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# Several Results for Holomorphic Mappings from $\mathbf{B}^n$ into $\mathbf{B}^N$

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*Dedicated to Professor F. Treves*

## §1. Introduction

Let $\mathbf{B}^n$ be the unit ball in $\mathbf{C}^n$. Write $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ for the collection of proper holomorphic maps $F : \mathbf{B}^n \to \mathbf{B}^N$. Write $\mathrm{Prop}_k(\mathbf{B}^n, \mathbf{B}^N) \subset \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ for proper holomorphic maps that are $C^k$-smooth up to the boundary; and denote by $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ the set of rational proper holomorphic maps from $\mathbf{B}^n$ into $\mathbf{B}^N$. Recall that $f, g \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ are said to be *equivalent* if there are automorphisms $\sigma \in Aut(\mathbf{B}^n)$ and $\tau \in Aut(\mathbf{B}^N)$ such that $f = \tau \circ g \circ \sigma$. When $f \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ is equivalent to the standard big circle embedding $z \to (z, 0)$, we call $f$ a linear map or a totally geodesic embedding. In all that follows, we always assume that $N \geq n > 1$.

There has been much work done in the past thirty years on the rigidity (linearity) and classification problems for elements in $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$. (See [Fo92] [H02] [H01] for references and historical discussions).

In [H02], the author assigned to each $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ an invariant integer $\kappa_0(F) \in \{0, 1, 2, ..., n-1\}$ which is called the *geometric rank* of $F$ (see [H02] or § 2 below for the precise definition). Using the language of geometric ranks, [H99, Theorem 4.2] is stated as follows: $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ has $\kappa_0(F) = 0$ if and only if $F$ is equivalent to the linear map. Therefore, to understand proper holomorphic mappings between balls, it suffices to study maps with geometric rank $\kappa_0 \geq 1$. Meanwhile, it was also shown in [H02, Lemma 3.2] that one always has $N \geq n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Namely, the least dimension of the target space is $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ to allow the existence of elements in $\mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ with geometric rank $\kappa_0 \geq 0$.

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The invariant integer $\kappa_0$ was introduced in [H02] to study a semi-rigidity problem for holomorphic maps, which we recall as follows:

**Definition 1.0** ([H02]): Let $F \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$. Then $F$ is called $k$-linear if for any point $p \in \mathbf{B}^n$ there is an affine complex subspace $S_p^a$ containing $p$ and of complex dimension $k$ such that the restriction of $F$ to $S_p^a$ is a linear fractional map.

**Theorem 1.1** ([Theorem 2.3, H02]): Suppose that $F \in \mathrm{Prop}_3(\mathbf{B}^n, \mathbf{B}^N)$ has geometric rank $\kappa_0 \leq n-2$. Then $F$ is $(n-\kappa_0)$-linear.

The first part of this paper is to study the rationality problem for proper holomorphic maps between balls, by making use of Theorem 1.1 and a result of Forstneric. We will prove the following:

**Theorem 1.2**: *Suppose that $F \in Prop_2(\mathbf{B}^n, \mathbf{B}^N)$ is $k$-linear with $k \geq 2$. Then $F$ is a rational map.*

**Corollary 1.3** ( to [Theorem 2.3, H02] and Theorem 1.2): *Suppose that $F \in Prop_3(\mathbf{B}^n, \mathbf{B}^N)$ has geometric rank $\kappa_0(F) < n-1$. Then $F$ is a rational map.*

**Corollary 1.4** (to [Lemma 3.2, H02] and Corollary 1.3): *Let $F \in Prop_3(\mathbf{B}^n, \mathbf{B}^N)$ with $N \leq \frac{n(n+1)}{2}$. Then $F$ is a rational map.*

Forstneric proved in his famous paper [Fo89] that any proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$ $(N \geq n \geq 2)$, that extends $C^{N-n+1}$-smoothly across some point at the boundary, is rational. For the proof of Theorem 1.2, we will show in §3 that when the map is partially linear, it then extends holomorphically across some boundary points. Thus the rationality of the map in Theorem 1.2 can be reduced to the theorem of Forstneric [Fo89]. There have also been important papers written by D'Angelo and his coauthors for the study of monomial mappings between balls. We refer the reader to [DA88], [DC], etc.

## § 2. Preliminaries

•**Maps between balls**: The ball $\mathbf{B}^n \subset \mathbf{C}^n$ is equivalent to the Siegel upper-half space $\mathbf{H}_n := \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \ \mathrm{Im}(w) > |z|^2\}$ by the Cayley transformation $\rho_n : \mathbf{H}_n \to \mathbf{B}^n$, $\rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right)$. We can similarly define the space $\mathrm{Rat}(\mathbf{H}_n, \mathbf{H}_N)$, $\mathrm{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$ and $\mathrm{Prop}(\mathbf{H}_n, \mathbf{H}_N)$.

We will identify a map $F \in \mathrm{Prop}_k(\mathbf{B}^n, \mathbf{B}^N)$ or $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ with the one in the space $\mathrm{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$ or $\mathrm{Rat}(\mathbf{H}_n, \mathbf{H}_N)$ by $\rho_N^{-1} \circ F \circ \rho_n$, respectively.

