat{F}_k(\xi, X)))^\lambda \right) \\ = \prod_{1\leq j\leq m} (1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\widehat{F}_j(\xi, X)))^\lambda, \quad z \in U,\ (\xi, X) \in \widehat{U}_0,\end{aligned} \tag{52}$$

to a neighborhood of (z^0, ξ^0, X^0). For our later discussions, we further define

$$\mathcal{M}_{\text{sing,z}} = \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}, \mathcal{M}_{\text{reg,z}} = \mathcal{M}\backslash\mathcal{M}_{\text{sing,z}};$$

$$\mathcal{M}_{\text{SING}} = \{(z, \xi) : \frac{\partial \rho}{\partial \xi_j} = 0, \forall j\} \cup \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}, \quad \mathcal{M}_{\text{REG}} = \mathcal{M}\backslash\mathcal{M}_{\text{SING}};$$

$\text{Pr}_z : \mathbb{C}^{2n} \to \mathbb{C}^n \quad (z, \xi) \mapsto (z)$ and $\text{Pr}_\xi : \mathbb{C}^{2n} \to \mathbb{C}^n \quad (z, \xi) \mapsto (\xi)$.

Notice that $\mathcal{M}_{\text{REG}}$ is a Zariski open subset of $\mathcal{M}$ and the restrictions of Pr_z,Pr_ξ to $\mathcal{M}_{\text{REG}}$ are open mappings. Also, for $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}}$, Q_{z^0} is smooth at ξ^0, and Q_{ξ^0} is smooth at z^0. By Proposition (III), $\mathcal{M}_{\text{reg,z}} \cap (Q_{\xi^0}, \xi^0)$ is Zariski open in (Q_{ξ^0}, ξ^0).

Lemma 4.6. *With the notations we have set up so far, there exists a point $(z^0, \xi^0, X^0) \in (U \times \mathbb{C}^n \times \mathbb{C}^K)$ with $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}} \cap (U \times \mathbb{C}^n)$ and $(\xi^0, X^0) \notin E$. Moreover, for each $j = 1, ..., m$, we can find $\beta_j^1, ..., \beta_j^N$ with $\beta_j^1 = (0, ..., 0)$ such that $\Lambda_{F_j}(\beta_j^1, ..., \beta_j^N)(z^0, \xi^0) \neq 0$.*

Proof of Lemma 4.6: This is an easy consequence of Propositions (I), (III), Lemma 4.4 and the Zariski openness of $\mathcal{M}_{\text{REG}}$ in $\mathcal{M}$. □

Let (z^0, ξ^0, X^0) be chosen as in Lemma 4.6. We then analytically continue the equation (52) to a neighborhood of the point (z^0, ξ^0, X^0) through a Jordan curve γ described above. We denote one of such neighborhoods by $V_1 \times V_2 \times V_3$, where V_1, V_2 and V_3 are chosen to be a small neighborhood of z^0, ξ^0, and X^0, respectively. It is clear, after shrinking V_1, V_2, V_3 if needed, that there exists a $j_0 \in \{1, ..., m\}$ such that

$$1 + \sum_{i=1}^{N} \psi_i(F_{j_0}(z))\psi_i(\widehat{F_{j_0}}(\xi, X)) = 0, \quad \text{for } (z, \xi) \in \mathcal{M} \cap (V_1 \times V_2), X \in V_3.$$

We next proceed to prove the algebraicity for $F_{j_0}(z)$.

Theorem 4.7. *$\widehat{F_{j_0}}(\xi, X)$, for $\xi \in V_2$, $X \in V_3$, is independent of X and is thus a Nash algebraic function of ξ. Hence F_{j_0} is an algebraic function of z. Moreover, the algebraic total degree of $\widehat{F_{j_0}}(\xi, X) = \overline{F_{j_0}}(\xi)$, and thus of $F_{j_0}(z)$, is uniformly bounded by a constant depending only on the manifold (X, ω) and the described canonical embedding.*

Before proceeding to the proof, we state a slightly modified version of a classical result of Hurwitz. We first give the following definition:

Definition 4.8. Suppose F is an algebraic function defined on $\xi \in \mathbb{C}^n$. The total degree of F is defined to be the total degree of its minimum polynomial. Namely, let $P(\xi; X)$ be an irreducible minimum polynomial of F, the total degree of F is defined as the degree of $P(\xi; X)$ as a polynomial in (ξ, X).

We next state some simple facts about algebraic functions, whose proof is more or less standard (see, for instance, [12]):

Lemma 4.9. *1. Suppose ϕ_1, ϕ_2 are algebraic functions defined in $\xi \in U \subset \mathbb{C}^n$ with total degree bounded by N. Then $\phi_1 \pm \phi_2, \phi_1\phi_2, 1/\phi_1$ (if $\phi_1 \not\equiv 0$) are algebraic functions and their degrees are bounded above by a constant depending only on N, n.*

2. Suppose $\phi_1(z_1, ..., z_n)$ is an algebraic function of total degree bounded by N, and suppose that $\psi_1(\xi_1, ..., \xi_m), ..., \psi_n(\xi_1, ..., \xi_m)$ are algebraic functions with total degree bounded by N as well. Let

$$A_0 = (\xi_1^0, \xi_2^0, ..., \xi_m^0) \in \mathbb{C}^m,$$

where $\psi_1, ..., \psi_n$ are holomorphic functions in a neighborhood of A_0 and let ϕ_1 be a holomorphic function in a neighborhood $U \subset \mathbb{C}^n$ of $(\psi_1(A_0), \psi_2(A_0), ..., \psi_m(A_0))$. Then the composition $\Phi(\xi_1, ..., \xi_m) = \phi_1(\psi_1(\xi_1, ..., \xi_m), \psi_2(\xi_1, ..., \xi_m), \psi_3(\xi_1, ..., \xi_m), ..., \psi_n(\xi_1, ..., \xi_m))$ is an algebraic function with total degree bounded by a constant $C(N, n, m)$ depending only on (N, n, m).

3. Suppose $P_1(z_1, z_2, ..., z_m, \xi_1, \xi_2, ..., \xi_n), ..., P_n(z_1, z_2, ..., z_m, \xi_1, \xi_2, ..., \xi_n)$ *are algebraic functions with total degrees bounded from above by* N *which are holomorphic in a neighborhood* $U \times V \subset \mathbb{C}^m \times \mathbb{C}^n$ *of* $A_0 = (z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_n^0)$*. Suppose that*

$$\begin{cases} P_1(z_1, z_2, ..., z_m, \xi_1, ..., \xi_n) = 0 \\ P_2(z_1, z_2, ..., z_m, \xi_1, ..., \xi_n) = 0 \\ ... \\ P_n(z_1, z_2, ..., z_m, \xi_1, ..., \xi_n) = 0 \end{cases}$$

has a solution at $A_0 = (z^0, \xi^0) = (z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_n^0)$ *and* $\frac{\partial(P_1, P_2, ..., P_n)}{\partial(\xi_1, \xi_2, ..., \xi_n)}(z_1^0, z_2^0, ..., z_m^0, \xi_1^0, ..., \xi_n^0) \neq 0$*. Then we can solve* $\xi_1 = \phi_1(z_1, z_2, ..., z_m)$, $\xi_2 = \phi_2(z_1, z_2, ..., z_m)$,..., $\xi_n = \phi_n(z_1, z_2, ..., z_m)$ *with* $\phi_j(z^0) = \xi^0$ *in a neighborhood of* $z^0 \in \tilde{U} \subset U \subset \mathbb{C}^m$*, where* $\phi_1, ..., \phi_n$ *are algebraic functions with total degree bounded by* $C(N, n, m)$*.*

We now state the following modified version of the classical Hurwitz theorem with a controlled total degree [3].

Theorem 4.10. *Let* $F(s, t, \xi_1, \xi_2, ..., \xi_m)$ *be holomorphic over* $U \times V \times W \subset \mathbb{C}^{m+2}$*. Suppose that for any fixed* $s \in U \subset \mathbb{C}$*,* F *is an algebraic function in* $(t, \xi_1, ..., \xi_m)$ *with its total degree uniformly bounded by N; and for any fixed* $t \in V \subset \mathbb{C}$*,* F *is an algebraic function of* $(s, \xi_1, ..., \xi_m)$ *with its total degree uniformly bounded by* N*. Then* F *is an algebraic function with total degree bounded by a constant depending only on* (m, N)*.*

The proof of Theorem 4.10 is more or less the same as in the classical setting [3]. (See, for example, the Ph. D. thesis of the first author [12].)

Proof of Theorem 4.7: By the choice of (z^0, ξ^0, X^0), there exist $\beta_{j_0}^1, ..., \beta_{j_0}^N$ such that

$$\Lambda_{F_{j_0}}(\beta_{j_0}^1, ..., \beta_{j_0}^N)(z^0, \xi^0) = \begin{vmatrix} \mathcal{L}^{\beta_{j_0}^1}\mathcal{F}_{j_0,1} & ... & \mathcal{L}^{\beta_{j_0}^N}\mathcal{F}_{j_0,N} \\ ... & ... & ... \\ \mathcal{L}^{\beta_{j_0}^N}\mathcal{F}_{j_0,1} & ... & \mathcal{L}^{\beta_{j_0}^N}\mathcal{F}_{j_0,N} \end{vmatrix}(z^0, \xi^0) \neq 0. \tag{53}$$

We can also assume that (z_0, ξ_0) satisfies the assumption in Proposition (II) after a slight perturbation of z_0 inside Q_{ξ_0} if needed. By Proposition (II), we can find $z^1 \in V_1 \cap Q_{\xi^0}$ such that Q_{z^0} intersects Q_{z^1} transversally at ξ^0. Moreover there exists a neighborhood B of ξ^0 and a biholomorphic parametrization of B: $(\xi_1, \xi_2, ..., \xi_n) = \mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n)$ with $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$. Here U_1, U_2 are as in Proposition (II). Moreover, $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}, \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$. Also, for $s \in U_1, t \in U_2$, $\mathcal{G}(\{\tilde{\xi}_0 = t\} \times U_2 \times ... \times U_n), \mathcal{G}(U_1 \times \{\tilde{\xi}_1 = s\} \times U_3 \times ... \times U_n)$ are open pieces of certain Segre varieties. Here $\mathcal{G}$ consists of algebraic functions with total algebraic degree uniformly bounded by M and the canonical embedding. Consider the equation:

$$1+\mathcal{F}_{j_0}(z)\cdot\widehat{\mathcal{F}_{j_0}}(\xi,X)=0,\ (z,\xi,X)\in V_1\times V_2\times V_3,(z,\xi)\in\mathcal{M}. \tag{54}$$

Since the holomorphic vector fields $\{\mathcal{L}_i\}_{i=1}^{n-1}$ are tangent to the Segre family, we have

$$\begin{pmatrix}\mathcal{L}^{\beta^1_{j_0}}\mathcal{F}_{j_0,1}(z,\xi) & \dots & \mathcal{L}^{\beta^1_{j_0}}\mathcal{F}_{j_0,N}(z,\xi)\\ \dots & \dots & \dots\\ \mathcal{L}^{\beta^N_{j_0}}\mathcal{F}_{j_0,1}(z,\xi) & \dots & \mathcal{L}^{\beta^N_{j_0}}\mathcal{F}_{j_0,N}(z,\xi)\end{pmatrix}\begin{pmatrix}\widehat{\mathcal{F}_{j_0,1}}(\xi,X)\\ \dots\\ \widehat{\mathcal{F}_{j_0,N}}(\xi,X)\end{pmatrix}=\begin{pmatrix}-1\\ \cdots\\ 0\end{pmatrix}, \tag{55}$$

where $(z,\xi)(\approx(z^0,\xi^0))\in\mathcal{M},\ X\approx X^0$.

By the Cramer's rule, we conclude that $\{\widehat{\mathcal{F}_{j_0,l}}(\xi,X)\}_{l=1}^N$ are rational functions of ξ with a uniformly bounded degree on an open piece of each Segre variety Q_z for $z\approx z^0$. By the previous modified Hurwitz Theorem (Theorem 4.10), we conclude the algebraicity of $\widehat{\mathcal{F}_{j_0,l}}(\xi,X)$ for $l=1,...,N$. Since in (55) the matrix $\left(\mathcal{L}^{\beta^\mu_{j_0}}\mathcal{F}_{j_0,\nu}(z,\xi)\right)_{1\le\mu,\nu\le N}$ and the right hand side are independent of X, these functions must also be independent of the X-variables. Moreover, by Lemma 4.9 and Theorem 4.10, the total algebraic degree of $\overline{F}_{j_0,l}(\xi)=\widehat{\mathcal{F}_{j_0,l}}(\xi,X)$, for $l=1,...,n$, is uniformly bounded. Since $\overline{F}$ is obtained by holomorphically continuing the conjugation function $\overline{F}$ of F, we conclude the algebraicity of $F_{j_0,l}$ for each $1\le l\le n$. Also the total algebraic degree of each $F_{j_0,l}$ is bounded by a constant depending only on (M,ω). □

Theorem 4.11. *Under the notations we have just set up, F_{j_0} is a rational map, whose degree depends only on the canonical embedding $M\hookrightarrow\mathbb{CP}^N$.*

For the proof Theorem 4.11, we first recall the following Lemma of [22]:

Lemma 4.12. *(Lemma 3.7 in [22]) Let $U\subset\mathbb{C}^n$ be a simply connected open subset and $\mathcal{S}\subset U$ be a closed complex analytic subset of codimension one. Then for $p\in U\setminus\mathcal{S}$, the fundamental group $\pi_1(U\setminus\mathcal{S},p)$ is generated by loops obtained by concatenating (Jordan) paths $\gamma_1,\gamma_2,\gamma_3$, where γ_1 connects p with a point arbitrarily close to a smooth point $q_0\in\mathcal{S}$, γ_2 is a loop around $\mathcal{S}$ near q_0 and γ_3 is γ_1 reversed.*

Proof of Theorem 4.11: We give a proof for the rationality of F_{j_0}. Once this is done, we then conclude that the degree of F_{j_0} is uniformly bounded, for we know the total algebraic degree of F is uniformly bounded by Theorem 4.7.

Suppose that F_{j_0} and thus $\overline{F_{j_0}}$ is not rational. Write $E\subset\mathbb{C}^n$ for a proper complex analytic variety containing the branch locus of $F_{j_0},\overline{F_{j_0}}$ and the zeros of the leading coefficients of the minimal polynomials of their components. We first notice that for $A\neq B\in\mathbb{C}^n$, $Q^*_A\neq Q^*_B$, by Lemma 2.1. Hence, for any proper complex analytic variety $V^1,V^2\subset\mathbb{C}^n$ and any point $(a,b)\in\mathcal{M}$, we can find $(a^1,b^1)\approx(a,b)$ such that $a^1\in Q_{b^1}\setminus V^1$ and $b^1\notin V^2$.

We choose (z^0,ξ^0) as above and assume further that $z^0,\xi^0\notin E$ (after a small perturbation if needed). We choose a sufficiently small neighborhood W of (z^0,ξ^0) in $\mathcal{M}_{\mathrm{REG}}$ such that for each $(z^1, \xi^1) \in W$, we can find, by Lemma 4.12, a loop of the form $\gamma = \gamma_1^{-1} \circ \gamma_2 \circ \gamma_1$ in $\mathbb{C}^n \setminus E$ with $\gamma(0) = \gamma(1) = \xi^1, \gamma_1(1) = q$. Here γ_1 is a simple curve connecting ξ^1 to q with q in a small ball B_p centered at a certain smooth point p of E such that the fundamental group of $B_p \setminus E$ is generated by γ_2; and γ_1^{-1} is the reverse curve of γ_1. Moreover, when $\overline{F_{j_0}}$ is holomorphically continued along γ, we end up with a different branch $\overline{F_{j_0}}^*$ of $\overline{F_{j_0}}$ near ξ^1. We pick p such that there is an $X_p \notin E$ with $(X_p, p) \in \mathcal{M}_{\text{reg,z}}$. (This follows from Proposition (III) and Lemma 2.1 as mentioned above.) Take a certain small neighborhood $\mathcal{W}$ of (X_p, p) in $\mathcal{M}_{\text{reg,z}}$. We assume, without loss of generality, that the piece $\mathcal{W}$ of $\mathcal{M}_{\text{reg,z}}$ is defined by a holomorphic function of the form $z_1 = \phi(z_2, \cdots, z_n, \xi)$. In particular, writing $X_p = (z_1^p, \cdots, z_n^p)$, we have $z_1^p = \phi(z_2^p, \cdots, z_n^p, p)$. Make B_p sufficiently small such that it is compactly contained in the image of the projection of $\mathcal{W}$ into the ξ-space. Write $X_q = (\phi(z_2^p, \cdots, z_n^p, q), z_2^p, \cdots, z_n^p)$ and define the loop $\gamma_2^*(t) = (\phi(z_2^p, \cdots, z_n^p, \gamma_2(t)), z_2^p, \cdots, z_n^p)$. Then γ_2^* is a loop whose base point is at X_q. Also, we have $(\gamma_2^*(t), \gamma_2(t)) \in \mathcal{M}$.

Notice that $X_p \notin E$. After shrinking B_p if needed, we assume that γ_2^* stays sufficiently close to X_p and is homotopically trivial in $\mathbb{C}^n \setminus E$.

Now we slightly thicken γ_1 to get a simply connected domain U_1 of $\mathbb{C}^n \setminus E$. Since $\mathcal{M}$ is irreducible over $\mathbb{C}^n \times U_1$, we can find a smooth simple curve $\widetilde{\gamma}_1 = ({\gamma_1}^*, \widehat{\gamma}_1)$ in $\mathcal{M} \setminus ((E \times \mathbb{C}^n) \cup (\mathbb{C}^n \times E))$ connecting (z^1, ξ^1) to (X_q, q). Then $\widehat{\gamma}_1$ is homotopic to γ_1 relatively to $\{\xi^1, q\}$ and ${\gamma_1}^*(1) = X_q$. Now replace γ by its homotopically equivalent loop $\widehat{\gamma}_1^{-1} \circ \gamma_2 \circ \widehat{\gamma}_1$ and define $\gamma^* = {\gamma^*}_1^{-1} \circ \gamma_2^* \circ {\gamma^*}_1$. Define $\Gamma = (\gamma^*, \gamma)$. Then the image of Γ lies inside $\mathcal{M} \setminus ((E \times \mathbb{C}^n) \cup (\mathbb{C}^n \times E))$. Continuing Equation (54) along Γ and noticing that it is independent of X now, we get both

$$1 + \mathcal{F}_{j_0}(z) \cdot \overline{\mathcal{F}}_{j_0}(\xi) = 0 \text{ and } 1 + \mathcal{F}_{j_0}(z) \cdot \overline{\mathcal{F}}^*_{j_0}(\xi) = 0 \ \forall (z, \xi) \in \mathcal{M} \cap ((V_1 \setminus E) \times (V_2 \setminus E)).$$

Now as before, applying the uniqueness for the solution of the linear system (55) (with an invertible coefficient matrix), we then conclude that $\overline{F_{j_0}}^* \equiv \overline{F_{j_0}}$. This is a contradiction. □

4.3. Isometric extension of F

For simplicity of notation, in the rest of this section, we denote the map F_{j_0} just by F. Now that all components of F are rational functions, it is easy to verify that F gives rise to a rational map $M \dashrightarrow M$. By the Hironaka theorem (see [17] and [27]), we have a (connected) complex manifold Y of the same dimension, holomorphic maps $\tau : Y \to M$, $\sigma : Y \to M$, and a proper complex analytic variety E_1 of M such that $\sigma : \ Y \setminus \sigma^{-1}(E_1) \to M \setminus E_1$ is biholomorphic; $F : M \setminus E_1 \to M$ is well-defined; and for any $p \in Y \setminus \sigma^{-1}(E_1)$, $F(\sigma(p)) = \tau(p)$.

Let E_2 be a proper complex analytic subvariety of M containing E_1, $M \setminus \mathcal{A}$ and let $E_3 \subset Y$ be the proper subvariety where τ fails to be biholomorphic. Write $E^* =$

$\tau(\sigma^{-1}(E_2) \cup E_3) \cup (M \setminus \mathcal{A})$ and $E = \sigma(\tau^{-1}(E^*))$. Then $F : \mathcal{A} \setminus E \to \mathcal{A} \setminus E^*$ is a holomorphic covering map. We first prove

Lemma 4.13. *Under the above notation, $F : \mathcal{A} \setminus E \to \mathcal{A} \setminus E^*$ is a biholomorphic map.*

Proof of Lemma 4.13: We first notice that since F is biholomorphic near 0 with $F(0) = 0$. We can assume that $0 \notin E$. Consider $F^2 = F \circ F$. Then $\overline{F^2} = \overline{F}^2$. Since $(F, \overline{F})$ maps $\mathcal{M}$ into $\mathcal{M}$ whenever it is defined, it is easy to see that $(F, \overline{F}) \circ (F, \overline{F}) = (F^2, \overline{F}^2)$ also maps $\mathcal{M}$ into $\mathcal{M}$ at the points where it is well-defined. Hence, we can repeat a similar argument for F to conclude that F^2, as a rational map, also has its degree bounded by a constant independent of F^2. Similarly, we can conclude that for any positive integer m, F^m is a rational map with degree bounded by a constant independent of m and F. Now, as for F, we can find complex analytic subvarieties $E^{(m)},\ E^{*(m)}$ of $\mathbb{C}^n$ such that F^m is a holomorphic covering map from $\mathcal{A} \setminus E^{(m)} \to \mathcal{A} \setminus E^{*(m)}$. Suppose $F : \mathcal{A} \setminus E \to \mathcal{A} \setminus E^*$ is a k to 1 covering map. It is easy to see that $F^m :\ \mathcal{A} \setminus E^{(m)} \to \mathcal{A} \setminus E^{*(m)}$ is a k^m to 1 covering map. However, since the degree F^m is independent of m, we conclude that $k = 1$ by the following Bezout theorem:

Theorem 4.14. ([42]) *The number of isolated solutions to a system of polynomial equations*

$$f_1(x_1, ..., x_n) = f_2(x_1, ..., x_n) = ... = f_n(x_1, ..., x_n) = 0$$

is bounded by $d_1 d_2 \cdots d_n$, where $d_i = \deg f_i$.

This proves the lemma. □

Now we prove that F extends to a global holomorphic isometry of (M, ω).

Theorem 4.15. *$F : (U, \omega|_U) \to (M, \omega)$ extends to a global holomorphic isometry of (M, ω).*

Proof of Theorem 4.15: By what we just achieved, we then have two proper complex analytic varieties $W_1,\ W_2$ of $\mathbb{C}^n$ such that $F : \mathbb{C}^n \backslash W_1 \to \mathbb{C}^n \backslash W_2$ is biholomorphic. Similarly we have two proper complex analytic subvarieties $W_1^*,\ W_2^*$ of $\mathbb{C}^n$ such that $\overline{F} : \mathbb{C}^n \setminus W_1^* \to \mathbb{C}^n \setminus W_2^*$ is a biholomorphic map. Hence

$$\mathfrak{F} = (F, \overline{F}) : \mathbb{C}^n \setminus W_1 \times \mathbb{C}^n \setminus W_1^* \to \mathbb{C}^n \setminus W_2 \times \mathbb{C}^n \setminus W_2^*$$

is biholomorphic. Let ρ be the defining function of the Segre family as described before. Since ρ is irreducible as a polynomial in (z, ξ), $\mathcal{M}$ is an irreducible complex analytic variety of $\mathcal{A}$. Since $\mathfrak{F}$ maps a certain open piece of $\mathcal{M}$ into an open piece of $\mathcal{M}$, by the uniqueness of holomorphic functions, we see that $\mathfrak{F} = (F, \overline{F})$ also gives a biholomorphic map from $(\mathbb{C}^n \backslash W_1 \times \mathbb{C}^n \backslash W_1^*) \cap \mathcal{M}$ to $(\mathbb{C}^n \backslash W_2 \times \mathbb{C}^n \backslash W_2^*) \cap \mathcal{M}$. Hence $\rho_F = \rho(F(z), \overline{F}(\xi))$ defines the same subvariety as ρ does over $\mathbb{C}^n \setminus W_1 \times \mathbb{C}^n \setminus W_1^*$. Since ρ_F is a rational function in (z, ξ) with denominator coming from the factors of the denominators of $F(z)$ and $\overline{F}(\xi)$, we can write

$$\rho_F(z, \xi) = (\rho(z, \xi))^l \frac{P_1^{i_1}(z, \xi) P_2^{i_2}(z, \xi) \cdots P_\tau^{i_\tau}(z, \xi)}{Q_1^{j_1}(z) \cdots Q_\mu^{j_\mu}(z) R_1^{k_1}(\xi) \cdots R_\nu^{k_\nu}(\xi)} \tag{56}$$

Here the zeros of $Q_j(z)$ and $R_j(\xi)$ stay in W_1 and W_1^*, respectively. All polynomials are irreducible and prime to each other. By what we just mentioned $P_j(z, \xi)$ can not have any zeros in $\mathbb{C}^n \setminus W_1 \times \mathbb{C}^n \setminus W_1^*$, for otherwise it must have ρ as its factor by the irreducibility of ρ. Hence the zeros of $P_j(z, \xi)$ must stay in $(W_1 \times \mathbb{C}^n) \cup (\mathbb{C}^n \times W_1^*)$. From this, it follows easily that $P_j(z, \xi) = P_{j,1}(z)$ or $P_j(z, \xi) = P_{j,2}(\xi)$. Namely, $P_j(z, \xi)$ depends either on z or on ξ. Since $\mathfrak{F}$ is biholomorphic, we see that $l = 1$. Thus replacing ξ by $\bar{z}$ and taking $i\partial\bar{\partial}\log$ to (56), we have $i\partial\bar{\partial}\log \rho_F(z, \bar{z}) = i\partial\bar{\partial}\log \rho(z, \bar{z})$. This shows that $F^*(\omega) = \omega$, or F is a local isometry. Now, by the Calabi Theorem (see [4]), F extends to a global holomorphic isometry of (M, ω). This proves Theorem 4.15. □

We now are ready to give a proof of Theorem 4.1. By what we have obtained, there is a component F_j for F in Theorem 4.1 that extends to a holomorphic isometry to (M, ω). Hence $F_j^*(d\mu) = d\mu$. Notice $\lambda_j < 1$ due to the positivity of all terms in the right hand side of the equation (40). After a cancellation, we reduce the theorem to the case with only $(m-1)$-maps. Then by an induction argument, we complete the proof of Theorem 4.1. □

5. Partial non-degeneracy: proof of Proposition (I)

In this section, we prove Proposition (I) for irreducible compact Hermitian spaces of compact type. Since the argument differs as its type varies, we do it on a case by case base. For convenience of the reader, we give a detailed proof here for the Grassmannians and Hyperquadrics. We will include the rest arguments in Appendix B.

5.1. Spaces of type I

With the same notations that we have set up in §2, Z is a $p \times q$ matrix $(p \leq q)$; $Z\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$ is the determinant of the submatrix of Z formed by its $i_1^{\text{th}}, \dots, i_k^{\text{th}}$ rows and $j_1^{\text{th}}, \dots, j_k^{\text{th}}$ columns; $z = (z_{11}, \dots, z_{1q}, z_{21}, \dots, z_{2q}, \dots, z_{p1}, \dots, z_{pq})$ is the coordinates of $\mathbb{C}^{pq} \cong \mathcal{A} \subset G(p, q)$. Let $0 \in U$ be a small neighborhood of 0 in $\mathbb{C}^{pq}$ and F be a biholomorphic map defined over U with $F(0) = 0$. For convenience of our discussions, we represent the map $F : U \to \mathcal{A}$ as a holomorphic matrix-valued map:

$$F = \begin{pmatrix} f_{11} & \dots & f_{1q} \\ \dots & \dots & \dots \\ f_{p1} & \dots & f_{pq} \end{pmatrix}.$$

Similar to $Z\left(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}\right)$, $F\left(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}\right)$ denotes the determinant of the submatrix formed by the $i_1^{\text{th}}, ..., i_k^{\text{th}}$ rows and $j_1^{\text{th}}, ..., j_k^{\text{th}}$ columns of the matrix F. Recall in (2), r_z is defined as

$$(\psi_1, \psi_2, ..., \psi_N) = (\cdots, Z\left(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}\right), \cdots), 1 \leq i_1 < ... < i_k \leq p, 1 \leq j_1 < ... < j_k \leq q,$$

$$1 \leq k \leq p.$$

Similarly, we define:

$$r_F := (\cdots, F\left(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}\right), \cdots), 1 \leq i_1 < ... < i_k \leq p, 1 \leq j_1 < ... < j_k \leq q, 1 \leq k \leq p.$$

Notice that $r_F = (\psi_1(F(z)), ..., \psi_N(F(z)))$. We define

$$\widetilde{z} := (z_{11}, ..., z_{1q}, z_{21}, ..., z_{2q}, ..., z_{p1}, ..., z_{p(q-1)}),$$

i.e., $\widetilde{z}$ is obtained from z by dropping the last component z_{pq}. Write $\frac{\partial^{|\alpha|}}{\partial \widetilde{z}^{\alpha}} = \frac{\partial^{|\alpha|}}{\partial z_{11}^{\alpha_{11}} \ldots \partial z_{p(q-1)}^{\alpha_{p(q-1)}}}$ for any $(pq-1)-$multiindex α, where $\alpha = (\alpha_{11}, ..., \alpha_{1p}, \alpha_{21}, ..., \alpha_{2q}, ..., \alpha_{p1}, ..., \alpha_{p(q-1)})$.

We apply the notion of the partial degeneracy defined in Definition 3.1 of §3 by letting $\psi = r_F$ and letting $\widetilde{z}$ be as just defined with $m = pq$. We prove the following proposition:

Proposition 5.1. *r_F are $\widetilde{z}$-nondegenerate near* 0. *More precisely,* $\text{rank}_{1+N-pq}(r_F, \widetilde{z}) = N$.

Proof of Proposition 5.1: If $p = 1, q = n \geq 1$ i.e., the Hermitian symmetric space $M = \mathbb{P}^n$, then it follows from Lemma 3.3 that $\text{rank}_1(r_F, \widetilde{z}) = N = n$. In the following we assume $p \geq 2$.

Suppose the conclusion is not true. Namely, assume that $\text{rank}_{1+N-pq}(r_F, \widetilde{z}) < N$. Since the hypothesis of Theorem 3.10 is satisfied, we see that there exist $c_{pq+1}, ..., c_N \in \mathbb{C}$ which are not all zero such that

$$\sum_{i=pq+1}^{N} c_i \psi_i(F)(z_{11}, ..., z_{pq-1}, 0) \equiv 0. \tag{57}$$

The next step is to show that (57) cannot hold in the setting of Proposition 5.1. This is obvious if we can prove the following:

Lemma 5.2. *Let*

$$H = \begin{pmatrix} h_{11} & \ldots & h_{1p} \\ \ldots & \ldots & \ldots \\ h_{p1} & \ldots & h_{pq} \end{pmatrix},$$

be a vector-valued holomorphic function in a neighborhood U *of* 0 *in* $\tilde{z} = (z_{11}, ..., z_{p(q-1)}) \in \mathbb{C}^{pq-1}$ *with* $H(0) = 0$*. Assume that* H *is of full rank at* 0*. Set*

$$(\phi_1, ..., \phi_m) := r_H = \left(\left(H\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \right)_{1 \le i_1 < ... < i_k \le p, 1 \le j_1 < ... < j_k \le q} \right)_{2 \le k \le p}. \tag{58}$$

Here

$$m = \binom{p}{2}\binom{q}{2} + ... + \binom{p}{p}\binom{q}{p}.$$

Let $a_1, ..., a_m$ *be complex numbers such that*

$$\sum_{i=1}^{m} a_i \phi_i(\tilde{z}) \equiv 0 \text{ for all } \tilde{z} \in U. \tag{59}$$

Then $a_i = 0$ *for each* $1 \le i \le m$*.*

Proof of Lemma 5.2: We start with the simple case $p = q = 2$, in which $m = 1$. Then by the assumption (59), $a_1 \phi_1 = 0$. Here

$$\phi_1 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}.$$

Note that $H = (h_{11}, h_{12}, h_{21}, h_{22})$ is of full rank at 0. We assume, without loss of generality, that $\tilde{H} := (h_{11}, h_{12}, h_{21})$ is a local biholomorphic map from $\mathbb{C}^3$ to $\mathbb{C}^3$. After an appropriate biholomorphic change of coordinates preserving 0, we can assume $h_{11} = z_{11}, h_{12} = z_{12}, h_{21} = z_{21}$, and still write the last component as h_{22}. Then we have

$$a_1 \phi_1 = a_1 (z_{11} h_{22} - z_{12} z_{21}) \equiv 0,$$

which easily yields that $a_1 = 0$.

We then prove the lemma for the case of $p = 2, q = 3$, in which $m = 3$. As before, without loss of generality, we assume that $\tilde{H} := (h_{11}, h_{12}, h_{13}, h_{21}, h_{22})$ is a local biholomorphic map near 0 from $\mathbb{C}^5$ to $\mathbb{C}^5$. After an appropriate biholomorphic change of coordinates, we assume that $\tilde{H} = (z_{11}, ..., z_{22})$. By (59), we have

$$a_1 \phi_1 + ... + a_3 \phi_3 = a_1 \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} + a_2 \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & h_{23} \end{vmatrix} + a_3 \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & h_{23} \end{vmatrix}. \tag{60}$$

The conclusion can be easily proved by checking the coefficients in the Taylor expansion at 0. Indeed, the quadratic terms $z_{13} z_{21}, z_{13} z_{22}$ only appear once in the last two determinants. This implies $a_2 = a_3 = 0$. Then trivially $a_1 = 0$.

We also prove the case $p = q = 3$. In this case $m = 10$. As before, without loss of generality, we assume that $\tilde{H} = (h_{11}, ..., h_{32})$ is a biholomorphic map from $\mathbb{C}^8$ to $\mathbb{C}^8$. After an appropriate biholomorphic change of coordinates, we can assume that $\tilde{H} = (z_{11}, ..., z_{32})$. Then by assumption, we have

$$\begin{aligned} & a_1\phi_1 + ... + a_{10}\phi_{10} = \\ & a_1 \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} + a_2 \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} + a_3 \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} + a_4 \begin{vmatrix} z_{11} & z_{12} \\ z_{31} & z_{32} \end{vmatrix} + a_5 \begin{vmatrix} z_{11} & z_{13} \\ z_{31} & h_{33} \end{vmatrix} \\ & \quad + a_6 \begin{vmatrix} z_{12} & z_{13} \\ z_{32} & h_{33} \end{vmatrix} + a_7 \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix} + a_8 \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & h_{33} \end{vmatrix} + a_9 \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & h_{33} \end{vmatrix} \\ & \quad + a_{10} \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & h_{33} \end{vmatrix} = 0. \end{aligned} \tag{61}$$

We then check the coefficients for each term in its Taylor expansion at 0. First it is easy to note that $a_5 = a_6 = a_8 = a_9 = 0$ by checking the coefficients of quadratic terms

$$z_{13}z_{31}, z_{13}z_{32}, z_{23}z_{31}, z_{23}z_{32},$$

respectively. Then by checking the coefficients of other quadratic terms, we see that $a_1 = a_2 = a_3 = a_4 = a_7 = 0$. Finally we check the coefficient of the cubic term $z_{13}z_{22}z_{31}$ to obtain that $a_{10} = 0$.

We now prove the general case: $q \geq p \geq 2$. As before, we assume without loss of generality that $\tilde{H} = (h_{11}, ..., h_{p(q-1)})$ is a biholomorphic map from $\mathbb{C}^{pq-1}$ to $\mathbb{C}^{pq-1}$. Furthermore, we have $\tilde{H} = (z_{11}, ..., z_{p(q-1)})$ after an appropriate biholomorphic change of coordinates. We again first consider the coefficients of the quadratic terms in (59). For that, we consider the 2×2 submatrix involving h_{pq}, i.e., $H\begin{pmatrix} l & p \\ k & q \end{pmatrix}, 1 \leq l < p, 1 \leq k < q$. Note that $z_{lq}z_{pk}$ only appears in this 2×2 determinant, which yields that the coefficient a_i associated to this 2×2 determinant is 0, for any $1 \leq i < p, 1 \leq j < q$. Then by checking the coefficients of other quadratic terms, we see that all coefficients $a_i's$ that are associated to 2×2 determinants $H\begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix}, 1 \leq l_1, l_2 \leq p, 1 \leq k_1, k_2 \leq q$, are 0.

We then consider the coefficients of cubic terms in (59). We first look at those 3×3 submatrix involving h_{pq}, i.e., $H\begin{pmatrix} l_1 & l_2 & p \\ k_1 & k_2 & q \end{pmatrix}, 1 \leq l_1 < l_2 < p, 1 \leq k_1 < k_2 < q$. Note that $z_{l_1q}z_{l_2k_2}z_{pk_1}$ only appears in this 3×3 matrix, which yields that the a_i associated to this 3×3 determinant is 0. Furthermore, we see that all coefficients a_i's that are associated to 3×3 determinants are 0.

Now the conclusion can be proved inductively. Indeed, assume that we have proved that all coefficients a_i's that are associated with the determinants of order up to $\mu\times\mu, 3 \leq \mu < p$ are zero. Then we will prove that the coefficients associated with $(\mu+1)\times(\mu+1)$ determinants are also 0. For this we consider all such determinants which involve h_{pq}, i.e., $H\begin{pmatrix} l_1 & ... & l_\mu & p \\ k_1 & ... & k_\mu & q \end{pmatrix}$ where $1 \le l_1 < ... < l_\mu < p, 1 \le k_1 < ... < k_\mu < q$. We conclude the a_i associated to it is 0 by noting that $z_{l_1 q} z_{l_2 k_\mu} ... z_{l_\mu k_2} z_{p k_1}$ only appears in this $(\mu+1) \times (\mu+1)$ determinant. Then we can show all coefficients that are associated with other $(\mu+1) \times (\mu+1)$ determinants, i.e.,

$$H\begin{pmatrix} l_1 & ... & l_\mu & l_{\mu+1} \\ k_1 & ... & k_\mu & k_{\mu+1} \end{pmatrix}, 1 \le l_1 < ... < l_{\mu+1} \le p, 1 \le k_1 < ... < k_{\mu+1} \le q,$$
$$(l_{\mu+1}, k_{\mu+1}) \neq (p, q).$$

are 0 by checking a term of the form $z_{l_1 k_1} ... z_{l_{\mu+1} k_{\mu+1}}$ that only appears once in the Taylor expansion of the left hand side of (57). This proves the lemma. □

We thus get a contradiction to the equation (57). This establishes Proposition 5.1. □

Remark 5.3. Let F be as in Proposition 5.1. There exist multiindices $\beta^1, ..., \beta^N$ with $|\beta^j| \le 1 + N - pq$ and

$$z^0 = \begin{pmatrix} z^0_{11} & ... & z^0_{1q} \\ ... & ... & ... \\ z^0_{p1} & ... & z^0_{pq} \end{pmatrix} \neq 0$$

such that z^0 is near 0 and

$$\Delta(\beta^1, ..., \beta^N) := \begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \bar{z}^{\beta^1}} & ... & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \bar{z}^{\beta^1}} \\ ... & ... & ... \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \bar{z}^{\beta^N}} & ... & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \bar{z}^{\beta^N}} \end{vmatrix} (z^0) \neq 0. \tag{62}$$

Perturbing z^0 if necessary, we can thus assume that $z^0_{pq} \neq 0$. Moreover, we can replace one of the $\beta^1, ..., \beta^N$ by $\beta = (0, ..., 0)$, because $(\psi_1(F), ..., \psi_N(F))$ are not identically zero (see also the proof of Theorem 3.4). Without lost of generality, we can assume that $\beta^1 = (0, ..., 0)$.