We write $L_j = 2i\overline{z_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ for $j = 1, \cdots, n-1$ and $T = \frac{\partial}{\partial u}$ where $w = u + iv$. Then $\{L_1, \cdots, L_{n-1}\}$ forms a global basis for the complex tangent bundle $\mathrm{T}^{(1,0)}\partial\mathbf{H}_n$ of $\partial\mathbf{H}_n$, and $T$ is a tangent vector field of $\partial\mathbf{H}_n$ transversal to $\mathrm{T}^{(1,0)}\partial\mathbf{H}_n \cup \mathrm{T}^{(0,1)}\partial\mathbf{H}_n$. Parameterize $\partial\mathbf{H}_n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of $z$ and $u$ to be 1 and 2, respectively. For a non negative integer $m$, a function $h(z, \overline{z}, u)$ defined over a small ball $U$ of 0 in $\partial\mathbf{H}_n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2u)}{|t|^m} \to 0$ uniformly for $(z, u)$ on any compact subset of $U$ as $t (\in \mathbf{R}) \to 0$. (In this case, we write $h = o_{wt}(m)$. By convention, we write $h = o_{wt}(0)$ if $h \to 0$ as $(z, \overline{z}, u) \to 0$).

• **Geometric rank of** $F$: Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a $C^2$-smooth CR map from an open piece $M$ of $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$. For each $p \in M$, we have an associated CR map $F_p$ from a small neighborhood $U$ of $0 \in \partial\mathbf{H}_n$ to $\partial\mathbf{H}_N$ with $F_p(0) = 0$, defined by

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p), \tag{2.1}$$

where for each $p = (z_0, w_0) \in M$, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbf{H}_n)$ for the map sending $(z, w)$ to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and we define $\tau_p^F \in \mathrm{Aut}(\mathbf{H}_N)$ by $\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle)$. For $F_p$, it associates a map $F_p^*$ which is equivalent to $F_p$ and satisfies certain normalization conditions (see [H99][H02] for the details).

**Lemma 2.1** ([H99, §2, Lemma 5.3]): Let $F$ be a $C^2$-smooth CR map from a connected open subset $M$ containing 0 in $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$, $2 \le n \le N$. For each $p \in \partial\mathbf{H}_n$, there are $\sigma \in \mathrm{Aut}_0(\partial\mathbf{H}_n)$ and $\tau \in \mathrm{Aut}_0(\partial\mathbf{H}_N)$ such that the map $F_p^{**} = \tau \circ F_p^* \circ \sigma$ satisfies the following normalization:

$$f_p^{**} = z + \frac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4), \tag{2.2}$$

with

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2. \tag{2.3}$$

From (2.3), we see that $a_p^{**(1)}(z) = z\mathcal{A}(p)$ and that $\mathcal{A}(p)$ is an $(n-1)\times(n-1)$ semi-positive Hermitian matrix. The rank of $\mathcal{A}(p) = -2i(P_j^l)_{1\le j,l\le(n-1)}$, which is denoted by $Rk_F(p)$, is called the geometric rank of $F$ at $p$. Notice that $P_j^l = \frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0$. By [(2.3.1), H02], $Rk_F(p)$ is a well-defined integer ([H02]), depending only on $F$ and $p$. (See [Definition 2.1, H02]). We define the *geometric rank* of $F$ to be $\kappa_0(F) = max_{p \in \partial \mathbf{H_n}} Rk_F(p)$. Notice that we always have $0 \le \kappa_0 \le n-1$. By [H02, Corollary 5.2], when $F \in \mathrm{Prop}_3(\mathbf{H}_n, \mathbf{H}_N)$, the set $\{p \in \partial \mathbf{H}_n, Rk_F(p) = \kappa_0\}$ is an open dense subset of $\mathbf{H}_n$. We define the geometric rank of $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbf{H}_n, \mathbf{H}_N)$. For such a map, we can similarly define the geometric rank $Rk_F(p)$ of $F$ at $p \in \partial \mathbf{B}^n$.

• **A normalization lemma**: The following normalization will be used later for the proof of theorem 1.5.

**Lemma 2.2** ([Lemma 3.2, H02]): Let $F$ be a $C^2$-smooth CR map from an open piece $M \subset \partial \mathbf{H}_n$ into $\partial \mathbf{H}_N$ with $Rk_F(p) = \kappa_0$. Let $P(n, \kappa_0) = \frac{\kappa_0(2n-\kappa_0-1)}{2}$. Then $N \ge n + P(n, \kappa_0)$ and there are $\sigma \in \mathrm{Aut}_0(\partial \mathbf{H}_n)$ and $\tau \in \mathrm{Aut}_0(\partial \mathbf{H}_N)$ such that $\tau \circ F_p \circ \sigma := (f, \phi, g)$, denoted by $F_p^{***}$, satisfies the following normalization condition:
(4.2)
$$\begin{aligned}
&f_j = z_j + \frac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \frac{\partial^2 f_j}{\partial w^2}(0) = 0, \ j = 1 \cdots, \kappa_0, \ \mu_j > 0, \\
&f_j = z_j + o_{wt}(3), \quad j = \kappa_0 + 1, \cdots, n-1 \\
&g = w + o_{wt}(4), \\
&\phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2), \text{ where } \ (j,l) \in \mathcal{S} \text{ with } \ \mu_{jl} > 0 \text{ for } \ (j,l) \in \mathcal{S}_0 \text{ and } \mu_{jl} = 0 \text{ otherwise.}
\end{aligned}$$

Moreover, $\mu_j \ge \mu_1 = 1$, $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j, l \le \kappa_0$ $j \ne l$; and $\mu_{jl} = \sqrt{\mu_j}$ if $j \le \kappa_0$ and $l > \kappa_0$ or if $j = l \le \kappa_0$. Here we label the components of $\phi$ by double indices $(j,l) \in \mathcal{S}$ with

$$\begin{aligned}
&\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le (n-1), j \le l\}, \\
&\mathcal{S} := \{(j,l) : \ (j,l) \in \mathcal{S}_0, \text{ or } \ j = \kappa_0 + 1, l \in \{\kappa_0 + 1, \cdots, \kappa_0 + N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\}\}.
\end{aligned}$$

• **Degree of a rational map**: For any rational holomorphic map $H = \frac{(P_1, \ldots, P_m)}{Q}$ on $\mathbf{C}^n$, where $P_j, Q$ are holomorphic polynomials and $(P_1, ..., P_m, Q) = 1$. We define

$$deg(H) = max\{deg(P_j), \ 1 \le j \le m, \ deg(Q)\}. \tag{2.5}$$

The following will also be used in our later discussion:

**Lemma 2.3** ([HJ01, Lemma 5.3 and 5.4]): Let $F \in \mathrm{Rat}(\mathbf{H_n}, \mathbf{H_N})$ and $F_p^{***}$ be as described in Lemma 2.2. If $deg(F_p^{***}(z,0)) \le l$ for any $p$ in an open neighborhood of 0 in $\partial\mathbf{H_n}$, then $deg(F) \le l$.