The defining function of the Segre family now is

$$\rho(z, \xi) = 1 + \sum_{k=1}^{p} \left(\sum_{1 \le i_1 < i_2 < ... < i_k \le p, 1 \le j_1 < j_2 < ... < j_k \le q} Z\begin{pmatrix} i_1 & ... & i_k \\ j_1 & ... & j_k \end{pmatrix} \Xi\begin{pmatrix} i_1 & ... & i_k \\ j_1 & ... & j_k \end{pmatrix} \right) \tag{63}$$

It is a complex manifold for any fixed ξ close enough to the point

$$\xi^0 = \begin{pmatrix} 0 & ... & 0 & 0 \\ 0 & ... & 0 & 0 \\ 0 & ... & 0 & \xi^0_{pq} \end{pmatrix} \in \mathbb{C}^{pq}, \quad \xi^0_{pq} = -\frac{1}{z^0_{pq}}.$$

Write for each $1 \le i \le p, 1 \le j \le q, (i, j) \neq (p, q)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z,\xi)}{\frac{\partial \rho}{\partial z_{pq}}(z,\xi)} \frac{\partial}{\partial z_{pq}}, \tag{64}$$

which is a well-defined holomorphic tangent vector field along $\mathcal{M}$ near (z^0, ξ^0). Here we note that $\frac{\partial \rho}{\partial z_{pq}}(z,\xi)$ is nonzero near (z^0, ξ^0). For any $(pq-1)$-multiindex $\beta = (\beta_{11}, ..., \beta_{p(q-1)})$, we write

$$\mathcal{L}^{\beta} = \mathcal{L}_{11}^{\beta_{11}} ... \mathcal{L}_{p(q-1)}^{\beta_{p(q-1)}}.$$

Now we define for any N collection of $(pq-1)-$multiindices $\{\beta^1, ..., \beta^N\}$,

$$\Lambda(\beta^1, ..., \beta^N)(z,\xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & ... & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix} (z,\xi). \tag{65}$$

Theorem 5.4. *There exist multiindices $\{\beta^1, ..., \beta^N\}$, such that*

$$\Lambda(\beta^1, ..., \beta^N)(z,\xi) \neq 0, \tag{66}$$

for (z,ξ) in a small neighborhood of (z^0, ξ^0). Moreover, we can require $\beta^1 = (0, ..., 0)$.

Proof of Theorem 5.4: First we observe that $\mathcal{L}_{ij}$ evaluating at (z^0, ξ^0) is just $\frac{\partial}{\partial z_{ij}}$. More generally, for any $(pq-1)-$multiindex β, by an easy computation, $\mathcal{L}^{\beta}$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \tilde{z}^{\beta}}$. Therefore, we can just choose the same $\beta^1, ..., \beta^N$ as in Remark 5.3. □

5.2. Spaces of type IV

In this subsection, we consider the hyperquadric case $M = Q^n$. This case is more subtle because the tangent vector fields of its Segre family are more complicated. Recall that Q^n is defined by

$$\left\{ [z_0, ..., z_{n+1}] \in \mathbb{CP}^{n+1} : \sum_{i=1}^{n} z_i^2 - 2z_0 z_{n+1} = 0 \right\},$$

where $[z_0, ..., z_{n+1}]$ is the homogeneous coordinates of $\mathbb{CP}^{n+1}$. The previously described minimal embedding $\mathbb{C}^n(\mathcal{A}) \to Q^n$ is given by

$$z := (z_1, ..., z_n) \mapsto [1, \psi_1(z), ..., \psi_{n+1}(z)] = [1, z_1, ..., z_n, \frac{1}{2}\sum_{i=1}^{n} z_i^2].$$

The defining function of the Segre family over $\mathcal{A} \times \mathcal{A}$ is $\rho(z,\xi) = 1 + r_z \cdot r_{\xi},$ where

$$r_z = (z_1, ..., z_n, \frac{1}{2}\sum_{i=1}^{n} z_i^2), r_\xi = (\xi_1, ..., \xi_n, \frac{1}{2}\sum_{i=1}^{n} \xi_i^2). \tag{67}$$

Let F be a local biholomorphic map at 0 with $F(0) = 0$. We write

$$F = (f_1, ..., f_n), \quad r_F = (f_1, ..., f_n, \frac{1}{2}\sum_{i=1}^{n} f_i^2). \tag{68}$$

Notice that

$$r_z = (\psi_1(z), ..., \psi_{n+1}(z)), r_F = (\psi_1(F), ..., \psi_{n+1}(F)).$$

We will need the following lemma:

Lemma 5.5. *For each fixed* $\mu_1, ..., \mu_{n-1}$ *with* $(\sum_{i=1}^{n-1} \mu_i^2) + 1 = 0$ *and each fixed* $(z_1, ..., z_n)$ *with* $(\sum_{i=1}^{n-1} \mu_i z_i) + z_n \neq 0$, *we can find* $(\xi_1, ..., \xi_n)$ *such that*

$$1 + z_1\xi_1 + ... + z_n\xi_n = 0; \quad \sum_{i=1}^{n} (\xi_i)^2 = 0, \quad \xi_j = \mu_j \xi_n, 1 \le j \le n-1, \quad \xi_n \neq 0. \tag{69}$$

Proof of Lemma 5.5: We just need to set

$$\xi_n = \frac{-1}{(\sum_{i=1}^{n-1} \mu_i z_i) + z_n}, \quad \xi_j = \mu_j \xi_n, 1 \le j \le n-1.$$

Then it is easy to verify that (69) is satisfied. □

Recall that in the type I case, the vector fields $\frac{\partial}{\partial \tilde{z}^\alpha}$ in $\mathbb{C}^{pq}$ are tangent vector fields of the particular hyperplane $\{z_{pq} = 0\}$. We can formulate the result in §3 in a more abstract way and extend it to a more general setting. For instance, it can be generalized to the complex hyperplane case. We briefly discuss this in more details as follows:

First fix $\mu_1, ..., \mu_{n-1}$ with $(\sum_{i=1}^{n-1} \mu_i^2) + 1 = 0$. Take the complex hyperplane $\mathbb{H}$: $z_n + \sum_{i=1}^{n-1} \mu_i z_i = 0$ in $(z_1, ..., z_n) \in \mathbb{C}^n$. Write

$$L_1 = \frac{\partial}{\partial z_1} - \mu_1 \frac{\partial}{\partial z_n}, ..., L_{n-1} = \frac{\partial}{\partial z_{n-1}} - \mu_{n-1} \frac{\partial}{\partial z_n}.$$

Then $\{L_i\}_{i=1}^{n-1}$ forms a basis of the tangent vector fields of $\mathbb{H}$. For any multiindex $\alpha = (\alpha_1, .., \alpha_{n-1})$, we write $L^\alpha = L_1^{\alpha_1} ... L_{n-1}^{\alpha_{n-1}}$. We define $L-$rank and $L-$nondegeneracy as in Definition 3.1 by using r_F in (68) and by using L^α instead of $\tilde{z}^\alpha$ with $m = n$. We write the kth L-rank defined in this setting as $\text{rank}_k(r_F, L)$. We now need to prove the following

Proposition 5.6. $\text{rank}_2(r_F, L) = n + 1.$

Proof of Proposition 5.6: Suppose not. By applying the same argument as in Section 3 and a linear change of coordinates, we can first obtain a modified version of Theorem 3.10:

Lemma 5.7. *There exist* $n+1$ *holomorphic functions* $g_1(w), ..., g_{n+1}(w)$ *which are defined near* 0 *on the* $w-$*plane with* $\{g_1(0), ..., g_{n+1}(0)\}$ *not all zero such that the following holds for all* $z \in U$.

$$\sum_{i=1}^{n+1} g_i(z_n + \mu_1 z_1 + ... + \mu_{n-1} z_{n-1})\psi_i(F(z)) \equiv 0. \tag{70}$$

Then one shows with a similar argument as in Section 3, by the fact that F has full rank at 0, that $g_1(0) = 0, ..., g_n(0) = 0$. Hence we obtain,

Lemma 5.8. *There exists a non-zero constant* $c \in \mathbb{C}$ *such that*

$$c\psi_{n+1}(F(z)) = \frac{c}{2}\sum_{i=1}^{n} f_i^2(z) \equiv 0, \tag{71}$$

for all $z \in U$ *when restricted on* $z_n + \sum_{i=1}^{n-1} \mu_i z_i = 0$.

We then just need to show that (71) cannot hold by applying the following lemma and a linear change of coordinates.

Lemma 5.9. *Let* $H = (h_1, ..., h_n)$ *be a vector-valued holomorphic function in a neighborhood* U *of* 0 *in* $\tilde{z} = (z_1, ..., z_{n-1}) \in \mathbb{C}^{n-1}$ *with* $H(0) = 0$. *Assume that* H *has full rank at* 0. *Assume that* a *is a complex number such that,*

$$a\sum_{i=1}^{n} h_i^2(\tilde{z}) \equiv 0, \tag{72}$$

then $a = 0$.

Proof of Lemma 5.9: Seeking a contradiction, suppose not. Notice that H has full rank at 0. We assume, without loss of generality, that $(h_1, ..., h_{n-1})$ gives a local biholomorphic map near 0 from $\mathbb{C}^{n-1}$ to $\mathbb{C}^{n-1}$. By a local biholomorphic change of coordinates, we assume $(h_1, ..., h_{n-1}) = (z_1, ..., z_{n-1})$, and still write the last component as h_n. Then equation (72) is reduced to

$$a(z_1^2 + ... + z_{n-1}^2 + h_n^2) = 0.$$

To cancel the z_1^2, z_2^2 terms, it yields that h_n has linear z_1, z_2 terms. But then h_n^2 would produce a $z_1 z_2$ term, which cannot be canceled out. This is a contradiction. □

This also establishes Proposition 5.6. □

Remark 5.10. By Proposition 5.6, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^{n+1}$ with $|\tilde{\beta}^j| \le 2$ and

$$z^0 = (z_1^0, ..., z_n^0) \text{ with } \sum_{i=1}^{n-1} \mu_i z_i^0 + z_n^0 \neq 0$$

such that

$$\begin{vmatrix} L^{\tilde{\beta}^1}(\psi_1(F)) & ... & L^{\tilde{\beta}^1}(\psi_{n+1}(F)) \\ ... & ... & ... \\ L^{\tilde{\beta}^{n+1}}(\psi_1(F)) & ... & L^{\tilde{\beta}^{n+1}}(\psi_{n+1}(F)) \end{vmatrix}(z^0) \neq 0. \tag{73}$$

We then choose $\xi^0 = (\xi_1^0, ..., \xi_n^0)$ as in Lemma 5.5. That is

$$1 + z_1^0\xi_1^0 + ... + z_n^0\xi_n^0 = 0; \quad \sum_{i=1}^{n}(\xi_i^0)^2 = 0, \quad \xi_j^0 = \mu_j \xi_n^0, 1 \le j \le n-1, \quad \xi_n^0 \neq 0.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M}$. We now define

$$\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z,\xi)}{\frac{\partial \rho}{\partial z_n}(z,\xi)} \frac{\partial}{\partial z_n}, 1 \le i \le n-1 \tag{74}$$

for $(z, \xi) \in \mathcal{M}$ near (z^0, ξ^0). They are well-defined holomorphic tangent vector fields along $\mathcal{M}$. Moreover, $\frac{\partial \rho}{\partial z_n}(z, \xi)$ is nonzero near (z^0, ξ^0).

We define for any multiindex $\alpha = (\alpha_1, .., \alpha_{n-1})$, $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1} ... \mathcal{L}_{n-1}^{\alpha_{n-1}}$. Then for any $(n+1)$ collection of $(n-1)-$multiindices, set $\{\beta^1, ..., \beta^N\}$,

$$\Lambda(\beta^1, ..., \beta^{n+1})(z,\xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_{n+1}(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^{n+1}}(\psi_1(F)) & ... & \mathcal{L}^{\beta^{n+1}}(\psi_{n+1}(F)) \end{vmatrix}(z, \xi). \tag{75}$$

By the fact that $\sum_{i=1}^n (\xi_i^0)^2 = 0$, one can check that, for any multiindex $\alpha = (\alpha_1, .., \alpha_n)$, $\mathcal{L}^\alpha = L^\alpha$ when evaluated at (z^0, ξ^0). Then we get the following:

Theorem 5.11. *There exist multiindices $\{\beta^1, ..., \beta^N\}$ such that*

$$\Lambda(\beta^1, ..., \beta^N)(z, \xi) \neq 0,$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0), where $\beta^1 = (0, 0, ..., 0)$.

Proofs for the other types are similar and will be left to Appendix B.

6. Transversality and flattening of Segre families: proof of Proposition (II)

In this section, we prove Proposition (II). We still use the notations we have set up so far. We equip the space M with the canonical Kähler-Einstein metric ω as described before. We start with the following lemma:

Lemma 6.1. *Let $\widehat{\sigma} : (M, \omega) \to (M, \omega)$ be a holomorphic isometry. In the affine space $\mathcal{A}$, its components consist of rational functions with its degree bounded only by a constant depending on (M, ω).*

Proof of Lemma 6.1: Notice that M has been isometrically embedded into $\mathbb{CP}^N$ through the canonical map defined before. Hence $\widehat{\sigma}$ is the restriction of a unitary transformation. Hence $\widehat{\sigma}$ can be identified with a map of the form:

$$(\tilde{\psi}_0, \tilde{\psi}_1, \tilde{\psi}_2, ..., \tilde{\psi}_N) = (\sum_{j=0}^{N} a_{0j}\psi_j, ..., \sum_{j=0}^{N} a_{ij}\psi_j, ..., \sum_{j=0}^{N} a_{Nj}\psi_j),$$

where $\psi_0 = 1$ and (a_{ij}) is a unitary matrix. Write

$$\Psi(z) : z(\in \mathcal{A}) \mapsto [1, \kappa_1 z_1, \cdots, \kappa_i z_i, \cdots, \kappa_n z_n, o(z^2)] \in \mathbb{CP}^N$$

for the embedding, where $\kappa_i = 1$ or $\sqrt{2}$. $\widehat{\sigma}$ induces a birational self-action σ of $\mathcal{A}$ such that $\Psi(\sigma(z)) = \widehat{\sigma}(\Psi(z))$. Then, from the special form of Ψ, $\sigma(z) = \left(\frac{\tilde{\psi}_1}{\kappa_1 \tilde{\psi}_0}, \frac{\tilde{\psi}_2}{\kappa_2 \tilde{\psi}_0}, ..., \frac{\tilde{\psi}_n}{\kappa_n \tilde{\psi}_0}\right)$. Apparently $\tilde{\psi}_0 \not\equiv 0$. $\square$

Theorem 6.2. *Suppose $\xi^0 \in \mathbb{C}^n \setminus \{0\}$. Then for a generic smooth point z^0 on the Segre variety Q_{ξ^0} and a small neighborhood $U \subset \mathbb{C}^n$ of z^0, there is a point $z^1 \in U \cap Q_{\xi^0}$, such that Q_{z^0} and Q_{z^1} are both smooth at ξ^0 and intersect transversally there. Moreover, there is a biholomorphic parametrization $\mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) = (\xi_1, \xi_2, ..., \xi_n)$, with $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$. Here when $1 \le j \le 2$, U_j is a small neighborhood of $1 \in \mathbb{C}$. When $3 \le j \le n$, U_j is a small neighborhood of $0 \in \mathbb{C}$ with $\mathcal{G}(1, 1, 0, \cdots, 0) = \xi^0$, such that $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}, \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$, and $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times ... \times U_n), \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times ... \times U_n), s \in U_1, t \in U_2$ are open pieces of Segre varieties. Also, $\mathcal{G}$ consists of algebraic functions with total degree bounded by a constant depending only on (M, ω).*

We first claim that, due to the invariance of the Segre family, we need only to prove the theorem for a special point $0 \ne \xi^0 \in \mathbb{C}^n \subset M$. Indeed, by the invariance property mentioned in §2, for an isometry σ, $(\sigma, \overline{\sigma})$ preserves the Segre family $\mathcal{M} \subset M \times M$. Here for $p \in \mathbb{CP}^N$, $\overline{\sigma}(p) := \overline{\sigma(\overline{p})}$ as before. Here, we mention that in the statement of the theorem, we assume that z^0 is a generic smooth point because under this transformation, some smooth points on Q_{ξ^0} may be mapped into the hyperplane of M at infinity, which can not be chosen as our z^0.

We now proceed to the proof of Theorem 6.2 by choosing a good point ξ^0. We only carry out the proof for the case of hyperquadrics and Grassmannian spaces here. The proof for the remaining cases is similar and will be included in Appendix C.

Proof of Theorem 6.2: **Case 1. Hyperquadrics**: Suppose M is the hyperquadric. Then the defining equation for the Segre family is

$$\rho(z,\xi) = 1 + \sum_{i=1}^{n} z_i\xi_i + \frac{1}{4}(\sum_{i=1}^{n} z_i^2)(\sum_{i=1}^{n} \xi_i^2) = 0.$$

We choose $\xi^0 = (1, 0, 0, ..., 0)$. Hence $Q_{\xi^0} = \{z : \rho(z, \xi^0) = 1 + z_1 + \frac{1}{4}(\sum_{i=1}^{n} z_i^2) = 0\}$. We compute the gradient of $\rho(z, \xi^0)$ as follows: $\nabla\rho(z, \xi^0) = (1 + \frac{1}{2}z_1, \frac{1}{2}z_2, ..., \frac{1}{2}z_n)$. Notice that Q_{ξ^0} is smooth except at $(-2, 0, ..., 0)$, namely, we have $\nabla\rho(z, \xi^0) \neq 0$ away from $(-2, 0, \cdots, 0)$. For a smooth point $z^0(\neq (-2, 0, \cdots, 0))$ of Q_{ξ^0}, we choose a neighborhood U of z^0 in $\mathbb{C}^n$ such that $U \cap Q_{\xi^0}$ is a smooth piece of Q_{ξ^0}. Pick also $z^1(\neq z^0) \in U \cap Q_{\xi_0}$ and compute the gradient of the defining function of Q_{z^0} and Q_{z^1} at $\xi^0 = (1, 0, ..., 0)$, respectively. Recall

$$Q_{z^s} = \{\xi | \rho(z^s, \xi) = 1 + \sum_{i=1}^{n} z_i^s\xi_i + \frac{1}{4}(\sum_{i=1}^{n}(z_i^s)^2)(\sum_{i=1}^{n} \xi_i^2) = 0\}, \text{ for } s = 0, 1.$$

$$\begin{aligned}\begin{pmatrix} \nabla\rho(z^0,\xi)|_{\xi^0=(1,0,\ldots,0)} \\ \nabla\rho(z^1,\xi)|_{\xi^0=(1,0,\ldots,0)} \end{pmatrix} &= \begin{pmatrix} z_1^0 + \frac{1}{2}\sum_{i=1}^{n}(z_i^0)^2 & z_2^0 & z_3^0 & \ldots & z_n^0 \\ z_1^1 + \frac{1}{2}\sum_{i=1}^{n}(z_i^1)^2 & z_2^1 & z_3^1 & \ldots & z_n^1 \end{pmatrix} \\ &= \begin{pmatrix} -2 - z_1^0 & z_2^0 & z_3^0 & \ldots & z_n^0 \\ -2 - z_1^1 & z_2^1 & z_3^1 & \ldots & z_n^1 \end{pmatrix}\end{aligned}$$

The second equality is simplified by making use of the fact that $z^0, z^1 \in Q_{\xi^0=(1,0,\ldots,0)}$, which implies that $0 = 1 + z_1^0 + \frac{1}{4}\sum_{i=1}^{n}(z_i^0)^2 = 1 + z_1^1 + \frac{1}{4}\sum_{i=1}^{n}(z_i^1)^2$. Hence,

$$\begin{aligned}\operatorname{rank}\begin{pmatrix} \nabla\rho(z^0,\xi)|_{\xi^0=(1,0,\ldots,0)} \\ \nabla\rho(z^1,\xi)|_{\xi^0=(1,0,\ldots,0)} \end{pmatrix} &= \operatorname{rank}\begin{pmatrix} -2 - z_1^0 & z_2^0 & \ldots & z_n^0 \\ -2 - z_1^1 & z_2^1 & \ldots & z_n^1 \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} -2 - z_1^0 & z_2^0 & \ldots & z_n^0 \\ -\Delta z_1^1 & \Delta z_2^1 & \ldots & \Delta z_n^1 \end{pmatrix}\end{aligned}$$

$$= \operatorname{rank}\begin{pmatrix} 2 + z_1^0 & z_2^0 & \ldots & z_n^0 \\ \Delta z_1^1 & \Delta z_2^1 & \ldots & \Delta z_n^1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} & & \nabla\rho(z,\xi^0)|_{z^0} & \\ \Delta z_1^1 & \Delta z_2^1 & \ldots & \Delta z_n^1 \end{pmatrix},$$

where $\Delta z_i^1 := z_i^1 - z_i^0$. Notice that z^0 is a smooth point on Q_{ξ_0}. Hence $\nabla\rho(z, \xi^0)$ is transversal to the tangent space of Q_{ξ^0} at z^0. If we choose $z^1 \in Q_{\xi^0}$ close enough to z^0, which ensures $(\Delta z_1^1, ..., \Delta z_n^1)$ close enough to tangent space of Q_{ξ^0} at z^0, we then get

$$\operatorname{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0=(1,0,\dots,0)}\\ \nabla\rho(z^1,\xi)|_{\xi^0=(1,0,\dots,0)}\end{pmatrix} = \operatorname{rank}\begin{pmatrix} & & \nabla\rho(z,\xi^0)|_{z^0} & \\ \Delta z_1^1 & \Delta z_2^1 & \dots & \Delta z_n^1\end{pmatrix} = 2.$$

We assume, without loss of generality, that $\frac{\partial(\rho(z^0,\xi),\rho(z^1,\xi))}{\partial(\xi_1,\xi_2)} \neq 0$ at ξ^0. Now we introduce new variables $\tilde{\xi}_1,\dots,\tilde{\xi}_n$ and consider the following system of equations:

$$\begin{cases} P_1 : 1+\sum_{i=1}^n(\tilde{\xi}_1 z_i^0)\xi_i + \frac{1}{4}(\sum_{i=1}^n(\tilde{\xi}_1)^2(z_i^0)^2)(\sum_{i=1}^n \xi_i^2) & = 0\\ P_2 : 1+\sum_{i=1}^n(\tilde{\xi}_2 z_i^1)\xi_i + \frac{1}{4}(\sum_{i=1}^n(\tilde{\xi}_2)^2(z_i^1)^2)(\sum_{i=1}^n \xi_i^2) & = 0\\ P_3 : \qquad \tilde{\xi}_3-\xi_3 & = 0\\ \dots & \dots\\ P_n : \qquad \tilde{\xi}_n-\xi_n & = 0\end{cases}$$

Then we have $\frac{\partial(P_1,\dots,P_n)}{\partial(\xi_1,\dots,\xi_n)}|_A \neq 0$ and $\frac{\partial(P_1,\dots,P_n)}{\partial(\tilde{\xi}_1,\dots,\tilde{\xi}_n)}|_A \neq 0$ where

$$A = (\tilde{\xi}_1,\dots,\tilde{\xi}_n;\xi_1,\dots,\xi_n) = (1,1,0,\dots,0;1,0,\dots,0).$$

By Lemma 4.9, we get the needed algebraic flattening with total degree bounded only by (M,ω). This completes the proof of Theorem 6.2 in the hyperquadric case.

Case 2. Grassmannians: Pick $\xi^0 = (\xi^0_{11},\xi^0_{12},\dots,\xi^0_{pq}) = (1,0,\dots,0)$. The defining function for the Segre family associated with this point is as follows:

$\rho(z,\xi) = 1 + z_{11}\xi_{11} + z_{12}\xi_{12} + \dots + z_{1q}\xi_{1q} + z_{21}\xi_{21} + \dots + z_{p1}\xi_{p1} + \sum_{i,j\neq 1} z_{ij}\xi_{ij} + \sum_{i,j\geq 2}(z_{11}z_{ij} - z_{i1}z_{1j})(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{(i,j),(k,l)\neq(1,1)}(z_{ij}z_{kl} - z_{il}z_{jk})(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{jk}) +$ higher order terms.

Then $Q_{\xi^0} = \{z|\rho(z,\xi^0) = 1 + z_{11} = 0\}, \nabla\rho(z,\xi^0) = (1,0,0,\dots,0)$. Hence Q_{ξ_0} is smooth. For $z\in Q_{\xi^0}$, we have $z = (-1,z_{12},\dots,z_{1q},z_{21},\dots,z_{p1},\dots,z_{ij},\dots,z_{pq})$. Pick $z^0,z^1\in Q_{\xi^0}$. Then $Q_{z^s} = \{\xi|0 = \rho(z^s,\xi) = 1 + z^s_{11}\xi_{11} + z^s_{12}\xi_{12} + \dots + z^s_{1q}\xi_{1q} + z^s_{21}\xi_{21} + \dots + z^s_{p1}\xi_{p1} + \sum_{i,j\neq 1} z^s_{ij}\xi_{ij} + \sum_{i,j\geq 2}(z^s_{11}z^s_{ij} - z^s_{i1}z^s_{1j})(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{(i,j),(k,l)\neq(1,1)}(z^s_{ij}z^s_{kl} - z^s_{il}z^s_{jk})(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{jk}) + \text{high order terms}\}$, for $s = 0,1$. We then compute their gradients as follows:

$$\begin{aligned}&\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\ \nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix}\\ &= \begin{pmatrix}\frac{\partial\rho(z^0,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{12}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{1q}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{21}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{p1}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{pq}}\\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{12}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{1q}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{21}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{p1}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{pq}}\end{pmatrix}\Bigg|_{\xi^0}\\ &= \begin{pmatrix}-1 & z^0_{12} & \dots & z^0_{1q} & z^0_{21} & \dots & z^0_{p1} & -z^0_{i1}z^0_{1j} & \dots\\ -1 & z^1_{12} & \dots & z^1_{1q} & z^1_{21} & \dots & z^1_{p1} & -z^1_{i1}z^1_{1j} & \dots\end{pmatrix}.\end{aligned}$$

Thus, we have

$$\operatorname{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)\big|_{\xi^0}\\ \nabla\rho(z^1,\xi)\big|_{\xi^0}\end{pmatrix}$$
$$=\operatorname{rank}\begin{pmatrix}-1 & z_{12}^0 & \dots & z_{p1}^0 & -z_{i1}^0 z_{1j}^0 & \dots\\ 0 & \Delta z_{12}^1 & \dots & \Delta z_{p1}^1 & (-z_{i1}^0\Delta z_{1j}^1 - z_{1j}^0\Delta z_{i1}^1 - \Delta z_{i1}^1\Delta z_{1j}^1) & \dots\end{pmatrix},$$

where $\Delta z_{ij}^1 = z_{ij}^1 - z_{ij}^0$. Hence, if we choose z^1 such that $z_{12}^1 \neq z_{12}^0$, then the rank equals to 2. Hence Q_{z^0} and Q_{z^1} are smooth and intersect transversally at ξ^0.

Without loss of generality, assume $\frac{\partial(\rho(z^0,\xi),\rho(z^1,\xi))}{\partial(\xi_{11},\xi_{12})} \neq 0$ at ξ^0. Now we introduce new variables $\tilde{\xi}_{11},\dots,\tilde{\xi}_{pq}$ and set up the system:

$$\begin{cases} P_{11}: & \rho(z^0,\tilde{\xi}_{11}\xi)=0\\ P_{12}: & \rho(z^1,\tilde{\xi}_{12}\xi)=0\\ P_{13}: & \tilde{\xi}_{13}-\xi_{13}=0\\ \dots & \dots\\ P_{pq}: & \tilde{\xi}_{pq}-\xi_{pq}=0\end{cases}$$

Then $\frac{\partial(P_{11},\dots,P_{pq})}{\partial(\xi_{11},\dots,\xi_{pq})}|_A$, $\frac{\partial(P_{11},\dots,P_{pq})}{\partial(\tilde{\xi}_{11},\dots,\tilde{\xi}_{pq})}|_A \neq 0$, where $A = (\tilde{\xi}_{11},\dots,\tilde{\xi}_{pq},\xi_{11},\dots,\xi_{pq}) = (1,1,0,\dots,0,1,0,\dots,0)$. By Lemma 4.9, we get the needed algebraic flattening.

The proof is similar in the other cases. We include a detailed argument for the remaining cases in Appendix C. □

7. Irreducibility of Segre varieties: proof of Proposition (III)

In this section we will establish Proposition (III). We prove results on the irreducibility of the potential function ρ, Segre varieties and the Segre family. We still adapt the previously used notation and assume that M is an irreducible Hermitian symmetric space of compact type of dimension n, which has been minimally embedded into a projective space as described before.

Lemma 7.1. *Each Segre variety is an irreducible algebraic subvariety.*

Proof of Lemma 7.1: For a minimally embedded Hermitian symmetric space, since all Segre varieties are unitarily equivalent, it suffices to prove the lemma for a single Segre variety. Without lost of generality, we take $z = (0,\dots,0) \in \mathcal{A} \subset M$. Therefore, the corresponding Segre variety Q_z^* is the hyperplane section $M \setminus \mathcal{A}$, which is of pure dimension. From the classical algebraic geometry [14], when M is an irreducible Hermitian symmetric space of compact type, the hyperplane section at infinity in the minimal canonical embedding case is a union of Schubert cells. Moreover as shown in [6], the top dimensional piece is equivalent to $\mathbb{C}^{n-1}$ and the others are of codimension at least two. Hence, the smooth points of Q_z are connected and thus Q_z is irreducible. □

As a corollary of this lemma, we conclude that for each $z \in \mathbb{C}^n$, the defining function $\rho(z, \cdot)$ of Q_z has to be a power of one irreducible factor. However, as in the proof of Theorem 6.2, for some $a(\neq 0) \in \mathbb{C}^n$, $d_\xi \rho(a, \xi)$ is not identically zero along Q_a. Next, we use this property and the symmetric property of M to prove the following:

Proposition 7.2. *For any $b \in \mathcal{A}$ with $b \neq (0, ..., 0)$, $\rho(b, \xi)$ ($\rho(z, b)$, respectively) is irreducible as a polynomial of ξ (as a polynomial in z, respectively).*

Proof of Proposition 7.2: Since $\rho(z, \xi) = \rho(\xi, z)$, we need just to verify the first statement. Let a be as above. For $b \in \mathcal{A}$, there is $\widehat{\sigma} \in \mathrm{Isom}(M, \omega) \cap SU(N+1, \mathbb{C})$ such that $\widehat{\sigma}(a) = b$. (Notice that $\widehat{\sigma}$ is represented by a unitary action.) By Lemma 6.1, let $\sigma = (\frac{l_1}{\kappa_1 l_0}, ..., \frac{l_n}{\kappa_n l_0})$ be the representation of $\widehat{\sigma}$ in $\mathcal{A}$ with $l_j's$ polynomials in z. Write $\Psi = [1, r_z]$ for the embedding of $\mathcal{A}$ in $\mathbb{P}^N$. Then from the definition of $\rho(z, \overline{z})$, we have

$$\rho(z, \overline{z}) = ||\Psi(z)||^2 = \Psi \cdot \overline{\Psi}^t = (\widehat{\sigma}\Psi) \cdot \overline{(\widehat{\sigma}\Psi)}^t.$$

Lemma 7.3. $(\widehat{\sigma}\Psi) \cdot \overline{(\widehat{\sigma}\Psi)}^t = |l_0(\Psi)|^2 \cdot ||\Psi(\sigma(z))||^2 = |l_0(\Psi)|^2 \cdot \rho(\sigma(z), \overline{\sigma(z)})$.

Proof. Writing $\Psi(z) = [1, r_z] = [1, \psi_1(z), \cdots, \psi_N(z)]$. Then the identity $\Psi(\sigma(z)) = \widehat{\sigma}(\Psi(z))$ obtained in the proof of Lemma 6.1 yields that,

$$(\psi_1(\sigma(z)), \cdots, \psi_N(\sigma(z))) = \left(\frac{\tilde{\psi}_1(\Psi(z))}{\tilde{\psi}_0(\Psi(z))}, \cdots, \frac{\tilde{\psi}_N(\Psi(z))}{\tilde{\psi}_0(\Psi(z))} \right).$$

Here $\tilde{\psi}_j = l_j$ for $0 \leq j \leq n$ and $\widehat{\sigma}(z) = [\tilde{\phi}_0, \cdots, \tilde{\phi}_N]$ as in the proof of Lemma 6.1. Then

$$\begin{aligned}(\hat{\sigma}\Psi) \cdot \overline{(\widehat{\sigma}\Psi)}^t &= \sum_{j=0}^{N} |\tilde{\psi}_j(\Psi(z))|^2 = \left(1 + \sum_{j=1}^{N} |\psi_j(\sigma(z))|^2\right) |\tilde{\psi}_0(\Psi(z))|^2 \\ &= |l_0(\Psi)|^2 \cdot ||\Psi(\sigma(z))||^2.\end{aligned}$$

This establishes the lemma. □

The Lemma 7.3 yields $\rho(z, \overline{z}) = |l_0(\Psi)|^2 \cdot \rho(\sigma(z), \overline{\sigma(z)})$. Complexifying the identity and substituting z by a, we have:

$$l_0(\Psi)(a) \cdot \overline{l_0(\Psi)}(\xi) \cdot \rho(b, \overline{\sigma}(\xi)) = \rho(a, \xi), \tag{76}$$

where $l_0(\Psi)(a) \neq 0$, $l_0(\Psi)(\xi)$, $\rho(a, \xi)$ are polynomials in ξ and $\sigma(\xi)$ is a rational map in ξ. Now supposing $\rho(b, \xi) = f^l(\xi), l \geq 2$, we have $\rho(b, \overline{\sigma}(\xi)) = (f(\overline{\sigma}(\xi)))^l = (\frac{f_1(\xi)}{f_2(\xi)})^l$, where f_1 and f_2 are coprime polynomials. Since $a, b \neq (0, ..., 0)$, f_1 is a non-constant polynomial. Therefore in (76), even after cancellation, we still have a factor $f_1^l(\xi)$. However as shown in §6, the right hand side of the identity (76) must be an irreducible polynomial, which is a contradiction. □

Proposition 7.4. *$\rho(z,\xi)$ is an irreducible polynomial over $\mathbb{C}^n \times \mathbb{C}^n$. Thus, the Segre family $\mathcal{M}$ restricted to $\mathbb{C}^n \times \mathbb{C}^n = \mathcal{A} \times \mathcal{A} \subset M \times M$ is an irreducible subvariety of dimension $2n-1$.*

We also have the following slightly strong version of the above proposition, which was used for applying a monodromy argument:

Proposition 7.5. *Suppose U is an connected open set in $\mathbb{C}^n \setminus \{0\}$. Then the Segre family $\mathcal{M}$ restricted to $U \times \mathbb{C}^n$ or restricted to $\mathbb{C}^n \times U$ is an irreducible analytic variety.*

Proof of Proposition 7.5: We need only to prove the first statement. Recall the notations we defined before: $\mathcal{M}_{\text{SING}} = \{(z,\xi) : \frac{\partial\rho}{\partial\xi_j} = 0, \forall j\} \cup \{(z,\xi) : \frac{\partial\rho}{\partial z_j} = 0, \forall j\}$, and $\mathcal{M}_{\text{REG}} = \mathcal{M} \backslash \mathcal{M}_{\text{SING}}$. Since $\rho(z,\xi)$ is an irreducible polynomial and $\frac{\partial\rho}{\partial\xi_j}, \frac{\partial\rho}{\partial z_j}, j = 1, ..., n$ are polynomials with lower degrees, $\frac{\partial\rho}{\partial\xi_j}, \frac{\partial\rho}{\partial z_j}, j = 1, ..., n$ are not identically zero on $\mathcal{M} = \{\rho(z,\xi) = 0\}$. Each of $\frac{\partial\rho}{\partial\xi_j}, \frac{\partial\rho}{\partial z_j}$ defines a proper subvariety inside $\mathcal{M}$. By Proposition 7.2, for each $\tilde{z}(\neq 0) \in \mathbb{C}^n$, there is a certain point $\tilde{\xi}$ on $Q_{\tilde{z}}$ such that a partial derivative of $\rho(\tilde{z},\xi)$ in ξ at $(\tilde{z},\tilde{\xi})$ does not vanish. Hence $\mathcal{M}_{\text{SING}}$ does not contain any Segre variety. Also the standard projection from $\mathcal{M}_{\text{REG}}$ into the z-space is a submersion. Since Q_z is irreducible for $z \in \mathbb{C}^n \backslash (0, ..., 0)$, $Q_z \cap \mathcal{M}_{\text{REG}}$ is connected. To prove the theorem, we just need to show that $\mathcal{M}_{\text{REG}}|_{U\times\mathbb{C}^n}$ is connected. Write the above projection map to the z-space as $\Phi : \mathcal{M}_{\text{REG}}|_{U\times\mathbb{C}^n} \to U$. Since it is a submersion, it is an open mapping. Suppose z^0 is a point in U. As mentioned above, we know that each fiber of Φ is connected. For any $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}}$ in the fiber above z^0, we can choose a connected neighborhood V of (z^0, ξ^0) on $\mathcal{M}_{\text{REG}}|_{U\times\mathbb{C}^n}$ such that $\Phi(V)$ is neighborhood of z_0. Hence, for any $z \in \Phi(V)$, any point in $Q_z \cap \mathcal{M}_{\text{REG}}$ can be connected by a smooth curve inside $\mathcal{M}_{\text{REG}}|_{V\times\mathbb{C}^n}$ to (z^0, ξ^0). Since U is connected, by a standard open-closeness argument, we see that $\mathcal{M}_{\text{REG}}|_{U\times\mathbb{C}^n}$ is connected. $\square$

Appendix A. Affine cell coordinate functions for two exceptional classes of the Hermitian symmetric spaces of compact type

Define the multiplication law of octonions with the standard basis $\{e_0 = 1, e_1, \cdots, e_7\}$ by the following table:

	e_1	e_2	e_4	e_7	e_3	e_6	e_5
e_1	-1	e_4	$-e_2$	$-e_3$	e_7	$-e_5$	e_6
e_2	$-e_4$	-1	e_1	$-e_6$	e_5	e_7	$-e_3$
e_4	e_2	$-e_1$	-1	$-e_5$	$-e_6$	e_3	e_7
e_7	e_3	e_6	e_5	-1	$-e_1$	$-e_2$	$-e_4$
e_3	$-e_7$	$-e_5$	e_6	e_1	-1	$-e_4$	e_2
e_6	e_5	$-e_7$	$-e_3$	e_2	e_4	-1	$-e_1$
e_5	$-e_6$	e_3	$-e_7$	e_4	$-e_2$	e_1	-1

♣1. **Case** M_{16}**:** Define

$$\begin{array}{rcl} x &=& (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7), \\ y &=& (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7). \end{array}$$

Define $A_j(x, y), j = 0, \dots, 7$, such that

$$x\bar{y} = \sum_{j=0}^{7} A_j(x, y)e_j, \text{ where } x = \sum_{j=0}^{7} x_j e_j \text{ and } y = \sum_{j=0}^{7} y_j e_j.$$

Define $B_j(x, y), j = 0, 1$ such that

$$x\bar{x} = B_0(x, y)e_0 \text{ and } y\bar{y} = B_1(x, y)e_0.$$

Then by computation, we have the following formulas:

$$\begin{array}{llcl} A_0 = & A_0(x,y) = & & y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4 + y_5x_5 + y_6x_6 + y_7x_7, \\ A_1 = & A_1(x,y) = & - & y_0x_1 + y_1x_0 - y_2x_4 + y_4x_2 - y_3x_7 + y_7x_3 - y_5x_6 + y_6x_5, \\ A_2 = & A_2(x,y) = & - & y_0x_2 + y_2x_0 - y_4x_1 + y_1x_4 - y_3x_5 + y_5x_3 - y_6x_7 + y_7x_6, \\ A_3 = & A_3(x,y) = & - & y_0x_3 + y_3x_0 + y_1x_7 - y_7x_1 + y_2x_5 - y_5x_2 - y_4x_6 + y_6x_4, \\ A_4 = & A_4(x,y) = & - & y_0x_4 + y_4x_0 - y_1x_2 + y_2x_1 + y_3x_6 - y_6x_3 - y_5x_7 + y_7x_5, \\ A_5 = & A_5(x,y) = & - & y_0x_5 + y_5x_0 + y_1x_6 - y_6x_1 - y_2x_3 + y_3x_2 + y_4x_7 - y_7x_4, \\ A_6 = & A_6(x,y) = & - & y_0x_6 + y_6x_0 - y_1x_5 + y_5x_1 + y_2x_7 - y_7x_2 - y_3x_4 + y_4x_3, \\ A_7 = & A_7(x,y) = & - & y_0x_7 + y_7x_0 - y_1x_3 + y_3x_1 - y_2x_6 + y_6x_2 - y_4x_5 + y_5x_4, \\ B_0 = & B_0(x,y) = & & x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2, \\ B_1 = & B_1(x,y) = & & y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2. \end{array}$$

Then the embedding functions of a Zariski open subset $\mathcal{A}$, which is identified with $\mathbb{C}^{16}$ with coordinates $z := (x_0, \cdots, x_7, y_0, \cdots, y_7)$, of $M_{16} := \frac{E_6}{SO(10)\times SO(2)}$ into $\mathbb{CP}^{26}$ are given by:

$$z \mapsto [1, x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7, B_0, B_1].$$

♣2. **Case** M_{27}**:** Similarly we define

$$\begin{array}{rcl} x &=& (x_1, x_2, x_3), \\ y &=& (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7), \\ t &=& (t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7), \\ \omega &=& (\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7). \end{array}$$

Define functions $A, B, C, D_0, \dots, D_7, E_0 \dots, E_7, F_0 \dots, F_7$ and G such that,

$$\mathrm{Com}(X) = X \times X = \begin{pmatrix} A & D & \overline{E} \\ \overline{D} & B & F \\ E & \overline{F} & C \end{pmatrix}, \ G = \det(X),$$

where $D = \sum_{j=0}^{7} D_j e_j$, $E = \sum_{j=0}^{7} E_j e_j$, $F = \sum_{j=0}^{7} F_j e_j$ and the matrix X corresponding to the point $(x, y, t, w) \in \mathbb{C}^{27}$ is given by

$$X = \begin{pmatrix} x_1 & y & \bar{t} \\ \overline{y} & x_2 & w \\ t & \overline{w} & x_3 \end{pmatrix} \in \mathcal{J}_3(\mathbb{O}).$$

Recall the formulas in [40], we have

$$X \times X = \begin{pmatrix} x_2x_3 - w\overline{w} & \overline{w}\bar{t} - x_3 y & yw - x_2\bar{t} \\ wt - x_3\overline{y} & x_3x_1 - t\bar{t} & \bar{t}\overline{y} - x_1 w \\ \overline{y}\overline{w} - x_2 t & ty - x_1\overline{w} & x_1x_2 - y\overline{y} \end{pmatrix} \in \mathcal{J}_3(\mathbb{O}),$$

$$\det(X) = x_1x_2x_3 - x_1 w\overline{w} - x_2 t\bar{t} - x_3 y\overline{y} + 2\Re^c(wty),$$

where $\Re^c(x) = x_0$ for any $x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$.