## §3. Proper maps with partial linearity

In this section, we give the proof of Theorem 1.2 and Corollaries 1.3-1.4. We mention that all these results and arguments are of purely local nature. However, we only focus on the global setting for simplicity of notation.

Let $F \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ with $N \ge n > 1$. For any integer $1 \le k \le n$, write $G_{n,k}(\mathbf{C})$ for the complex Grassmannian manifold consisting of complex $k$-planes in $\mathbf{C}^n$. Define

$$\mathcal{V}_F := \{(Z, S_Z) \in \mathbf{B}^n \times G_{n,k}(\mathbf{C}),\ F \text{ is linear fractional when restricted to } S_Z + Z\}. \tag{3.1}$$

Then, as in [Lemma 5.1, H02], $\mathcal{V}_F$ is a complex analytic variety with $\pi : \mathcal{V}_F \to \mathbf{B}^n$ as its proper holomorphic projection. In particular, this implies that if there is a subset $E \subset \mathbf{B}^n$ of Hausdorff dimension greater than $2n-2$ such that for any $Z \in E$, there is an affine complex subspace of dimension $k$ through $Z$ along which $F$ is linear, then $F$ is $k$-linear over $\mathbf{B}^n$. We define the quantity $\kappa^0(F)$ such that $F$ is $\kappa^0(F)$-linear, but not $(\kappa(F)^0+1)$-linear. Then Theorem 2.3 of [H02] states that when $F$ is $C^3$-smooth up to the boundary and when $\kappa_0(F) < n-1$, it holds

$$\kappa^0(F) = n - \kappa_0(F). \tag{3.2}$$

We next recall the following Lemma from [H02]:

**Lemma 3.1**([Lemma 5.3, H02]): Let $M$ be a connected open subset of $\partial\mathbf{H}_n$. Let $F$ be a $C^2$ CR map from $M$ into $\partial\mathbf{H}_N$ with $N \ge n > 1$ and with constant geometric rank $\kappa_0 < n-1$. Assume that $F$ extends holomorphically to a sub-domain $\Omega$ of $\mathbf{H}_n$, which has $M$ as part of its smooth boundary. Assume that $F$ is $(n-\kappa_0)$-linear over $\Omega$. Let $p_0 \in M$. Then for $Z (\in \Omega \setminus E) \approx p_0$ with $E$ a certain complex analytic variety of positive codimension, there is a unique complex subspace $S_Z$ of dimension $(n-\kappa_0)$ such that $F$, when restricted to $S_Z + Z$, is linear fractional. Moreover $S_Z$, as elements in $G_{n,n-\kappa_0}(\mathbf{C})$, depends holomorphically on $Z(\approx p_0) \in \Omega \setminus E$ and extends holomorphically across $E$.

**Proposition 3.2**: Let $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ with $N \ge n > 1$. Assume that $\kappa_0(F) < n-1$. Let $\mathcal{V}_F$ be as defined in (3.1) with $k = n - \kappa_0(F) = \kappa^0(F)$. Then it has a unique irreducible component of dimension $n$, denoted by $\mathcal{V}_F^0$, and there is a complex analytic variety $E_F$ of positive codimension in $\mathbf{B}^n$ such that the following holds: (i) $\pi$ is surjective from $\mathcal{V}_F^0$ to $\mathbf{B}^n$ ; (ii) $\pi$ is one-to-one from $\mathcal{V}_F \setminus \pi^{-1}(E_F)$ to $\mathbf{B}^n \setminus E_F$; (iii) $\mathcal{V}_F \setminus \pi^{-1}(E_F) = \mathcal{V}_F^0 \setminus \pi^{-1}(E_F)$.

*Proof of Proposition 3.2*: By Lemma 3.1, there is an open subset $U$ of the ball such that $\pi$ is biholomorphic from $\mathcal{V}_F \cap \pi^{-1}(U)$ to $U$. Now write $\mathcal{V}_F^0$ for the irreducible component of $\mathcal{V}_F$ which contains $\pi^{-1}(U)$ as an open piece. Then $\mathcal{V}_F^0$ has complex dimension $n$. Moreover, for each irreducible component $V$ of $\mathcal{V}_F$, either $V \equiv \mathcal{V}_F^0$ or $\pi(V)$ must be a proper complex analytic variety of $\mathbf{B}^n$. Let $E_F$ be the union of such $\pi(V)$'s with $V$ different from $\mathcal{V}_F^0$. Then we see the conclusion of the statements in the proposition. ■

We next prove the following result:

**Proposition 3.3**: Let $F$ be a holomorphic map from $\Omega \subset \mathbf{B}^n$ into $\Omega' \subset \mathbf{B}^N$ with $n \geq 2$, that is $k$-linear ($k > 1$) over $\Omega$. Assume that $\Omega$ ($\Omega'$, respectively) has a connected open piece $M \subset \partial\mathbf{B}^n$ ($M' \subset \partial\mathbf{B}^N$, respectively) as part of its smooth boundary such that when $Z \in \Omega$ goes to a point in $M$, the limit set $\lim_Z F(Z) \subset M'$. Also, assume that there is an open subset $U$ of $\Omega$, sufficiently close to $M$, such that for each $Z \in U$, there is a unique affine subspace $S_Z^a$ of dimension $k$ passing through $Z$ such that $F|_{S_Z^a \cap \Omega}$ is linear fractional. Then $F$ is rational.