By further computation, we have the explicit expressions as follows:

$$\begin{array}{rcl}
A = A(x,y,t,\omega) & = & x_2x_3 - (\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2), \\
B = B(x,y,t,\omega) & = & x_1x_3 - (t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2), \\
C = C(x,y,t,\omega) & = & x_1x_2 - (y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2), \\
D_0 = D_0(x,y,t,\omega) & = & t_0\omega_0 + t_1\omega_1 + t_2\omega_2 + t_3\omega_3 + t_4\omega_4 + t_5\omega_5 + t_6\omega_6 + t_7\omega_7 - x_3y_0, \\
D_1 = D_1(x,y,t,\omega) & = - & t_0\omega_1 + t_1\omega_0 - t_2\omega_4 + t_4\omega_2 - t_3\omega_7 + t_7\omega_3 - t_5\omega_6 + t_6\omega_5 - x_3y_1, \\
D_2 = D_2(x,y,t,\omega) & = - & t_0\omega_2 + t_2\omega_0 - t_4\omega_1 + t_1\omega_4 - t_3\omega_5 + t_5\omega_3 - t_6\omega_7 + t_7\omega_6 - x_3y_2, \\
D_3 = D_3(x,y,t,\omega) & = - & t_0\omega_3 + t_3\omega_0 + t_1\omega_7 - t_7\omega_1 + t_2\omega_5 - t_5\omega_2 - t_4\omega_6 + t_6\omega_4 - x_3y_3, \\
D_4 = D_4(x,y,t,\omega) & = - & t_0\omega_4 + t_4\omega_0 - t_1\omega_2 + t_2\omega_1 + t_3\omega_6 - t_6\omega_3 - t_5\omega_7 + t_7\omega_5 - x_3y_4, \\
D_5 = D_5(x,y,t,\omega) & = - & t_0\omega_5 + t_5\omega_0 + t_1\omega_6 - t_6\omega_1 - t_2\omega_3 + t_3\omega_2 + t_4\omega_7 - t_7\omega_4 - x_3y_5, \\
D_6 = D_6(x,y,t,\omega) & = - & t_0\omega_6 + t_6\omega_0 - t_1\omega_5 + t_5\omega_1 + t_2\omega_7 - t_7\omega_2 - t_3\omega_4 + t_4\omega_3 - x_3y_6, \\
D_7 = D_7(x,y,t,\omega) & = - & t_0\omega_7 + t_7\omega_0 - t_1\omega_3 + t_3\omega_1 - t_2\omega_6 + t_6\omega_2 - t_4\omega_5 + t_5\omega_4 - x_3y_7, \\
E_0 = E_0(x,y,t,\omega) & = & y_0\omega_0 - y_1\omega_1 - y_2\omega_2 - y_3\omega_3 - y_4\omega_4 - y_5\omega_5 - y_6\omega_6 - y_7\omega_7 - x_2t_0, \\
E_1 = E_1(x,y,t,\omega) & = & y_0\omega_1 + y_1\omega_0 + y_2\omega_4 - y_4\omega_2 + y_3\omega_7 - y_7\omega_3 + y_5\omega_6 - y_6\omega_5 - x_2t_1, \\
E_2 = E_2(x,y,t,\omega) & = & y_0\omega_2 + y_2\omega_0 + y_4\omega_1 - y_1\omega_4 + y_3\omega_5 - y_5\omega_3 + y_6\omega_7 - y_7\omega_6 - x_2t_2, \\
E_3 = E_3(x,y,t,\omega) & = & y_0\omega_3 + y_3\omega_0 - y_1\omega_7 + y_7\omega_1 - y_2\omega_5 + y_5\omega_2 + y_4\omega_6 - y_6\omega_4 - x_2t_3, \\
E_4 = E_4(x,y,t,\omega) & = & y_0\omega_4 + y_4\omega_0 + y_1\omega_2 - y_2\omega_1 - y_3\omega_6 + y_6\omega_3 + y_5\omega_7 - y_7\omega_5 - x_2t_4, \\
E_5 = E_5(x,y,t,\omega) & = & y_0\omega_5 + y_5\omega_0 - y_1\omega_6 + y_6\omega_1 + y_2\omega_3 - y_3\omega_2 - y_4\omega_7 + y_7\omega_4 - x_2t_5, \\
E_6 = E_6(x,y,t,\omega) & = & y_0\omega_6 + y_6\omega_0 + y_1\omega_5 - y_5\omega_1 - y_2\omega_7 + y_7\omega_2 + y_3\omega_4 - y_4\omega_3 - x_2t_6, \\
E_7 = E_7(x,y,t,\omega) & = & y_0\omega_7 + y_7\omega_0 + y_1\omega_3 - y_3\omega_1 + y_2\omega_6 - y_6\omega_2 + y_4\omega_5 - y_5\omega_4 - x_2t_7, \\
F_0 = F_0(x,y,t,\omega) & = & y_0t_0 + y_1t_1 + y_2t_2 + y_3t_3 + y_4t_4 + y_5t_5 + y_6t_6 + y_7t_7 - x_1\omega_0, \\
F_1 = F_1(x,y,t,\omega) & = & y_0t_1 - y_1t_0 - y_2t_4 + y_4t_2 - y_3t_7 + y_7t_3 - y_5t_6 + y_6t_5 - x_1\omega_1, \\
F_2 = F_2(x,y,t,\omega) & = & y_0t_2 - y_2t_0 - y_4t_1 + y_1t_4 - y_3t_5 + y_5t_3 - y_6t_7 + y_7t_6 - x_1\omega_2, \\
F_3 = F_3(x,y,t,\omega) & = & y_0t_3 - y_3t_0 + y_1t_7 - y_7t_1 + y_2t_5 - y_5t_2 - y_4t_6 + y_6t_4 - x_1\omega_3, \\
F_4 = F_4(x,y,t,\omega) & = & y_0t_4 - y_4t_0 - y_1t_2 + y_2t_1 + y_3t_6 - y_6t_3 - y_5t_7 + y_7t_5 - x_1\omega_4, \\
F_5 = F_5(x,y,t,\omega) & = & y_0t_5 - y_5t_0 + y_1t_6 - y_6t_1 - y_2t_3 + y_3t_2 + y_4t_7 - y_7t_4 - x_1\omega_5, \\
F_6 = F_6(x,y,t,\omega) & = & y_0t_6 - y_6t_0 - y_1t_5 + y_5t_1 + y_2t_7 - y_7t_2 - y_3t_4 + y_4t_3 - x_1\omega_6, \\
F_7 = F_7(x,y,t,\omega) & = & y_0t_7 - y_7t_0 - y_1t_3 + y_3t_1 - y_2t_6 + y_6t_2 - y_4t_5 + y_5t_4 - x_1\omega_7.
\end{array}$$

$$\begin{aligned}
G = G(x,y,t,\omega) = {} & x_1x_2x_3 - x_1(\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2) \\
& - x_2(t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2) \\
& - x_3(y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2) \\
& + 2\{(y_0\omega_0 - y_1\omega_1 - y_2\omega_2 - y_3\omega_3 - y_4\omega_4 - y_5\omega_5 - y_6\omega_6 - y_7\omega_7)t_0 \\
& + (y_0\omega_1 + y_1\omega_0 + y_2\omega_4 - y_4\omega_2 + y_3\omega_7 - y_7\omega_3 + y_5\omega_6 - y_6\omega_5)t_1
\end{aligned}$$

$$
\begin{aligned}
&+ (y_0\omega_2 + y_2\omega_0 + y_4\omega_1 - y_1\omega_4 + y_3\omega_5 - y_5\omega_3 + y_6\omega_7 - y_7\omega_6)t_2 \\
&+ (y_0\omega_3 + y_3\omega_0 - y_1\omega_7 + y_7\omega_1 - y_2\omega_5 + y_5\omega_2 + y_4\omega_6 - y_6\omega_4)t_3 \\
&+ (y_0\omega_4 + y_4\omega_0 + y_1\omega_2 - y_2\omega_1 - y_3\omega_6 + y_6\omega_3 + y_5\omega_7 - y_7\omega_5)t_4 \\
&+ (y_0\omega_5 + y_5\omega_0 - y_1\omega_6 + y_6\omega_1 + y_2\omega_3 - y_3\omega_2 - y_4\omega_7 + y_7\omega_4)t_5 \\
&+ (y_0\omega_6 + y_6\omega_0 + y_1\omega_5 - y_5\omega_1 - y_2\omega_7 + y_7\omega_2 + y_3\omega_4 - y_4\omega_3)t_6 \\
&+ (y_0\omega_7 + y_7\omega_0 + y_1\omega_3 - y_3\omega_1 + y_2\omega_6 - y_6\omega_2 + y_4\omega_5 - y_5\omega_4)t_7\}.
\end{aligned}
$$

Hence the embedding functions of a Zariski open subset $\mathcal{A}$, which is identified with $\mathbb{C}^{27}$ with coordinates $z := (x, y, t, \omega) = (x_1, x_2, x_3, y_0 \cdots, y_7, t_0, \cdots, t_7, \omega_0, \cdots, \omega_7)$, of $M_{27} := \frac{E_7}{E_6 \times SO(2)}$ into $\mathbb{CP}^{55}$ are given by: $z \mapsto [1, x, y, t, \omega, A, B, C, D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7, E_0, E_1, E_2, E_3, E_4, E_5, E_6, E_7, F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, G]$. The detailed discussions related to this Appendix can be found in [6], [13] and [40].

Appendix B. Proof of Proposition (I) for other types

In this Appendix, we complete the proof of Proposition (I) for spaces of the other type.

B.1. Spaces of type II

In this subsection, we establish Proposition (I) for the orthogonal Grassmannians $G_{II}(n, n)$. As shown in §2, we have a Zariski open affine chart $\mathcal{A} \subset G_{II}(n, n)$ of elements of the form:

$$
(I_{n\times n} \quad Z) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -z_{12} & 0 & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -z_{1n} & -z_{2n} & \cdots & 0 \end{pmatrix}
$$

Here $z = (z_{12}, z_{13}, ..., z_{(n-1)n})$ is the local coordinates for $\mathcal{A} \cong \mathbb{C}^{\frac{n(n-1)}{2}}$. Its conjugate $\mathcal{A}^* \subset (G_{II}(n, n))^*$ is also a copy of $\mathbb{C}^{\frac{n(n-1)}{2}}$. We write the local coordinates for A^* as $\xi = (\xi_{12}, ..., \xi_{(n-1)n})$.

The canonical embedding is given by

$$
(1, ..., \mathrm{pf}(Z_\sigma), ...).
$$

The defining function for the Segre family (in the product of such affine pieces) is given by

$$
\rho(z, \xi) = 1 + \sum_{\substack{\sigma \in S_k, \\ 2 \le k \le n, 2|k}} \mathrm{Pf}(Z_\sigma)\mathrm{Pf}(\Xi_\sigma)
$$

Write

$$r_Z = \left(\mathrm{Pf}(Z_\sigma)_{\sigma\in S_k}\right)_{2\le k\le n, 2|k}. \tag{77}$$

The local biholomorphic map F defined near $0 \in U$ with $F(0) = 0$ can be represented as a matrix:

$$F = \begin{pmatrix} 0 & f_{12} & \dots & f_{1n} \\ -f_{12} & 0 & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ -f_{1n} & \dots & \dots & 0 \end{pmatrix}.$$

Let r_F be

$$r_F = \left(\mathrm{pf}((F)_\sigma)_{\sigma\in S_k}\right)_{2\le k\le n, 2|k}. \tag{78}$$

Under the notation of §2, it is easy to see $r_Z = (\psi_1, ..., \psi_N)$, $r_F = (\psi_1(F), ..., \psi_N(F))$.

We write $\widetilde{z}$ for the z with the last component $z_{(n-1)n}$ dropped. More precisely,

$$\widetilde{z} = (z_{12}, ..., z_{1n}, z_{23}, ..., z_{2n}, ..., z_{(n-2)(n-1)}, z_{(n-2)n}). \tag{79}$$

Recall z has $n' = n(n-1)/2$ independent variables. Thus $\widetilde{z}$ has $(n'-1)$ components. We define the $\widetilde{z}-$rank and $\widetilde{z}-$nondegeneracy as in Definition 3.1 using $\psi = r_F$ in (78) and $\widetilde{z}$ as in (79) with $m = n'$, respectively. We now prove the following:

Proposition B.1. *r_F is $\widetilde{z}$–nondegenerate near 0. More precisely,* $\mathrm{rank}_{1+N-n'}(r_F, \widetilde{z}) = N$.

Proof of Proposition B.1: Suppose not. Without loss of generality, we assume that

$$\mathrm{rank}_{1+N-n'}(r_F, \widetilde{z}) < N.$$

As a consequence of Theorem 3.10, there exist $c_{\sigma,k} \in \mathbb{C}, 4 \le k \le n, 2|k, \sigma \in S_k$, which are not all zero, such that

$$\sum_{4\le k\le n, 2|n} \sum_{\sigma\in S_k} c_{\sigma,k}\ \mathrm{pf}((F)_\sigma)(z_{12}, ..., z_{(n-2)n}, 0)) \equiv 0. \tag{80}$$

However, (80) cannot hold by the following lemma, which gives a contradiction:

Lemma B.2. *Let*

$$H = \begin{pmatrix} 0 & h_{12} & \dots & h_{1n} \\ -h_{12} & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -h_{1n} & \dots & \dots & 0 \end{pmatrix}$$

be an anti-symmetric matrix-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (z_{12}, ..., z_{(n-2)n}) \in \mathbb{C}^{n'-1}$ with $H(0) = 0$. Assume that H is of full rank at 0. Set r_H similar to the definition of r_F,

$$r_H = \left(\mathrm{pf}(H_\sigma)_{\sigma\in S_k}\right)_{2\le k\le n, 2|k}. \tag{81}$$

Assume that $a_{\sigma,k}, \sigma \in S_k, 4 \le k \le n$, are complex numbers such that

$$\sum_{4\le k\le n, 2|k} \sum_{\sigma\in S_k} a_{\sigma,k}\ \mathrm{pf}(H_\sigma)(z_{12}, ..., z_{(n-2)n})) \equiv 0 \text{ for all } \widetilde{z} \in U. \tag{82}$$

Then

$$a_{\sigma,k} = 0$$

for all $\sigma \in S_k, 4 \le k \le n, 2|k$.

Proof of Lemma B.2: Suppose not. We will prove the lemma by seeking a contradiction. Note that H has full rank at 0. Hence there exist $(n' - 1)$ components $\widehat{H}$ of H that forms a local biholomorphism from $\mathbb{C}^{n'-1}$ to $\mathbb{C}^{n'-1}$. We assume that these $(n' - 1)$ components $\widehat{H}$ are H with $h_{i_0 j_0}$ being dropped, where $i_0 < j_0$. Without loss of generality, we assume $i_0 = n - 1, j_0 = n$. By a local biholomorphic change of coordinates, we assume $\widehat{H} = \widetilde{z} = (z_{12}, ..., z_{(n-2)n})$. We still write the missing component as $h_{(n-1)n}$. Now we assume $2(m + 1), m \ge 1$, is the least number k such that $\{a_{\sigma,k}\}_{\sigma\in S_k}$ are not all zero. We then consider $\{a_{\sigma,2(m+1)}\}_{\sigma\in S_{2(m+1)}}$. We first claim that $a_{\sigma,2(m+1)} = 0$ for those $\sigma \in S_{2(m+1)}$ such that $\mathrm{pf}(H_\sigma)$ involves $h_{(n-1)n}$. More precisely, if $\mathrm{pf}(H_\sigma), \sigma \in S_{2(m+1)}$ involves $h_{(n-1)n}$, then $\sigma = \{i_1, ..., i_{2m}, (n-1), n\}$ for some $1 \le i_1 < ... < i_{2m} \le n-2$. Suppose its coefficient is not zero. Then $\mathrm{pf}(H_\sigma)$ will produce the monomial $z_{i_1 i_2} z_{i_3 i_4} ... z_{i_{2m-3} i_{2m-2}} z_{i_{2m-1}(n-1)} z_{i_{2m} n}$. This term can only be canceled by the terms of form: $z_{i_{2m-1}(n-1)} h_{(n-1)n} Q$ or $z_{i_{2m} n} h_{(n-1)n} Q$. But neither of them can appear in any other Pfaffians. This is a contradiction. Once we know there are no $h_{(n-1)n}$ involved, then the remaining Pfaffians have only terms consisting of the product of some of $z_{12}, ..., z_{(n-2)n}$. Their sum cannot be zero unless their coefficients are all zero. This is a contradiction. We thus establish Lemma B.2. □

We thus also get a contradiction to equation (80). This establishes Proposition B.1. □

Remark B.3. By Proposition B.1, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^N$ with all $|\tilde{\beta}^j| \le 1 + N - n'$, and there is a point

$$z^0 = \begin{pmatrix} 0 & z^0_{12} & ... & z^0_{1(n-1)} & z^0_{1n} \\ -z^0_{12} & 0 & ... & z^0_{2(n-1)} & z^0_{2n} \\ ... & ... & ... & ... & ... \\ -z^0_{1(n-1)} & -z^0_{2(n-1)} & ... & 0 & z^0_{(n-1)n} \\ -z^0_{1n} & -z^0_{2n} & ... & -z^0_{(n-1)n} & 0 \end{pmatrix}, z^0_{(n-1)n} \neq 0;$$

near 0 such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \bar{z}^{\tilde{\beta}^1}} & ... & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \bar{z}^{\tilde{\beta}^1}} \\ ... & ... & ... \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \bar{z}^{\tilde{\beta}^N}} & ... & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \bar{z}^{\tilde{\beta}^N}} \end{vmatrix} (z^0) \neq 0. \tag{83}$$

We set

$$\xi^0 = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & \xi^0_{(n-1)n} \\ 0 & 0 & ... & -\xi^0_{(n-1)n} & 0 \end{pmatrix} \in \mathbb{C}^{n^2}, \xi^0_{(n-1)n} = -\frac{1}{z^0_{(n-1)n}}.$$

Then it is easy to see that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z,\xi) = 0\}$.