*Proof of Proposition 3.3*: Without loss of generality, we assume that when $Z = 0$, $S_0^a = \{z_{k+1} = \cdots = z_n = 0\}$ and $F|_{S_0^a} = \mathrm{Id}$. Also, to simplify the notation, we assume that $\Omega = \mathbf{B}^n$, $\Omega' = \mathbf{B}^N$. And $\pi$ is biholomorphic from a neighborhood of $\pi^{-1}(0)$ in $\mathcal{V}_F$ to a neighborhood of $0$ in $\mathbf{B}^n$.

Notice that $S_0^a = S_0 = \mathrm{span}\{e_1, \cdots, e_{n-k}\}$, where $e_j$ is the $n$-tuple whose component at the $l^{th}$ position is $\delta_j^l$. We will use the standard local coordinates for the Grassmannian $G_{n,k}(\mathbf{C})$ near $S_0$. Namely, for any $S$ near $S_0$, we associate it uniquely with the coordinates $(\xi_{jl})$ where $j$ runs from 1 to $k$ and $l$ runs from $k+1$ to $n$ such that $S = \mathrm{span}\{e_1(S), \cdots, e_k(S)\}$. Here $e_j(S) = (0, \cdots, 1, \cdots, 0, \xi_{j(k+1)} \cdots, \xi_{jn})$. Now, for each $Z \approx 0$, $S_Z^a$ associated with $F$ can be parameterized by $k(n-k)$ holomorphic functions $\xi_{jl}(Z)$, where $j = 1, \cdots, k$ and $l = k+1, \cdots, n$, in the manner such that $S_Z^a = Z + span_j\{e_j(S_Z^a - Z)\}$. Then the assumption above shows that $\xi_{jl}(Z)$ as functions in $Z$ are holomorphic near 0 for each $(j, l)$.

Consider the holomorphic map $\Psi$ which sends $(t, \tau) := (t_1, \cdots, t_k, \tau_1, \cdots, \tau_{n-k})$ to

$$\left(t_1, \cdots, t_k, \sum_{j=1}^{k} \xi_{j(k+1)}(0, \tau)t_j + \tau_1, \cdots, \sum_{j=1}^{k} \xi_{jn}(0, \tau)t_j + \tau_{n-k}\right).$$

Then $\Psi(t,\tau) = (t,\tau) + (0, O(\tau)|t|)$ is holomorphic from a neighborhood $U_\epsilon$ of $\{\sum_{j=1}^k |t_j|^2 < 1+\epsilon\} \times \{|\tau| < \epsilon\}$ for a certain positive number $\epsilon << 1$. Moreover, $\Psi$ is the identical map when restricted to $S_0 \cap \overline{\mathbf{B}^n}$ and has non zero Jacobian there. Moreover, $\Psi$ sends $(t,\tau)$ into $S^a_{(t,\tau)}$. Hence $F \circ \Psi$ is linear fractional in $t$ for each fixed $\tau \approx 0$ by the assumption. Therefore, we have

$$F \circ \Psi(t,\tau) = \frac{F(\tau) + \sum_{j=1}^k A_j(\tau)t_j}{1 + \sum_{j=1}^k b_j(\tau)t_j}. \tag{3.3}$$

Now, we claim that $A_j(\tau), b_j(\tau), F(\tau)$ are holomorphic for $\tau \approx 0$ (See [Lemma 5.1, H01]). For this purpose, write

$$F \circ \Psi(t,\tau) = \sum_\alpha C_\alpha(\tau)t^\alpha.$$

Then $C_\alpha$ depends holomorphically on $\tau$ for $\tau \approx 0$. Multiplying $(1+\sum_j b_j(\tau)t_j)$ of both sides of (3.3) and then considering the Taylor expansion in $t$ at the origin, we see that

$$C_\alpha + \sum_{j=1}^k b_j C_{\alpha - e'_j} \equiv 0 \text{ for } |\alpha| \geq 2, \tag{3.4}$$

$C_0(\tau) = F(\tau)$, $C_{e'_j}(\tau) = D_{t_j}(F\circ\Psi)(t,\tau)|_{t=0}$ and $A_j(\tau) = (F\circ\Psi)(\tau)b_j(0,\tau) + C_{e'_j}(\tau)$. Here $e'_j$ is the vector in $\mathbf{C}^k$ defined as for $e_j$.

By the Alexander theorem [A77], since $F|_{S^a_Z}$ must be a linear embedding, we see that $\{C_{e'_j}\}_{j=1}^{n-\kappa_0}$ are linearly independent vectors. Hence, we can holomorphically solve $b_j(\tau)'s$ in (3.4) in terms of $C_\alpha(\tau)$ with $|\alpha| = 2$. Hence $A_j(\tau), b_j(\tau), F(\tau)$ are holomorphic for $\tau \approx 0$.

Notice that $b_j(0) = 0$ by our normalization that $F|_{S^a_0} = \mathrm{id}$. It is clear that $F \circ \Psi$ extends holomorphically to a neighborhood $U'_{\epsilon'}(\subset\subset U_\epsilon)$ of $\{\sum_{j=1}^k |t_j|^2 < 1+\epsilon'\} \times \{|\tau| < \epsilon'\}$ for a certain positive number $\epsilon' < \epsilon$.

Now, as mentioned before, one can find a point $Z_0 \in U'_{\epsilon'}$ such that $\Phi(Z_0)$ is on the unit sphere and $\Psi$ is locally biholomorphic near $Z_0$. It thus follows that near $\Phi(Z_0)$, $F = (F\circ\Phi)\circ\Phi^{-1}$ extends holomorphically to a neighborhood of $\Psi(Z_0)$. By a result of Forstneric [Fo89], we conclude the rationality of $F$. This completes the proof of Proposition 3.3. ■

*Proof of Theorem 1.2, Corollaries 1.3-1.4*: Theorem 1.2 now follows from Propositions 3.2-3.3. Corollary 1.3 follows from Theorem 1.2 and [Theorem 2.3, H02]. Corollary 1.4 follows from Corollary 1.3 and [Lemma 3.2, H02] for $N < \frac{n(n+1)}{2}$. when $N = \frac{n(n+1)}{2}$, with an argument identical to that in [Corollary 2.1, H01], we also see the proof of Corollary 1.4. ■

As mentioned at the beginning of the section, one similarly has the local version of all these results. For instance, one has the following

**Corollary 3.4**: *Let $M$ be a connected open subset of $\partial\mathbf{H}_n$. Let $F$ be a non-constant $C^3$-smooth CR map from $M$ into $\partial\mathbf{H}_N$ with $N \geq n > 1$ and with constant geometric rank $\kappa_0 < n-1$. Then $F$ is rational. In particular, any $C^3$ CR map from an open piece of $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$ with $N = \frac{n(n+1)}{2}$ is rational.*

**Remark 3.5**: It is known from the work of Forstneric [Fo89] that $\mathrm{Prop}_{N-n+1}(\mathbf{B}^n, \mathbf{B}^N) = \mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ for $N \geq n > 1$. It has been conjectured (see [Fo92] [Hu01]) that

$$\mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N) = \mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N) \quad \text{for } N \geq n > 1. \tag{3.5}$$

In Corollary 1.4, we need three times differentiability to conclude the rationality because the partial linearity was only obtained in [Hu02] for maps which are $C^3$ regular up to the boundary. We believe that Theorem 1.1 of [Hu02] should also hold for maps which are merely $C^2$-smooth up to the boundary, which, if proved, would provide, as an immediate application, a solution to the above mentioned conjecture for $N \leq \frac{n(n+1)}{2}$. (Namely, (3.5) then would hold for $N \leq \frac{n(n+1)}{2}$.) The work in [Hu02] and in the present paper may suggest that the following special case of the above mentioned conjecture should be more workable, by suitably extending the notion of the geometric ranks and then generalizing the partial-rigidities obatined in [Hu02].

**Conjecture 3.6**: Let $T_{0,n} = 1$. For $k \geq 1$, let $T_{k,n}$ be the complex dimension of the vector space (over $\mathbf{C}$) spanned by monomials of degree $k$ with $(n-1)$ complex variables. Then if $f \in \mathrm{Prop}_{N^*}(\mathbf{B}^n, \mathbf{B}^N)$ with $N \leq \sum_{j=0}^{N^*} T_{j,n}$ and $n > 1$, $f$ must be rational.

## § 4. A degree estimate

In this section, we let $n = 3$, $N = \frac{n(n+1)}{2} = 6$. We will consider rational proper maps from $\mathbf{B}^3$ into $\mathbf{B}^6$ and give the following degree estimate.

**Theorem 4.1**: *Let $F \in Rat(\mathbf{B}^3, \mathbf{B}^6)$ with $\kappa_0(F) = 2$. Then $deg(F) \leq 4$.*

By the discussions in §2, for the proof of Theorem 4.1, we can assume that $F \in \mathrm{Rat}(\mathbf{H}_3, \mathbf{H}_6)$ with $\kappa_0 = 2$. By Lemma 2.2, for any $p \in U \subset \partial\mathbf{H}_3$, there are $\sigma_0 \in \mathrm{Aut}_0(\mathbf{H}_3)$ and $\tau_0 \in \mathrm{Aut}_0(\mathbf{H}_6)$ such that $F_p^{***} = \tau_0 \circ \tau_p^F \circ F \circ \sigma_p^0 \circ \sigma_0 := (f_1, f_2, \phi_{11}, \phi_{12}, \phi_{22}, g) = (f, \phi, g) = (\widetilde{f}, g)$ satisfies the following conditions:

$$
\begin{aligned}
&f_1 = z_1 + \frac{i}{2} z_1 w + o_{wt}(3), \ \frac{\partial^2 f_1}{\partial w^2}(0) = 0, \\
&f_2 = z_2 + \frac{i\mu_2}{2} z_2 w + o_{wt}(3), \ \frac{\partial^2 f_2}{\partial w^2}(0) = 0, \ \mu_2 \geq 1 \\
&\phi_{11} = z_1^2 + o_{wt}(2), \quad \phi_{12} = \sqrt{1+\mu_2} z_1 z_2 + o_{wt}(2), \\
&\phi_{22} = \sqrt{\mu_2} z_2^2 + o_{wt}(2), \quad g = w + o_{wt}(4).
\end{aligned} \tag{4.1}
$$