Write for each $1 \leq i < j \leq n, (i,j) \neq (n-1,n)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z,\xi)}{\frac{\partial \rho}{\partial z_{(n-1)n}}(z,\xi)} \frac{\partial}{\partial z_{(n-1)n}} \tag{84}$$

which are holomorphic tangent vector fields along $\mathcal{M}$ near (z^0, ξ^0). Here we note that $\frac{\partial \rho}{\partial z_{(n-1)n}}(z,\xi)$ is nonzero near (z^0, ξ^0). For any $(n'-1)$-multiindex $\beta = (\beta_{12}, ..., \beta_{(n-2)n})$, we write

$$\mathcal{L}^\beta = \mathcal{L}_{12}^{\beta_{12}} ... \mathcal{L}_{(n-2)n}^{\beta_{(n-2)n}}.$$

Now we define for any N collection of $(n'-1)-$multiindices $\{\beta^1, ..., \beta^N\}$,

$$\Lambda(\beta^1, ..., \beta^N)(z,\xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & ... & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix} (z,\xi). \tag{85}$$

Note that for any multiindex $\beta, \mathcal{L}^\beta$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \bar{z}^\beta}$. We thus again have

Theorem B.4. *There exist multiindices $\{\beta^1, ..., \beta^N\}$, such that*

$$\Lambda(\beta^1, ..., \beta^N)(z,\xi) \neq 0,$$

for (z, ξ) *in a small neighborhood of* (z^0, ξ^0) *and* $\beta^1 = (0, ..., 0)$.

B.2. Spaces of type III

Let F be a local biholomorphic map at 0. In this case, both Z and F are $n \times n$ symmetric matrices. We write

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{12} & z_{22} & \dots & z_{2n} \\ \dots & \dots & \dots & \dots \\ z_{1n} & z_{2n} & \dots & z_{nn} \end{pmatrix}, \quad z = (z_{11}, z_{12}, z_{13}, ..., z_{nn}).$$

Similar notations are used for F.

Recall from (13) of ♣3 in §2:

$$r_z = \left(\psi^1_1(z), ..., \psi^1_{N_1}(z), \psi^2_1(z), ..., \psi^2_{N_2}(z), ..., \psi^{n-1}_1(z), ..., \psi^{n-1}_{N_{n-1}}(z), \psi^n(z)\right), \tag{86}$$

where ψ^k_j is a homogeneous polynomial of degree $k, 1 \leq j \leq N_k$. ψ^n is a homogeneous polynomial of degree n. Moreover, the components of r_z are linearly independent.

We write the number of components in r_z to be $N = N_1 + ... + N_n$, where we set $N_n = 1$. We will also sometimes write $\psi^n_{N_n} = \psi^n$.

We emphasize that for each fixed $k, \psi^k_1, ..., \psi^k_{N_k}$ are linearly independent. Moreover, each ψ^k_j is a certain linear combination of the determinants of $k \times k$ submatrices of Z. This will be crucial for our argument later.

We define r_F as the composition of r_z with the map F:

$$r_F = \left(\psi^1_1(F), ..., \psi^1_{N_1}(F), \psi^2_1(F), ..., \psi^2_{N_2}(F), ..., \psi^{n-1}_1(F), ..., \psi^{n-1}_{N_{n-1}}(F), \psi^n(F)\right). \tag{87}$$

In what follows, we write also $z_{ij} = z_{ji}$. We write $\det(A)$ as the determinant of A when A is a square matrix.

Let $P, \widetilde{P}$ be monomials in z'_{ij}s, and h a polynomial in z'_{ij}s. Let a, b be two complex numbers. In the following lemmas, when we say h always has the terms $aP, b\widetilde{P}$, we mean h has the term aP if and only if it has the term $b\widetilde{P}$.

Lemma B.5. *Fixing* $1 \leq i, j < n$, *let* $P = z_{in}z_{nj}Q$ *and* $\widetilde{P} = z_{ij}z_{nn}Q$ *with* Q *a monomial in* z'_{ij}*s. The following statements are true.*

- *Let* A *be a square submatrix of* Z. *If* $z_{ij} \nmid Q$, *then* $\det(A)$ *always has monomials of the form* $cP, -c\widetilde{P}$ *for some* $c \in \mathbb{C}$ *depending on the submatrix* A. *(If* $\det(A)$ *does not have any multiple of* P, *it does not have any multiple of* $\widetilde{P}$, *either; vice versa.) If* $z_{ij} | Q$, *then* $\det(A)$ *always has monomials* $cP, -(c/2)\widetilde{P}$ *for some* $c \in \mathbb{C}$ *depending on* A.

- *Let $k \geq 1$. Let $\psi_l^k(z)$ be as defined in (86), $1 \leq l \leq N_k$. If $z_{ij} \nmid Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -\lambda \widetilde{P}$ for some $\lambda \in \mathbb{C}$. If $z_{ij}|Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -(\lambda/2)\widetilde{P}$ for some $\lambda \in \mathbb{C}$.*

Proof of Lemma B.5: The first part is a consequence of the Laplace expansion of a determinant by complementary minors. The second part is due to the fact that ψ_j^k is a linear combination of the determinants of submatrices of Z of order k. □

Similarly, one can prove in a similar way Lemmas B.6–B.8.

Lemma B.6. *Fixing $1 \leq j < n-1$, let $P = z_{jn}z_{(n-1)(n-1)}Q$ and $\widetilde{P} = z_{j(n-1)}z_{(n-1)n}Q$ with Q a monomial in $z_{ij}'s$.*

- *Let A be a square submatrix of Z. If $z_{jn} \nmid Q$, then $\det(A)$ always has monomials $cP, -c\widetilde{P}$ for some $c \in \mathbb{C}$. If $z_{jn}|Q$, then $\det(A)$ always has monomials $cP, -2c\widetilde{P}$ for some $c \in \mathbb{C}$.*
- *Let $k \geq 1$. Let $\psi_l^k(z)$ be as defined in (86), $1 \leq l \leq N_k$. If $z_{jn} \nmid Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -\lambda \widetilde{P}$ for some $\lambda \in \mathbb{C}$. If $z_{jn}|Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -2\lambda\widetilde{P}$ for some $\lambda \in \mathbb{C}$.*

Lemma B.7. *Fixing $1 \leq i < n-1$, let $P = z_{i(n-1)}z_{ni}Q$ and $\widetilde{P} = z_{ii}z_{(n-1)n}Q$ with Q a monomial in $z_{ij}'s$.*

- *Let A be a square submatrix of Z. If $z_{(n-1)n} \nmid Q$, then $\det(A)$ always has monomials $cP, -c\widetilde{P}$ for some $c \in \mathbb{C}$. If $z_{(n-1)n}|Q$, then $\det(A)$ always has monomials $cP, -(c/2)\widetilde{P}$ for some $c \in \mathbb{C}$.*
- *Let $k \geq 1$. Let $\psi_l^k(z)$ be as defined in (86), $1 \leq l \leq N_k$. If $z_{(n-1)n} \nmid Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -\lambda\widetilde{P}$ for some $\lambda \in \mathbb{C}$. If $z_{(n-1)n}|Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -(\lambda/2)\widetilde{P}$ for some $\lambda \in \mathbb{C}$.*

Lemma B.8. *Fixing $1 \leq i < n-1, 1 \leq j < n-1, i \neq j$, let $P_1 = z_{i(n-1)}z_{nj}Q, P_2 = z_{in}z_{j(n-1)}Q$, and $\widetilde{P} = z_{ij}z_{(n-1)n}Q$ with Q a monomial in $z_{ij}'s$.*

- *Let A be a square submatrix of Z. If $z_{ij} \nmid Q, z_{(n-1)n} \nmid Q$, then $\det(A)$ always has terms $c_1P_1 + c_2P_2, -(c_1+c_2)\widetilde{P}$ for some $c_1, c_2 \in \mathbb{C}$. If $z_{ij} \nmid Q, z_{(n-1)n}|Q$, or $z_{ij}|Q, z_{(n-1)n} \nmid Q$, then $\det(A)$ always has terms $c_1P_1 + c_2P_2, -\frac{c_1+c_2}{2}\widetilde{P}$ for some $c_1, c_2 \in \mathbb{C}$. If $z_{ij}|Q, z_{(n-1)n}|Q$, then $\det(A)$ always has terms $c_1P_1 + c_2P_2, -\frac{c_1+c_2}{4}\widetilde{P}$.*
- *Let $k \geq 1$. Let $\psi_l^k(z)$ be as defined in (86), $1 \leq l \leq N_k$. If $z_{ij} \nmid Q$ and $z_{(n-1)n} \nmid Q$, then $\psi_l^k(z)$ always has terms $\lambda_1P_1 + \lambda_2P_2, -(\lambda_1+\lambda_2)\widetilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. If $z_{ij} \nmid Q, z_{(n-1)n}|Q$, or $z_{ij}|Q, z_{(n-1)n} \nmid Q$, then $\psi_l^k(z)$ always has terms $\lambda_1P_1 + \lambda_2P_2, -\frac{\lambda_1+\lambda_2}{2}\widetilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. If $z_{ij}|Q, z_{(n-1)n}|Q$, then $\psi_l^k(z)$ always has terms $\lambda_1P_1 + \lambda_2P_2, -\frac{\lambda_1+\lambda_2}{4}\widetilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$.*

We write $\widetilde{z}$ for z with the last components z_{nn} being dropped. More precisely,

$$\widetilde{z} = (z_{11}, ..., z_{1n}, z_{22}, ..., z_{2n}, ..., z_{(n-1)(n-1)}, z_{(n-1)n}). \tag{88}$$

Recall z has $n' = n(n+1)/2$ independent variables. Thus $\widetilde{z}$ has $(n'-1)$ components. We define $\widetilde{z}-$rank and $\widetilde{z}-$nondegeneracy in the same way as before, using r_F in (87) and $\widetilde{z}$ in (88) with $m = n'$. We now need to prove the following:

Proposition B.9. *r_F is $\widetilde{z}-$nondegenerate at 0. More precisely,* $\mathrm{rank}_{1+N-n'}(r_F, \widetilde{z}) = N$*.*

Proof of Proposition B.9: Suppose not. Then one easily verifies that the hypothesis of Theorem 3.10 is satisfied. As a consequence of Theorem 3.10, there exist $c_j^k \in \mathbb{C}, 2 \leq k \leq n, 1 \leq j \leq N_k$, which are not all zero such that

$$\sum_{k=2}^{n} \sum_{j=1}^{N_k} c_j^k \psi_j^k(F(z_{11}, ..., z_{(n-1)n}, 0)) \equiv 0. \tag{89}$$

Here as before, we write $N_n = 1, \psi_{N_n}^n = \psi^n$.

Then we just need to show it can not happen by the following lemma:

Lemma B.10. *Let*

$$H = \begin{pmatrix} h_{11} & h_{12} & ... & h_{1n} \\ h_{12} & h_{22} & ... & h_{2n} \\ ... & ... & ... & ... \\ h_{1n} & ... & ... & h_{nn} \end{pmatrix}$$

be a symmetric matrix-valued holomorphic function near 0 *in* $\widetilde{z} = (z_{11}, ..., z_{1n}, z_{22}, ..., z_{2n}, ..., z_{(n-1)n}) \in \mathbb{C}^{n'-1}$ *with* $H(0) = 0$*. Assume that* H *is of full rank at* 0*. Set* r_H *in a similar way as in (36):*

$$r_H = \Big(\psi_1^1(H), ..., \psi_{N_1}^1(H), \psi_1^2(H), ..., \psi_{N_2}^2(H), ..., \psi_1^{n-1}(H), ..., \psi_{N_{n-1}}^{n-1}(H), \psi^n(H)\Big)$$

Again we write $N_n = 1, \psi^n = \psi_{N_n}^n$*. Assume that* $a_j^k, 2 \leq k \leq n, 1 \leq j \leq n$ *are complex numbers such that*

$$\sum_{k=2}^{n} \sum_{j=1}^{N_k} a_j^k \psi_j^k(H(\widetilde{z})) \equiv 0 \quad \text{for} \ \ \widetilde{z} \in U. \tag{90}$$

Then

$$a_j^k = 0$$

for each $2 \leq k \leq n, 1 \leq j \leq N_k$*.*

Proof of Lemma B.10: Suppose not. We will prove the lemma by seeking a contradiction. Notice that H has full rank at 0. Hence there exist $(n'-1)$ components $\widehat{H}$ of H that gives a local biholomorphism from $\mathbb{C}^{n'-1}$ to $\mathbb{C}^{n'-1}$. We assume these $(n'-1)$ components $\widehat{H}$ are H with $h_{i_0 j_0}$ being dropped, where $i_0 \leq j_0$. Then we split our argument into two parts in terms of $i_0 = j_0$ or $i_0 < j_0$.

Case I: Assume that $i_0 = j_0$. Without loss of generality, we assume $i_0 = j_0 = n$. By a local biholomorphic change of coordinates, we assume $\widehat{H} = \widetilde{z} = (z_{11}, ..., z_{n(n-1)})$. We still write the last component as h_{nn}. Now we assume $m \geq 2$ is the least number k such that $\{a_1^k, ..., a_{N_k}^k\}$ are not all zero. For any holomorphic g, we define $T_l(g)$ to be the homogeneous part of degree l in the Taylor expansion of g at 0. Now the assumption in (90) yields:

$$T_m \left(\sum_{j=1}^{N_m} a_j^m \psi_j^m (H(\widetilde{z})) \right) \equiv 0. \tag{91}$$

We first compute

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m (H) = \sum_{j=1}^{N_m} a_j^m \psi_j^m (z_{11}, ..., z_{(n-1)n}, h_{nn})$$

formally. Namely, we regard h_{nn} as a formal variable and only conduct formal cancellations. We write formally

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m (z_{11}, ..., z_{(n-1)n}, h_{nn}) = P_1 + h_{nn} P_2. \tag{92}$$

Here $P_1 = P_1(z_{11}, ..., z_{(n-1)n})$ is a homogeneous polynomial of degree m, and $P_2 = P_2(z_{11}, ..., z_{(n-1)n})$ is a homogeneous polynomial of degree $m-1$. We claim $P_2 \neq 0$. Otherwise,

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m (z_{11}, ..., z_{(n-1)n}, h_{nn}) = P_1.$$

This implies that $\sum_{j=1}^{N_m} a_j^m \psi_j^m (z_{11}, ..., z_{(n-1)n}, h_{nn})$ does not depend on h_{nn} formally. Then we can replace h_{nn} by z_{nn}. That is,

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m (z_{11}, ..., z_{(n-1)n}, z_{nn}) = \sum_{j=1}^{N_m} a_j^m \psi_j^m (z_{11}, ..., z_{(n-1)n}, h_{nn}(\widetilde{z})) = P_1. \tag{93}$$

By (91), we see that (93) is identically zero. This is a contradiction to the fact that $\{\psi_1^m, ..., \psi_{N_m}^m\}$ is linearly independent.

Now since $P_2 \neq 0$, thus by (92), $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn})$ has a monomial of the form $\mu\widetilde{P} = \mu z_{ij} h_{nn} Q$ of degree m for some $1 \leq i, j < n, \mu \neq 0$ and some monomial Q. By Lemma B.5, we get that $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn})$ has also the term $-\mu P$ or $-2\mu P$, where $P = z_{in} z_{nj} Q$. This is a contradiction to (91). Indeed, P can be only canceled by the terms of the forms: $z_{in} h_{nn} \widetilde{Q}$ or $z_{nj} h_{nn} \widetilde{Q}$, where $\widetilde{Q}$ is of degree $m-2$. But they cannot appear in determinant of any submatrix of H as z_{in}(or z_{nj}) can not appear with h_{nn}.

Case II: Assume that $i_0 \neq j_0$. Without loss of generality, we assume $i_0 = (n-1), j_0 = n$. Then $\widehat{H} = (h_{11}, ..., h_{(n-1)(n-1)}, h_{nn})$ is a local biholomorphism. By a local biholomorphic change of coordinates, we assume $\widehat{H} = \widetilde{z} = (z_{11}, ..., z_{(n-1)n})$. We will still write the remaining component as $h_{(n-1)n} = h_{n(n-1)}$. Note that the fact we are using only is that $\{z_{11}, ..., z_{(n-1)n}\}$ are independent variables. Hence, to make our notation easier, we will write

$$\widehat{H} = (z_{11}, ..., z_{(n-1)n}) = (w_{11}, ..., w_{1n}, w_{22}, ..., w_{2n}, ..., w_{(n-1)(n-1)}, w_{nn})$$

such that they have the same indices as h's in $\widehat{H}$. Now we assume m is the least number k such that $\{a_1^k, ..., a_{N_k}^k\}$ are not all zero. Then again assumption (90) yields that

$$T_m\left(\sum_{j=1}^{N_m} a_j^m \psi_j^m(H(\widetilde{Z}))\right) \equiv 0. \tag{94}$$

Again we formally compute that

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(w_{11}, ..., h_{(n-1)n}, w_{nn}) = Q_1 + h_{(n-1)n} Q_2. \tag{95}$$

Here $Q_1 = Q_1(w_{11}, ..., w_{(n-1)(n-1)}, w_{nn})$ is a homogeneous polynomial of degree m. Similarly, we can show that $Q_2 \neq 0$. We claim that (95) does not have a monomial of the form $h_{(n-1)n} h_{(n-1)n} Q$. Otherwise, by Lemma B.5, we get that (95) has also a monomial of degree m of the form: $w_{(n-1)(n-1)} w_{nn} Q$. But note that in (95) it can be canceled only by $h_{(n-1)n} h_{(n-1)n} Q$. Then $h_{(n-1)n}$ will have a linear term $w_{(n-1)(n-1)}$. But then $h_{(n-1)n} h_{(n-1)n} Q$ will produce the term $w_{(n-1)(n-1)} w_{(n-1)(n-1)} Q$. This cannot be canceled out by any other terms.

Now since $Q_2 \neq 0$, (95) has a monomial of the form $w_{ij} h_{(n-1)n} Q$, where Q is another monomial in w's. Here $1 \leq i, j \leq n$. Moreover, $(i, j) \neq (n-1, n-1), (n-1, n), (n, n-1)$ or (n, n). We first assume $1 \leq i, j < n-1, i \neq j$. Then by Lemma B.8, (95) has either P_1 or P_2, where $P_1 = w_{i(n-1)} w_{nj} Q, P_2 = w_{in} w_{j(n-1)} Q$. Note P_1, P_2 can only be canceled by the terms $w_{i(n-1)} h_{(n-1)n} Q, w_{nj} h_{(n-1)n} Q, w_{in} h_{(n-1)n} Q, w_{j(n-1)} h_{(n-1)n} Q$. So one of them will appear in (95). Whichever case it is, by Lemmas B.5, B.6, (95) will have either $P = w_{ln} w_{(n-1)(n-1)} Q$, or $\widehat{P} = w_{l(n-1)} w_{nn} Q$ for some $1 \leq l < n$. We assume, for instance,

(95) has the monomial P. Then it also has the monomial $\widetilde{P} = w_{l(n-1)}h_{(n-1)n}Q$ by Lemma B.6. Note that the only term that can cancel P and appear in some determinant is $w_{ln}h_{n(n-1)}Q$. Hence $h_{n(n-1)}$ has a linear $w_{(n-1)(n-1)}$ term. Then $\widetilde{P}$ will have the monomial $w_{l(n-1)}w_{(n-1)(n-1)}Q$, which can not be canceled by any other terms. This is a contradiction. The other cases can be proved similarly. □

This establishes Proposition B.9. □

Remark B.11. By Proposition B.9, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^N$ with $|\tilde{\beta}^j| \leq 1 + N - pq$, and there exist

$$z^0 = \begin{pmatrix} z^0_{11} & ... & z^0_{1n} \\ ... & ... & ... \\ z^0_{1n} & ... & z^0_{nn} \end{pmatrix}, \; z^0_{nn} \neq 0,$$

near 0 such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{Z}^{\tilde{\beta}^1}} & ... & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \tilde{Z}^{\tilde{\beta}^1}} \\ ... & ... & ... \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \tilde{Z}^{\tilde{\beta}^N}} & ... & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \tilde{Z}^{\tilde{\beta}^N}} \end{vmatrix} (z^0) \neq 0. \tag{96}$$

Here we simply write $r_F = (\psi_1(F), ..., \psi_N(F))$.