By the argument in [HJ01, Lemma 5.2], we have

$$
\overline{\widetilde{f}(\zeta, 0)^t} = \begin{pmatrix} I & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix} \begin{pmatrix} \overline{\zeta}^t \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\zeta}^t \\ -B^{-1}A\overline{\zeta}^t \end{pmatrix}, \tag{4.2}
$$

where

$$
A = \begin{pmatrix} A_{2\times 2} \\ A_{1\times 2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1\mathcal{L}_1(f_1) & \mathcal{L}_1\mathcal{L}_1(f_2) \\ \mathcal{L}_1\mathcal{L}_2(f_1) & \mathcal{L}_1\mathcal{L}_2(f_2) \\ \mathcal{L}_2\mathcal{L}_2(f_1) & \mathcal{L}_2\mathcal{L}_2(f_2) \end{pmatrix} |_{(0,0,0,\zeta_1,\zeta_2,0)} = \begin{pmatrix} -2\overline{\zeta}_1 & 0 \\ -\overline{\zeta}_2 & -\mu_2\overline{\zeta}_1 \\ 0 & -2\mu_2\overline{\zeta}_2 \end{pmatrix}, \tag{4.3}
$$

and

$$
B = \begin{pmatrix} \mathcal{L}_1\mathcal{L}_1(\phi_{11}) & \mathcal{L}_1\mathcal{L}_1(\phi_{12}) & \mathcal{L}_1\mathcal{L}_1(\phi_{22}) \\ \mathcal{L}_1\mathcal{L}_2(\phi_{11}) & \mathcal{L}_1\mathcal{L}_2(\phi_{12}) & \mathcal{L}_1\mathcal{L}_2(\phi_{22}) \\ \mathcal{L}_2\mathcal{L}_2(\phi_{11}) & \mathcal{L}_2\mathcal{L}_2(\phi_{12}) & \mathcal{L}_2\mathcal{L}_2(\phi_{22}) \end{pmatrix} |_{(0,0,0,\zeta_1,\zeta_2,0)}. \tag{4.4}
$$

Here we denote by $\mathcal{L}_j = 2i\overline{\zeta_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ the complexification of $L_j$. By (4.1), we have

$$
\mathcal{L}_j\mathcal{L}_l(f_k)|_{(0,0,0,\zeta_1,\zeta_2,0)} = (2i\overline{\zeta_j}\frac{\partial^2 f_k}{\partial z_l \partial w} + 2i\overline{\zeta_l}\frac{\partial^2 f_k}{\partial z_j \partial w})|_{(0,0,0,\zeta_1,\zeta_2,0)}
$$

and

$$
\mathcal{L}_j\mathcal{L}_l(\phi_{kt})|_{(0,0,0,\zeta_1,\zeta_2,0)} = (\frac{\partial^2 \phi_{kt}}{\partial z_j \partial z_l} + 2i\overline{\zeta_j}\frac{\partial^2 \phi_{kt}}{\partial z_l \partial w} + 2i\overline{\zeta_l}\frac{\partial^2 \phi_{kt}}{\partial z_j \partial w} - 4\overline{\zeta_j\zeta_l}\frac{\partial^2 \phi_{kt}}{\partial w^2})|_{(0,0,0,\zeta_1,\zeta_2,0)}.
$$

Hence

$$
\begin{aligned}
&\mathcal{L}_1\mathcal{L}_1(\phi_{11})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2 + 4i\overline{\zeta_1}b^{(11)}_{101} - 8\overline{\zeta_1}^2 b^{(11)}_{002}, \\
&\mathcal{L}_1\mathcal{L}_1(\phi_{12})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_1}b^{(12)}_{101} - 8\overline{\zeta_1}^2 b^{(12)}_{002}, \\
&\mathcal{L}_1\mathcal{L}_1(\phi_{22})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_1}b^{(22)}_{101} - 8\overline{\zeta_1}^2 b^{(22)}_{002}, \\
&\mathcal{L}_1\mathcal{L}_2(\phi_{11})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2i\overline{\zeta_1}b^{(11)}_{011} + 2i\overline{\zeta_2}b^{(11)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(11)}_{002}, \\
&\mathcal{L}_1\mathcal{L}_2(\phi_{12})|_{(0,0,0,\zeta_1,\zeta_2,0)} = \sqrt{1+\mu_2} + 2i\overline{\zeta_1}b^{(12)}_{011} + 2i\overline{\zeta_2}b^{(12)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(12)}_{002}, \\
&\mathcal{L}_1\mathcal{L}_2(\phi_{22})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2i\overline{\zeta_1}b^{(22)}_{011} + 2i\overline{\zeta_2}b^{(22)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(22)}_{002}, \\
&\mathcal{L}_2\mathcal{L}_2(\phi_{11})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_2}b^{(11)}_{011} - 8\overline{\zeta_2}^2 b^{(11)}_{002}, \\
&\mathcal{L}_2\mathcal{L}_2(\phi_{12})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_2}b^{(12)}_{011} - 8\overline{\zeta_2}^2 b^{(12)}_{002}, \\
&\mathcal{L}_2\mathcal{L}_2(\phi_{22})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2\sqrt{\mu_2} + 4i\overline{\zeta_2}b^{(22)}_{011} - 8\overline{\zeta_2}^2 b^{(22)}_{002}
\end{aligned} \tag{4.5}
$$

where we write $\phi_{kt} = \sum b_{jls}^{(kt)} z_1^j z_2^l w^s$.

**Lemma 4.2**: Let $B$ be the matrix in (4.4). Then

$$B^{-1} = \frac{1}{det(B)} \begin{pmatrix} b_{11}^{-1} & -b_{12}^{-1} & b_{13}^{-1} \\ -b_{21}^{-1} & b_{22}^{-1} & -b_{23}^{-1} \\ b_{31}^{-1} & -b_{32}^{-1} & b_{33}^{-1} \end{pmatrix} \tag{4.6}$$

where

$$\begin{aligned}
b_{11}^{-1} &= A_{00} + A_{10}\overline{\zeta_1} + A_{01}\overline{\zeta_2} + A_{11}\overline{\zeta_1\zeta_2} + A_{02}\overline{\zeta_2}^2 + A_{12}\overline{\zeta_1\zeta_2}^2 + A_{03}\overline{\zeta_2}^3, \\
b_{12}^{-1} &= B_{10}\overline{\zeta_1} + B_{20}\overline{\zeta_1}^2 + B_{11}\overline{\zeta_1\zeta_2} + B_{21}\overline{\zeta_1}^2\overline{\zeta_2} + B_{12}\overline{\zeta_1\zeta_2}^2, \\
b_{13}^{-1} &= C_{10}\overline{\zeta_1} + C_{20}\overline{\zeta_1}^2 + C_{30}\overline{\zeta_1}^3 + C_{21}\overline{\zeta_1}^2\overline{\zeta_2}, \\
b_{21}^{-1} &= D_{10}\overline{\zeta_1} + D_{01}\overline{\zeta_2} + D_{11}\overline{\zeta_1\zeta_2} + D_{02}\overline{\zeta_2}^2 + D_{12}\overline{\zeta_1\zeta_2}^2 + D_{03}\overline{\zeta_2}^3, \\
b_{22}^{-1} &= E_{00} + E_{10}\overline{\zeta_1} + E_{01}\overline{\zeta_2} + E_{20}\overline{\zeta_1}^2 + E_{11}\overline{\zeta_1\zeta_2} + E_{02}\overline{\zeta_2}^2 + E_{21}\overline{\zeta_1}^2\overline{\zeta_2} + E_{12}\overline{\zeta_1\zeta_2}^2, \\
b_{23}^{-1} &= F_{10}\overline{\zeta_1} + F_{01}\overline{\zeta_2} + F_{20}\overline{\zeta_1}^2 + F_{11}\overline{\zeta_1\zeta_2} + F_{30}\overline{\zeta_1}^3 + F_{21}\overline{\zeta_1}^2\overline{\zeta_2}, \\
b_{31}^{-1} &= G_{01}\overline{\zeta_2} + G_{02}\overline{\zeta_2}^2 + G_{12}\overline{\zeta_1\zeta_2}^2 + G_{03}\overline{\zeta_2}^3, \\
b_{32}^{-1} &= H_{01}\overline{\zeta_2} + H_{11}\overline{\zeta_1\zeta_2} + H_{02}\overline{\zeta_2}^2 + H_{21}\overline{\zeta_1}^2\overline{\zeta_2} + H_{12}\overline{\zeta_1\zeta_2}^2, \\
b_{33}^{-1} &= I_{00} + I_{10}\overline{\zeta_1} + I_{01}\overline{\zeta_2} + I_{20}\overline{\zeta_1}^2 + I_{11}\overline{\zeta_1\zeta_2} + I_{30}\overline{\zeta_1}^3 + I_{21}\overline{\zeta_1}^2\overline{\zeta_2}.
\end{aligned}$$

with

$$\begin{aligned}
&A_{00} = 2\sqrt{\mu_2}\sqrt{1+\mu_2},\ A_{10} = 4i\sqrt{\mu_2}b_{011}^{(12)},\ A_{01} = 4i\sqrt{1+\mu_2}b_{011}^{(22)} + 4i\sqrt{\mu_2}b_{101}^{(12)}, \\
&A_{02} = -8\sqrt{1+\mu_2}b_{002}^{(22)} + 8b_{011}^{(12)}b_{101}^{(22)} - 8b_{101}^{(12)}b_{011}^{(22)},\ A_{11} = -16\sqrt{\mu_2}b_{002}^{(12)}, \\
&A_{03} = -16ib_{101}^{(12)}b_{002}^{(22)} + 16ib_{002}^{(12)}b_{101}^{(22)},\ A_{12} = -16ib_{002}^{(12)}b_{011}^{(22)} + 16ib_{011}^{(12)}b_{002}^{(22)};
\end{aligned}$$

$$\begin{aligned}
&B_{10} = 8i\sqrt{\mu_2}b_{101}^{(12)},\ B_{11} = -16b_{101}^{(12)}b_{011}^{(22)} + 16b_{011}^{(12)}b_{101}^{(22)},\ B_{20} = -16\sqrt{\mu_2}b_{002}^{(12)}, \\
&B_{12} = -32ib_{101}^{(12)}b_{002}^{(22)} + 32ib_{002}^{(12)}b_{101}^{(22)},\ B_{21} = -32ib_{002}^{(12)}b_{011}^{(22)} + 32ib_{011}^{(12)}b_{002}^{(22)};
\end{aligned}$$

$$\begin{aligned}
&C_{10} = -4i\sqrt{1+\mu_2}b_{101}^{(22)},\ C_{20} = -8b_{101}^{(12)}b_{011}^{(22)} + 8\sqrt{1+\mu_2}b_{002}^{(22)} + 8b_{011}^{(12)}b_{101}^{(22)}, \\
&C_{21} = 16ib_{002}^{(12)}b_{101}^{(22)} - 16ib_{101}^{(12)}b_{002}^{(22)},\ C_{30} = -16ib_{002}^{(12)}b_{011}^{(22)} + 16ib_{011}^{(12)}b_{002}^{(22)};
\end{aligned}$$

$$\begin{aligned}
&D_{10} = 4i\sqrt{\mu_2}b_{011}^{(11)},\ D_{11} = -16\sqrt{\mu_2}b_{002}^{(11)}\ D_{02} = -8b_{101}^{(11)}b_{011}^{(22)} + 8b_{011}^{(11)}b_{101}^{(22)}, \\
&D_{12} = -16ib_{002}^{(11)}b_{011}^{(22)} + 16ib_{011}^{(11)}b_{002}^{(22)},\ D_{03} = -16ib_{101}^{(11)}b_{002}^{(22)} + 16ib_{002}^{(11)}b_{101}^{(22)};
\end{aligned}$$