We then set

$$\xi^0 = \begin{pmatrix} 0 & ... & 0 & 0 \\ 0 & ... & 0 & 0 \\ 0 & ... & 0 & \xi^0_{nn} \end{pmatrix} \in \mathbb{C}^{n^2}, \xi^0_{nn} = -\frac{1}{z^0_{nn}}.$$

It is easy to verify that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$.

Write for each $1 \leq i \leq j \leq n, (i,j) \neq (n,n)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z,\xi)}{\frac{\partial \rho}{\partial z_{nn}}(z,\xi)} \frac{\partial}{\partial z_{nn}}, \tag{97}$$

which are holomorphic tangent vector fields along $\mathcal{M}$ near (z^0, ξ^0). Here we note that $\frac{\partial \rho}{\partial z_{nn}}(z,\xi)$ is nonzero near (z^0, ξ^0). For any $(n'-1)$-multiindex $\beta = (\beta_{11}, ..., \beta_{(n-1)n})$, we write

$$\mathcal{L}^\beta = \mathcal{L}^{\beta_{11}}_{11} ... \mathcal{L}^{\beta_{(n-1)n}}_{(n-1)n}.$$

Now we define for any N collection of $(n'-1)-$multiindices $\{\beta^1, ..., \beta^N\}$,

$$\Lambda(\beta^1, ..., \beta^N)(z,\xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & ... & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix} (z,\xi). \tag{98}$$

Note $\mathcal{L}^\beta$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \widetilde{Z}^\beta}$. We have

Theorem B.12. *There exist multiindices* $\{\beta^1, ..., \beta^N\}$ *such that* $\Lambda(\beta^1, ..., \beta^N)(z, \xi) \neq 0$ *for* (z, ξ) *in a small neighborhood of* (z^0, ξ^0) *and* $\beta^1 = (0, 0, ..., 0)$.

B.3. The exceptional class M_{27}

In this setting, we use the coordinates

$$z = (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7) \in \mathbb{C}^{27}.$$

The defining function of the Segre family described in (17) is:

$$\rho(z, \xi) = 1 + r_z \cdot r_\xi = 1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi), \quad \text{where } N = 55 \text{ and}$$

$$r_z = (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7, A, B, C, D_0, ...D_7, E_0, ..., E_7, F_0, ..., F_7, G). \tag{99}$$

Here A, B, C, D_i, E_i, F_i are homogeneous quadratic polynomials in z and G is a homogeneous cubic polynomial in z:

$$A = x_2 x_3 - \sum_{i=0}^{7} w_i^2, B = x_1 x_3 - \sum_{i=0}^{7} t_i^2, C = x_1 x_2 - \sum_{i=0}^{7} y_i^2. \tag{100}$$

For the expressions for D_i, E_i, F_i, G, see Appendix A. Let F be a local biholomorphic map near 0. We write

$$F = (\phi_1, \phi_2, \phi_3, f_{10}, ..., f_{17}, f_{20}, ..., f_{27}, f_{30}, ..., h_{37}).$$

Also define r_F to be the composition of r_z with F:

$$r_F = r_z \circ F = (\phi_1, \phi_2, \phi_3, f_{10}, ..., f_{17}, f_{20}, ..., f_{27}, f_{30}, ..., f_{37}, A(F), B(F), C(F), ..., G(F)). \tag{101}$$

Notice that r_F has 55 components. We will also write

$$r_F = (\psi_1(F), ..., \psi_{55}(F)).$$

We write $\widetilde{z}$ for z with x_3 being dropped. Namely,

$$\widetilde{z} = (x_1, x_2, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7). \tag{102}$$

We define the $\widetilde{z}-$rank and $\psi-$nondegeneracy as in Definition 3.1 using r_F in (101) and $\widetilde{z}$ in (102) with $m = 27$.

Proposition B.13. *F is $\widetilde{z}-$nondegenerate near 0. More precisely,* $\mathrm{rank}_{29}(F, \widetilde{z}) = 55$.

Proof of Proposition B.13: Suppose not. As a consequence of Theorem 3.10, there exist $c_1, ..., c_{28} \in \mathbb{C}$ that are not all zero, such that

$$c_1 A(F(x_1, x_2, 0, y_0, ..., w_7)) + ... + c_{28} G(F(x_1, x_2, 0, y_0, ..., w_7)) \equiv 0. \tag{103}$$

We will show that (103) cannot hold by the following lemma:

Lemma B.14. *Let $H = (\psi_1, \psi_2, \psi_3, h_{10}, ..., h_{17}, h_{20}, ..., h_{27}, h_{30}, ..., h_{37})$ be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (x_1, x_2, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7) \in \mathbb{C}^{26}$ with $H(0) = 0$. Assume that H has full rank at 0. Assume that $a_1, ..., a_{28}$ are complex numbers such that*

$$a_1 A(H(\widetilde{z})) + ... + a_{28} G(H(\widetilde{z})) = 0 \text{ for all } \widetilde{z} \in U. \tag{104}$$

Then $a_i = 0$ for all $1 \leq i \leq 28$.

Proof of Lemma B.14: Suppose not. Notice that H has full rank at 0. Hence there exist 26 components $\widehat{H}$ of H that give a local biholomorphism from $\mathbb{C}^{26}$ to $\mathbb{C}^{26}$. We assume these 26 components $\widehat{H}$ are the H with η dropped, where $\eta \in \{\psi_1, \psi_2, \psi_3, h_{10}, ..., h_{17}, h_{20}, ..., h_{27}, h_{30}, ..., h_{37}\}$. By a local biholomorphic change of coordinates, we assume

$$\widehat{H} = (x_1, x_2, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7).$$

We still write the remaining components as η.

Case I: If $\eta \in \{\psi_1, \psi_2, \psi_3\}$, without loss of generality, we can assume $\eta = \psi_3$. We first claim that the coefficients of A, B, i.e., a_1, a_2 are zero. This is due to the fact that $t_i^2, w_i^2, 0 \leq i \leq 7$ can only be canceled by $t_i\psi_3, w_i\psi_3$, which do not appear in the expressions of $A(H), ..., G(H)$. We then claim the coefficients of C are zero, for x_1x_2 can not be canceled. Then the coefficients of all D's have to be zero, for each t_iw_j is unique and can not be canceled. Then it follows trivially that all other coefficients are zero.

Case II: If $\eta \in \{h_{10}, ..., h_{17}, h_{20}, ..., h_{27}, h_{30}, ..., h_{37}\}$, without loss of generality, we assume $\eta = h_{37}$. Notice that the only fact we are using about $\widehat{H}$ is that its components are independent variables. For simplicity of notation, we will write

$$\widehat{H} = (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_6).$$

We first claim that the coefficient of A is zero. This is due to the fact that x_2x_3 cannot be canceled. We also claim that the coefficient of B is zero. Suppose not. Notice that t_i^2 can only be canceled by t_ih_{37}. Then the coefficient of each D_i is non zero for $0 \leq i \leq 7$. Moreover, x_1x_3 can only be canceled by x_1h_{37}. This implies h_{37} has a linear x_3-term.

Then, in particular, the t_7h_{37} term in D_0 will produce a t_7x_3 term. It cannot be canceled by any other terms. This is a contradiction. Similarly, one can show that the coefficient of C is zero. Then we claim the coefficient of D_0 is zero. Otherwise, to cancel the x_3y_0 term, h_{37} needs have a linear x_3 term. Then the term t_7h_{37} in D_0 will produce a t_7x_3 term, which cannot be canceled by any other term. By the same argument, one can show that the coefficients of all $D_i, 0 \le i \le 7$, are zero. Similarly, we can obtain the coefficients of all $E_i, 0 \le i \le 7$, are zero. Then we claim the coefficients of all F's have to be zero. This is because each y_it_j is unique. It can not be canceled out. Finally we get the coefficient of G to be zero. □

This also establishes Proposition B.13. □

Remark B.15. By Proposition B.13, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^{55}$ with $|\tilde{\beta}^j| \le 29$, and there exist

$$z^0 = (x_1^0, x_2^0, x_3^0, y_0^0, ..., y_7^0, t_0^0, .., t_7^0, w_0^0, ..., w_7^0), \quad x_3^0 \ne 0,$$

such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{z}^{\tilde{\beta}^1}} & ... & \frac{\partial^{|\beta^1|}(\psi_{55}(F))}{\partial \tilde{z}^{\tilde{\beta}^1}} \\ ... & ... & ... \\ \frac{\partial^{|\beta^{55}|}(\psi_1(F))}{\partial \tilde{z}^{\tilde{\beta}^{55}}} & ... & \frac{\partial^{|\beta^{55}|}(\psi_{55}(F))}{\partial \tilde{z}^{\tilde{\beta}^{55}}} \end{vmatrix} (z^0) \ne 0.$$

Then we set $\xi^0 = (0, 0, \xi_3^0, 0, ...0, 0, ..., 0, 0, ..., 0), \xi_3^0 = -\frac{1}{x_3^0}$. It is easy to see that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$. Write

$$\mathcal{L}_i = \frac{\partial}{\partial x_i} - \frac{\frac{\partial \rho}{\partial x_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 1 \le i \le 2;$$

$$\mathcal{L}_{3+i} = \frac{\partial}{\partial y_i} - \frac{\frac{\partial \rho}{\partial y_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 0 \le i \le 7;$$

$$\mathcal{L}_{11+i} = \frac{\partial}{\partial t_i} - \frac{\frac{\partial \rho}{\partial t_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 0 \le i \le 7;$$

$$\mathcal{L}_{19+i} = \frac{\partial}{\partial w_i} - \frac{\frac{\partial \rho}{\partial w_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 0 \le i \le 7.$$

For any 26-multiindex $\beta = (\beta_1, ..., \beta_{26})$, we write $\mathcal{L}^\beta = \mathcal{L}_1^{\beta_1} ... \mathcal{L}_{26}^{\beta_{26}}$. Now we define for any 55 collection of 26−multiindices $\{\beta^1, ..., \beta^{55}\}$,

$$\Lambda(\beta^1, ..., \beta^{55})(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_{55}(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^{55}}(\psi_1(F)) & ... & \mathcal{L}^{\beta^{55}}(\psi_{55}(F)) \end{vmatrix} (z, \xi). \tag{105}$$

Note that for any multiindex, $\mathcal{L}^\beta$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \tilde{Z}^\beta}$. We have,

Theorem B.16. *There exist multiindices* $\{\beta^1, ..., \beta^{55}\}$*, such that*

$$\Lambda(\beta^1, ..., \beta^{55})(z, \xi) \neq 0$$

for (z, ξ) *in a small neighborhood of* (z^0, ξ^0) *and* $\beta^1 = (0, ..., 0)$.

B.4. The exceptional class M_{16}

This case is very similar to the hyperquadric setting. In this case, we write the coordinates of $\mathbb{C}^{16}$ as

$$z := (x_0, ..., x_7, y_0, ..., y_7).$$

The defining function of the Segre family as described in (16) is

$$\rho(z, \xi) = 1 + r_z \cdot r_\xi = 1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi), \qquad \text{where } N = 26 \text{ and}$$
$$r_z = (x_0, ..., x_7, y_0, ..., y_7, A_0, ...A_7, B_0, B_1). \tag{106}$$

Here $A_i, 0 \leq i \leq 7, B_0, B_1$ are homogeneous quadratic polynomials in z. For instance,

$$B_0 = \sum_{i=0}^{7} x_i^2, B_1 = \sum_{i=0}^{7} y_i^2.$$

For the expressions for A_i, see Appendix A.

Let F be as before. We write

$$F = (f_0, ..., f_7, \widetilde{f}_0, ...\widetilde{f}_7).$$

And define r_F as the composition of r_z with F:

$$r_F = r_z \circ F = (f_0, ..., f_7, \widetilde{f}_0, ...\widetilde{f}_7, A_0(F), ...A_7(F), B_0(F), B_1(F)). \tag{107}$$

Notice that r_F has 26 components.

We will need the following lemma:

Lemma B.17. *For each fixed* $\mu_0, ..., \mu_6$ *with* $(\sum_{i=0}^{6} \mu_i^2) + 1 = 0$ *and fixed* $(y_0, ..., y_7)$ *with* $(\sum_{i=0}^{6} \mu_i y_i) + y_7 \neq 0$*, we can always find* $(\xi_0, ..., \xi_7)$ *such that*

$$1 + y_0\xi_0 + ... + y_7\xi_7 = 0; \quad \sum_{i=0}^{7} (\xi_i)^2 = 0, \quad \xi_j = \mu_j \xi_7, 0 \leq j \leq 6, \quad \xi_7 \neq 0.$$

Proof of Lemma B.17: The proof is similar to that as in the hyperquadric case. □

Take the complex hyperplane $\mathbb{H} : y_7 + \sum_{j=0}^{6} \mu_j y_j = 0$ in $(x_0, ..., x_7, y_0, ..., y_7) \in \mathbb{C}^{16}$. Write $L_0 = \frac{\partial}{\partial x_0}, ..., L_7 = \frac{\partial}{\partial x_7}; L_8 = \frac{\partial}{\partial y_0} - \mu_1 \frac{\partial}{\partial y_7}, ..., L_{14} = \frac{\partial}{\partial y_6} - \mu_6 \frac{\partial}{\partial y_7}$.

Then $\{L_i\}_{i=0}^{14}$ forms a basis of the tangent vector fields of $\mathbb{H}$. For any multiindex $\alpha = (\alpha_0, .., \alpha_{14})$, we write $L^\alpha = L_0^{\alpha_0} ... L_{14}^{\alpha_{14}}$. We define the notion of $L-$rank and $L-$nondegeneracy as in Definition 3.1 using r_F in (107) and L^α instead of $\widetilde{z}^\alpha$. We write the kth L-rank defined in this setting as $\text{rank}_k(r_F, L)$. We now need to prove the following:

Proposition B.18. *F is $L-$nondegenerate near 0. More precisely,* $\text{rank}_{11}(r_F, L) = 26$.

Proof of Proposition B.18: Suppose not. As in the hyperquadric case, by a modified version of Theorem 3.10, we have that there exist 26 holomorphic functions $g_0(w), ..., g_{25}(w)$ defined near 0 on the $w-$plane with $\{g_0(0), ..., g_{25}(0)\}$ not all zero such that the following holds for $z \in U$:

$$\sum_{i=0}^{25} g_i(y_7 + \mu_0 y_0 + ... + \mu_6 y_6)\psi_i(F(z)) \equiv 0. \tag{108}$$

Then since F has full rank at 0, one can similarly prove that $g_0(0) = 0, ..., g_{15}(0) = 0$. Hence we obtain:

Lemma B.19. *There exist $c_0, ..., c_9 \in \mathbb{C}$ that are not all zero such that*

$$c_0 A_0(F(Z)) + ... + c_7 A_7(F(Z)) + c_8 B_0(F(Z)) + c_9 B_1(F(Z)) \equiv 0, \tag{109}$$

for all $Z \in U$ when restricted on $y_7 + \sum_{i=0}^{6} \mu_i y_i = 0$.

We then just need to show that (109) can not hold by the following lemma after applying a linear change of coordinates.

Lemma B.20. *Let $H = (h_0, ..., h_7, g_0, ..., g_7)$ be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (x_0, ..., x_7, y_0, ..., y_6) \in \mathbb{C}^{15}$ with $H(0) = 0$. Assume that H has full rank at 0. Assume that $a_0, ..., a_9$ are complex numbers such that*

$$a_0 A_1(H(\widetilde{z})) + ... + a_7 A_7(H(\widetilde{z})) + a_8 B_0(H(\widetilde{z})) + a_9 B_1(H(\widetilde{z})) = 0 \text{ for all } \widetilde{z} \in U. \tag{110}$$

Then $a_i = 0$ for $1 \leq i \leq 10$.

Proof of Lemma B.20: Suppose not. Notice that H has full rank at 0. Hence there exist 15 components $\widehat{H}$ of H that gives a local biholomorphism from $\mathbb{C}^{15}$ to $\mathbb{C}^{15}$. We assume these 15 components $\widehat{H}$ are H with η being dropped, where $\eta \in \{h_0, ..., h_7, g_0, ..., g_7\}$. By a local biholomorphic change of coordinates, we assume $\widehat{H} = (x_0, ..., x_7, y_0, ..., y_6)$. We still write the remaining component as η. Without loss of generality, we assume $\eta = g_7$.

First we claim the coefficient a_9 of B_1 is zero. Suppose not. Note that y_1^2, y_2^2 can be only canceled by g_7^2. Then g_7 will have linear y_1, y_2 terms. Hence g_7^2 will produce a $y_1 y_2$ term. It cannot be canceled by any other terms. This is a contradiction. Now we consider the coefficients of $A_0, ..., A_7$. We claim $a_i = 0, 0 \leq i \leq 7$. Suppose not. We write

$$y_7(\widetilde{Z}) = \lambda_0 y_0 + ... + \lambda_6 y_6 + \mu_0 x_0 + ... + \mu_7 x_7 + O(2),$$

for some $\lambda_i, \mu_j \in \mathbb{C}, 0 \leq i \leq 6, 0 \leq j \leq 7$. By collecting the terms of the form $x_0 y_i$ in the Taylor expansion of (110) we get

$$a_i + a_7 \lambda_i = 0, 0 \leq i \leq 6. \tag{111}$$

By collecting the terms of the form $x_1 y_i, 0 \leq i \leq 6$, we get,

$$a_1 + a_3\lambda_0 = 0, -a_0 + a_3\lambda_1 = 0, -a_4 + a_3\lambda_2 = 0, -a_7 + a_3\lambda_3 = 0,$$
$$a_2 + a_3\lambda_4 = 0, -a_6 + a_3\lambda_5 = 0, a_5 + a_3\lambda_6 = 0.$$

By collecting the terms of the form $x_2 y_i, 0 \leq i \leq 6$, we get,

$$a_2 + a_6\lambda_0 = 0, a_4 + a_6\lambda_1 = 0, -a_0 + a_6\lambda_2 = 0, -a_5 + a_6\lambda_3 = 0.$$
$$-a_1 + a_6\lambda_4 = 0, a_3 + a_6\lambda_5 = 0, -a_7 + a_6\lambda_6 = 0.$$

One can further write down all the coefficients for $x_i y_j, 0 \leq i \leq 7, 0 \leq j \leq 6$. Once this is done, one easily sees that $a_i \neq 0$ for any $0 \leq i \leq 7$. Otherwise, all $a_i = 0,\ 0 \leq i \leq 7$.

Then by the above equations, we see that the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_1 & -a_0 & -a_4 & -a_7 & a_2 & -a_6 & a_5 \\ a_2 & a_4 & -a_0 & -a_5 & -a_1 & a_3 & -a_7 \end{pmatrix} \tag{112}$$

is of rank one. Then one can get a contradiction by, for instance, carefully checking the determinants of its 2×2 submatrices. Hence $a_i = 0, 0 \leq i \leq 7$. Finally we easily get the coefficient a_8 of B_0 is zero. $\square$

This then establishes Proposition B.18. $\square$

Remark B.21. First fix $\mu_0, ..., \mu_6$ with $(\sum_{i=0}^{6} \mu_i^2) + 1 = 0$. By Proposition B.18, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^{26}$ with $|\tilde{\beta}^j| \leq 11$, and

$$Z^0 = (x_0^0, ..., x_7^0, y_0^0, ..., y_7^0) \text{ with } \sum_{i=0}^{6} \mu_i y_i + y_7 \neq 0,$$

such that

$$\begin{vmatrix} L^{\tilde{\beta}^1}(\psi_1(F)) & ... & L^{\tilde{\beta}^1}(\psi_{26}(F)) \\ ... & ... & ... \\ L^{\tilde{\beta}^{26}}(\psi_1(F)) & ... & L^{\tilde{\beta}^{26}}(\psi_{26}(F)) \end{vmatrix} (Z^0) \neq 0.$$

We then let $\xi^0 = (0, ..., 0, \xi_0^0, ..., \xi_7^0)$, where $(\xi_0^0, ..., \xi_7^0)$ is chosen as in Lemma B.17 associated with $(y_0^0, ..., y_7^0)$. That is

$$1 + y_0^0\xi_0^0 + ... + y_7^0\xi_7^0 = 0; \quad \sum_{i=0}^{7}(\xi_i^0)^2 = 0, \quad \xi_j^0 = \mu_j\xi_7^0, 0 \leq j \leq 6, \quad \xi_7^0 \neq 0.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M}$.

We now define

$$\mathcal{L}_i = \frac{\partial}{\partial x_i} - \frac{\frac{\partial \rho}{\partial x_i}(z,\xi)}{\frac{\partial \rho}{\partial y_7}(Z,\xi)}\frac{\partial}{\partial y_7}, 0 \leq i \leq 7; \tag{113}$$

$$\mathcal{L}_{8+i} = \frac{\partial}{\partial y_i} - \frac{\frac{\partial \rho}{\partial y_i}(z,\xi)}{\frac{\partial \rho}{\partial y_7}(Z,\xi)}\frac{\partial}{\partial y_7}, 0 \leq i \leq 6; \tag{114}$$

for $(z, \xi) \in \mathcal{M}$ near (z^0, ξ^0). They are tangent vector fields along $\mathcal{M}$. Moreover, $\frac{\partial \rho}{\partial y_n}(z, \xi)$ is nonzero near (z^0, ξ^0).

We define for any multiindex $\alpha = (\alpha_0, .., \alpha_{14})$, $\mathcal{L}^\alpha = \mathcal{L}_0^{\alpha_0}...\mathcal{L}_{14}^{\alpha_{14}}$. Define for any 26 collection of 15-multiindices $\{\beta^1, ..., \beta^{26}\}$,

$$\Lambda(\beta^1, ..., \beta^{26})(z,\xi) = \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_{26}(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^{26}}(\psi_1(F)) & ... & \mathcal{L}^{\beta^{26}}(\psi_{26}(F)) \end{vmatrix} (z,\xi). \tag{115}$$

By the fact that $\sum_{i=0}^{7}(\xi_i^0)^2 = 0$, one can check that, for any multiindex $\alpha = (\alpha_0, .., \alpha_{14})$, $\mathcal{L}^\alpha = L^\alpha$ when evaluated at (z^0, ξ^0). Then as before, we get the following:

Theorem B.22. *There exist multiindices $\{\beta^1, ..., \beta^{26}\}$ such that*

$$\Lambda(\beta^1, ..., \beta^{26})(z,\xi) \neq 0,$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0) and $\beta^1 = (0, 0, ..., 0)$.

Appendix C. Transversality and flattening of Segre families for the remaining cases

In this appendix, we will complete the proof of Theorem 6.2 for the remaining cases.

Continuation of the proof of Theorem 6.2: By the same method used before, we first establish the second part of Theorem 6.2 by assuming the first part of Theorem 6.2 is true.

Namely, suppose $\xi^0 \in \mathbb{C}^n \setminus \{0\}$ and z^0 and z^1 are smooth points on the Segre variety Q_{ξ^0} such that Q_{z^0} and Q_{z^1} are both smooth at ξ^0 and intersect transversally there. We shall prove that there is a biholomorphic parametrization $\mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) = (\xi_1, \xi_2, ..., \xi_n)$, with $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$. Here when $1 \le j \le 2$, U_j is a small neighborhood of $1 \in \mathbb{C}$. When $3 \le j \le n$, U_j is a small neighborhood of $0 \in \mathbb{C}$ with $\mathcal{G}(1, 1, 0, \cdots, 0) = \xi^0$, such that $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}, \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$, and $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times ... \times U_n), \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times ... \times U_n), s \in U_1, t \in U_2$ are open pieces of Segre varieties. Also, $\mathcal{G}$ consists of algebraic functions with total degree bounded by a constant depending only on (M, ω). By the first part of Theorem 6.2, we have

$$\operatorname{rank} \begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = 2.$$

Without loss of generality, we assume $\frac{\partial(\rho(z^0,\xi),\rho(z^1,\xi))}{\partial(\xi_1,\xi_2)} \neq 0$ at ξ^0. Now we introduce new variables $\tilde{\xi}_1, ..., \tilde{\xi}_n$ and set up the system:

$$\begin{cases} P_1: & \rho(z^0, \tilde{\xi}_1\xi) = 0 \\ P_2: & \rho(z^1, \tilde{\xi}_2\xi) = 0 \\ P_3: & \tilde{\xi}_3 - \xi_3 = 0 \\ ... & ... \\ P_n: & \tilde{\xi}_n - \xi_n = 0 \end{cases}$$

Then $\frac{\partial(P_1,...,P_n)}{\partial(\xi_1,...,\xi_n)}|_A$, $\frac{\partial(P_1,...,P_n)}{\partial(\tilde{\xi}_1,...,\tilde{\xi}_n)}|_A \neq 0$, where $A = (\tilde{\xi}_1, ..., \tilde{\xi}_n, \xi_1, ..., \xi_n) = (1, 1, 0, ..., 0, 1, 0, ..., 0)$. By Lemma 4.9, we get the needed algebraic flattening with the bound total degree.

Next, we proceed to prove the first part of Theorem 6.2. It suffices to find a sufficiently close point z^1 to z^0 such that

$$\operatorname{rank} \begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = 2.$$

We shall establish the above equation case by case as follows:

Case 3. Symplectic Grassmannians: Pick $\xi_0 = (1, 0, 0, ..., 0)$. The defining equation of the Segre family is $\rho = 1 + \sum_{i=1}^n z_{ii}\xi_{ii} + 2\sum_{i<j} z_{ij}\xi_{ij} + 2\sum_{2\le i<j}(z_{11}z_{ij} - z_{1j}z_{i1})(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{i=2}^n (z_{11}z_{ii} - z_{1i}^2)(\xi_{11}\xi_{ii} - \xi_{1i}^2) + \sum_{i<k,j<l,(i,j)\neq(1,1)} (z_{ij}z_{kl} - z_{il}z_{kj})(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{kj})$ + high order terms, where $z_{ji} := z_{ij}$ for $j > i$.

$Q_{\xi^0} = \{z|\rho(z, \xi^0) = 1 + z_{11} = 0\}, \nabla\rho(z, \xi^0) = (1, 0, ..., 0)$. Hence Q_{ξ^0} is smooth, and for $z \in Q_{\xi^0}$ we have $z = (-1, z_{12}, z_{22}, z_{13}, ..., z_{(n-1)n})$. Pick $z^0, z^1 \in Q_{\xi^0}$. Then $Q_{z^s} = \{\xi|0 = \rho(z^s, \xi) = 1 + \sum_{i=1}^n z_{ii}^s\xi_{ii} + 2\sum_{i<j} z_{ij}^s\xi_{ij} + 2\sum_{2\le i<j}(z_{11}^s z_{ij}^s - z_{1j}^s z_{i1}^s)(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{i=2}^n (z_{11}^s z_{ii}^s - (z_{1i}^s)^2)(\xi_{11}\xi_{ii} - \xi_{1i}^2) + \sum_{i<k,j<l,(i,j)\neq(1,1)} (z_{ij}^s z_{kl}^s - z_{il}^s z_{kj}^s)(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{kj})$ + high order terms$\}$, for $s = 0, 1$.

$$\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\ \nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix}=\begin{pmatrix}\frac{\partial\rho(z^0,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{12}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{1n}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{ij}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{nn}}\\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{12}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{1n}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{ij}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{nn}}\end{pmatrix}\Big|_{\xi^0}$$

$$=\begin{pmatrix}-1 & 2z^0_{12} & 2z^0_{13} & \dots & 2z^0_{1n} & -(z^0_{12})^2 & -2z^0_{12}z^0_{13} & \dots & -(2-\delta_{ij})z^0_{1j}z^0_{1i} & \dots\\ -1 & 2z^1_{12} & 2z^1_{13} & \dots & 2z^1_{1n} & -(z^1_{12})^2 & -2z^1_{12}z^1_{13} & \dots & -(2-\delta_{ij})z^1_{1j}z^1_{1i} & \dots\end{pmatrix}.$$

Hence, we have

$$\begin{aligned}&\operatorname{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\ \nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix}\\ &=\operatorname{rank}\begin{pmatrix}-1 & 2z^0_{12} & 2z^0_{13} & \dots & 2z^0_{1n} & -(z^0_{12})^2 & -2z^0_{12}z^0_{13} & \dots & -(2-\delta_{ij})z^0_{1j}z^0_{1i} & \dots\\ -1 & 2z^1_{12} & 2z^1_{13} & \dots & 2z^1_{1n} & -(z^1_{12})^2 & -2z^1_{12}z^1_{13} & \dots & -(2-\delta_{ij})z^1_{1j}z^1_{1i} & \dots\end{pmatrix}\\ &=\operatorname{rank}\begin{pmatrix}-1 & 2z^0_{12} & 2z^0_{13} & \dots & 2z^0_{1n} & -(2-\delta_{ij})z^0_{1j}z^0_{1i} & \dots\\ 0 & 2\Delta z^1_{12} & 2\Delta z^1_{13} & \dots & 2\Delta z^1_{1n} & (2-\delta_{ij})\{z^1_{1j}\Delta z^1_{1i}+\Delta z^1_{1j}z^1_{1i}-\Delta z^1_{1j}\Delta z^1_{1i}\} & \dots\end{pmatrix}.\end{aligned}$$

where $\Delta z^1_{ij}=z^1_{ij}-z^0_{ij}$. If we pick $z^1_{12}\neq z^0_{12}$, then the above rank is 2.

Case 4. Orthogonal Grassmannians: Here we use the Pfaffian embedding stated in §2. Fixing $\xi^0=(\xi^0_{12},\xi^0_{13},\xi^0_{23},...,\xi^0_{(n-1)n})=(1,0,...,0)$, the defining function of the Segre family is given by $\rho=1+\sum_{i<j}z_{ij}\xi_{ij}+\sum_{2<i<j}(z_{12}z_{ij}-z_{1i}z_{2j}+z_{1j}z_{2i})(\xi_{12}\xi_{ij}-\xi_{1i}\xi_{2j}+\xi_{1j}\xi_{2i})+\sum_{i<j<k<l,\{1,2\}\not\subset\{i,j,k,l\}}(z_{ij}z_{kl}-z_{ik}z_{jl}+z_{il}z_{jk})(\xi_{ij}\xi_{kl}-\xi_{ik}\xi_{jl}+\xi_{il}\xi_{jk})+$ high order terms. Note here we use the notation $z_{ji}:=-z_{ij}$ for $j>i$.

Note $Q_{\xi^0}=\{z|0=\rho(z,\xi^0)=1+z_{12}\}$. Hence it is smooth. Since $z\in Q_{\xi^0}$, we have $z=(-1,z_{13},...,z_{(n-1)n})$. Pick $z^0,\ z^1\in Q_{\xi^0}$. Then $Q_{z^s}=\{\xi|0=\rho(z^s,\xi)=1+\sum_{i<j}z^s_{ij}\xi_{ij}+\sum_{2<i<j}(z^s_{12}z^s_{ij}-z^s_{1i}z^s_{2j}+z^s_{1j}z^s_{2i})(\xi_{12}\xi_{ij}-\xi_{1i}\xi_{2j}+\xi_{1j}\xi_{2i})+\sum_{i<j<k<l,\{1,2\}\not\subset\{i,j,k,l\}}(z^s_{ij}z^s_{kl}-z^s_{ik}z^s_{jl}+z^s_{il}z^s_{jk})(\xi_{ij}\xi_{kl}-\xi_{ik}\xi_{jl}+\xi_{il}\xi_{jk})+\ h.o.t.s.\}$, for $s=0,1$.

$$\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\ \nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix}=\begin{pmatrix}\frac{\partial\rho(z^0,\xi)}{\partial\xi_{12}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{13}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{1n}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{ij}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{(n-1)n}}\\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_{12}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{13}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{1n}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{ij}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{(n-1)n}}\end{pmatrix}\Big|_{\xi^0}$$

$$=\begin{pmatrix}-1 & z^0_{13} & \dots & z^0_{1n} & \dots & z^0_{2n} & (-z^0_{13}z^0_{24}+z^0_{14}z^0_{23})a & \dots & (-z^0_{1i}z^0_{2j}+z^0_{1j}z^0_{2i})a & \dots\\ -1 & z^1_{13} & \dots & z^1_{1n} & \dots & z^1_{2n} & (-z^1_{13}z^1_{24}+z^1_{14}z^1_{23})a & \dots & (-z^1_{1i}z^1_{2j}+z^1_{1j}z^1_{2i})a & \dots\end{pmatrix}.$$

Hence,

$$\operatorname{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\ \nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix}=\operatorname{rank}\begin{pmatrix}-1 & z^0_{13} & \dots & z^0_{1n} & \dots & z^0_{2n} & \dots\\ 0 & \Delta z^1_{13} & \dots & \Delta z^1_{1n} & \dots & \Delta z^1_{2n} & \dots\end{pmatrix}.$$

Here $\Delta z^1_{ij}=z^1_{ij}-z^0_{ij}$. If we choose $z^1_{13}\neq z^0_{13}$, then the rank is 2.

Case 5. M_{16}: Pick $\xi^0=(\kappa^0_0,\kappa^0_1,...,\kappa^0_7,\eta^0_0,\eta^0_1,...,\eta^0_7)=(1,0,...,0),\ z^0\in Q_{\xi^0}$. The defining equation of the Segre family is $1+x_0\kappa_0+x_1\kappa_1+...+x_7\kappa_7+y_0\eta_0+y_1\eta_1+...+$

$y_7\eta_7 + (x_0y_0 + x_1y_1 + ...)(\kappa_0\eta_0 + \kappa_1\eta_1 + ...) + (-y_0x_1 + y_1x_0 + ...)(-\eta_0\kappa_1 + \eta_1\kappa_0 + ...) + ... + (x_0^2 + x_1^2 + ... + x_7^2)(\kappa_0{}^2 + \kappa_1{}^2 ... + \kappa_7{}^2) + (y_0^2 + y_1^2 + ... + y_7^2)(\eta_0^2 + \eta_1^2 + ... + \eta_7^2) = 0$.

$Q_{\xi^0} = \{z|\rho(z,\xi^0) = 1 + x_0 + (x_0^2 + x_1^2 + ... + x_7^2) = 0\}$, and $\nabla\rho(z,\xi^0)|_{z^0} = (1 + 2x_0, 2x_1, ..., 2x_7^0, 0, ..., 0)$. Hence Q_{ξ^0} is smooth. Pick $z^0, z^1 \in Q_{\xi^0}$. Then $Q_{z^s} = \{\xi|0 = \rho(z^s,\xi) = 1 + x_0^s\kappa_0 + x_1^s\kappa_1 + ... + x_7^s\kappa_7 + y_0^s\eta_0 + y_1^s\eta_1 + ... + y_7^s\eta_7 + (x_0^sy_0^s + x_1^sy_1^s + ...)(\kappa_0\eta_0 + \kappa_1\eta_1 + ...) + (-y_0^sx_1^s + y_1^sx_0^s + ...)(-\eta_0\kappa_1 + \eta_1\kappa_0 + ...) + ... + ((x_0^s)^2 + (x_1^s)^2 + ... + (x_7^s)^2)(\kappa_0{}^2 + \kappa_1{}^2 + ... + \kappa_7{}^2) + ((y_0^s)^2 + (y_1^s)^2 + ... + (y_7^s)^2)(\eta_0^2 + \eta_1^2 + ... + \eta_7^2)\}$, for $s = 0, 1$.

$$\operatorname{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\ \nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix} \geq \operatorname{rank}\begin{pmatrix}\frac{\partial\rho(z^0,\xi)}{\partial\kappa_0} & \frac{\partial\rho(z^0,\xi)}{\partial\kappa_1} & ... & \frac{\partial\rho(z^0,\xi)}{\partial\kappa_7}\\ \frac{\partial\rho(z^1,\xi)}{\partial\kappa_0} & \frac{\partial\rho(z^1,\xi)}{\partial\kappa_1} & ... & \frac{\partial\rho(z^1,\xi)}{\partial y_7}\end{pmatrix}\Big|_{\xi^0}$$

$$= \operatorname{rank}\begin{pmatrix}-2 - x_0^0 & x_1^0 & x_2^0 & \cdots & x_7^0\\ -2 - x_0^1 & x_1^1 & x_2^1 & \cdots & x_7^1\end{pmatrix}. \qquad (C)$$

Since $(-2 - x_0^0, x_1^0, x_2^0, \cdots, x_7^0) \neq (0, ..., 0)$, we can pick z^1 sufficiently close to z^0, such that the above rank is 2. That is because Q_{ξ^0} is irreducible and the subvarieties, defined by 2×2 minors of the last matrix in (C), are thin subsets of Q_{ξ^0}.

Case 6. M_{27}: Take $\xi^0 = (\xi_1^0, \xi_2^0, \xi_3^0, \eta_0^0, \eta_1^0, ..., \eta_7^0, \kappa_0^0, \kappa_1^0, ..., \kappa_7^0, \tau_0^0, \tau_1^0, ..., \tau_7^0) = (1, 0, ..., 0)$. The defining function of the Segre family is $1 + r_z \cdot r_\xi$ where

$$r_z = (x_1, x_2, x_3, y_0, ..., y_7, z_0, ..., z_7, w_0, ..., w_7, A, B, C, D_0, ...D_7, E_0, ..., E_7, F_0, ..., F_7, G)$$

$$r_\xi = (\xi_1, \xi_2, \xi_3, ..., \eta_7, ..., \kappa_7, ..., \tau_7, A(\xi), B(\xi), C(\xi), ..., D_7(\xi), ..., E_7(\xi), ..., G(\xi)).$$

Here A, B, C, D_i, E_i, F_i are homogeneous quadratic polynomials; G is a homogeneous cubic polynomial defined in Appendix A.

For our purpose here, we present terms only involving ξ_1, ξ_2, and omit those involving $\xi_3, \eta_0, \eta_1, ..., \eta_7, \kappa_0, \kappa_1, ..., \kappa_7, \tau_0, \tau_1, ..., \tau_7$ as follows: $\rho(z,\xi) = 1 + x_1\xi_1 + x_2\xi_2 + ... + (x_1x_2 - (\sum_{i=0}^7 y_i^2))(\xi_1\xi_2 - (\sum_{i=0}^7(\tau_i)^2)) + \cdots$.

$Q_{\xi^0} = \{z|\rho(z,\xi^0) = 1 + x_1 = 0\}, \nabla\rho(z,\xi^0) = (1, 0, 0, ..., 0)$. Hence Q_{ξ^0} is smooth and for $z \in Q_{\xi^0}$, we have $z = (-1, x_2, x_3, ...,)$. Pick $z^0, z^1 \in Q_{\xi^0}$. Then $Q_{z^s} = \{\xi|0 = \rho(z^s,\xi) = 1 + x_1^s\xi_1 + x_2^s\xi_2 + ... + (x_1^sx_2^s - (\sum_{i=0}^7(y_i^s)^2))(\xi_1\xi_2 - (\sum_{i=0}^7(\tau_i)^2)) + ...\}$, for $s = 0, 1$.

$$\operatorname{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\ \nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix}\frac{\partial\rho(z^0,\xi)}{\partial\xi_1} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_2} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_3} & ... & \frac{\partial\rho(z^0,\xi)}{\partial\eta_7} & ... & \frac{\partial\rho(z^0,\xi)}{\partial\kappa_7} & ... & \frac{\partial\rho(z^0,\xi)}{\partial\tau_7}\\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_1} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_2} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_3} & ... & \frac{\partial\rho(z^1,\xi)}{\partial\eta_7} & ... & \frac{\partial\rho(z^1,\xi)}{\partial\kappa_7} & ... & \frac{\partial\rho(z^1,\xi)}{\partial\tau_7}\end{pmatrix}\Big|_{\xi^0}$$

$$\geq \operatorname{rank}\begin{pmatrix}\frac{\partial\rho(z^0,\xi)}{\partial\xi_1} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_2}\\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_1} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_2}\end{pmatrix}\Big|_{\xi^0} = \operatorname{rank}\begin{pmatrix}-1 & -(\sum_{i=0}^7(y_i^0)^2)\\ -1 & -(\sum_{i=0}^7(y_i^1)^2)\end{pmatrix}\Big|_{\xi^0} \geq 2,$$

for those z^1's such that $\sum_{i=0}^{7}(y_i^1)^2 \neq \sum_{i=0}^{7}(y_i^0)^2$. This can be done in any small neighborhood of z^0; for $\{z|\sum_{i=0}^{7}(y_i)^2 = B\}$ is a thin set in $\{z|0 = 1 + x_1\}$ for each fixed $B \in \mathbb{C}$.

This completes the proof of the flattening theorem. □

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