$$\begin{aligned}
&E_{00} = 4\sqrt{\mu_2},\ E_{10} = 8i\sqrt{\mu_2}b_{101}^{(11)},\ E_{01} = 8ib_{011}^{(22)}, \\
&E_{02} = -16b_{002}^{(22)},\ E_{20} = -16\sqrt{\mu_2}b_{002}^{(11)},\ E_{11} = -16b_{101}^{(11)}b_{011}^{(22)} + 16b_{011}^{(11)}b_{101}^{(22)}, \\
&E_{21} = -32ib_{002}^{(11)}b_{011}^{(22)} + 32ib_{011}^{(11)}b_{002}^{(22)},\ E_{12} = -32ib_{101}^{(11)}b_{002}^{(22)} + 32ib_{002}^{(11)}b_{101}^{(22)};
\end{aligned}$$

$$F_{11} = -16b_{002}^{(22)},\ F_{20} = -8b_{101}^{(11)}b_{011}^{(22)} + 8b_{011}^{(11)}b_{101}^{(22)},$$
$$F_{30} = -16ib_{002}^{(11)}b_{011}^{(22)} + 16ib_{011}^{(11)}b_{002}^{(22)},\ F_{21} = -16ib_{101}^{(11)}b_{002}^{(22)} + 16ib_{002}^{(11)}b_{101}^{(22)};$$

$$G_{01} = -4i\sqrt{1+\mu_2}b_{011}^{(11)},\ G_{02} = -8b_{101}^{(11)}b_{011}^{(12)} + 8b_{011}^{(11)}b_{101}^{(12)} + 8\sqrt{1+\mu_2}b_{002}^{(11)},$$
$$G_{12} = 16ib_{011}^{(11)}b_{002}^{(12)} - 16ib_{002}^{(11)}b_{011}^{(12)},\ G_{03} = -16ib_{101}^{(11)}b_{002}^{(12)} + 16ib_{002}^{(11)}b_{101}^{(12)};$$

$$H_{01} = 8ib_{011}^{(12)},\ H_{11} = -16b_{101}^{(11)}b_{011}^{(12)} + 16b_{011}^{(11)}b_{101}^{(12)},\ H_{02} = -16b_{002}^{(12)},$$
$$H_{21} = -32ib_{002}^{(11)}b_{011}^{(12)} + 32ib_{011}^{(11)}b_{002}^{(12)},\ H_{12} = -32ib_{101}^{(11)}b_{002}^{(12)} + 32ib_{002}^{(11)}b_{101}^{(12)};$$

$$I_{00} = 2\sqrt{1+\mu_2},\ I_{01} = 4ib_{101}^{(12)},\ I_{20} = -8b_{101}^{(11)}b_{011}^{(12)} + 8b_{011}^{(11)}b_{101}^{(12)} - 8\sqrt{1+\mu_2}b_{002}^{(11)}$$
$$I_{10} = 4ib_{011}^{(12)} + 4i\sqrt{1+\mu_2}b_{101}^{(11)},\ I_{11} = -16b_{002}^{(12)},$$
$$I_{21} = -16ib_{002}^{(12)}b_{101}^{(11)} + 16ib_{002}^{(11)}b_{101}^{(12)},\ I_{30} = -16ib_{002}^{(11)}b_{011}^{(12)} + 16ib_{011}^{(11)}b_{002}^{(12)}.$$

*Proof of Lemma 4.2*: Denote by $B = (b_{ij})_{3\times 3}$. By (4.5)

$$\begin{aligned}
b_{11}^{-1} &= det\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33}\end{pmatrix} = b_{22}b_{33} - b_{32}b_{23} \\
&= \left(\sqrt{1+\mu_2} + 2i\overline{\zeta_1}b_{011}^{(12)} + 2i\overline{\zeta_2}b_{101}^{(12)} - 8\overline{\zeta}_1\overline{\zeta_2}b_{002}^{(12)}\right)\left(2\sqrt{\mu_2} + 4i\overline{\zeta_2}b_{011}^{(22)} - 8\overline{\zeta_2}^2 b_{002}^{(22)}\right) \\
&\quad - \left(4i\overline{\zeta_2}b_{011}^{(12)} - 8\overline{\zeta_2}^2 b_{002}^{(12)}\right)\left(2i\overline{\zeta_1}b_{011}^{(22)} + 2i\overline{\zeta_2}b_{101}^{(22)} - 8\overline{\zeta}_1\overline{\zeta_2}b_{002}^{(22)}\right) \\
&= A_{00} + A_{10}\overline{\zeta_1} + A_{01}\overline{\zeta_2} + A_{11}\overline{\zeta_1\zeta_2} + A_{02}\overline{\zeta_2}^2 + A_{12}\overline{\zeta_1\zeta_2}^2 + A_{03}\overline{\zeta_2}^3
\end{aligned}$$

where the $A_j$ are as above. Other formulas are obtained by the similar computation. ∎

*Proof of Theorem 4.1*: From (4.6) (4.3) and (4.2), we have

$$\overline{\phi_{11}(\zeta,0)} = \frac{\Phi_{11}}{det(B)},\ \overline{\phi_{12}(\zeta,0)} = \frac{\Phi_{12}}{det(B)},\ \overline{\phi_{22}(\zeta,0)} = \frac{\Phi_{22}}{det(B)}, \tag{4.7}$$

where

$$\begin{aligned}
\Phi_{11}(\overline{\zeta_1},\overline{\zeta_2}) &= 2\overline{\zeta_1}^2 b_{11}^{-1} - (1+\mu_2)\overline{\zeta_1\zeta_2}b_{12}^{-1} + 2\mu_2\overline{\zeta_2}^2 b_{13}^{-1} \\
&= 2\overline{\zeta_1}^2\left(A_{00} + A_{10}\overline{\zeta_1} + A_{01}\overline{\zeta_2} + A_{11}\overline{\zeta_1\zeta_2} + A_{02}\overline{\zeta_2}^2 + A_{12}\overline{\zeta_1\zeta_2}^2 + A_{03}\overline{\zeta_2}^3\right) \\
&\quad - (1+\mu_2)\overline{\zeta_1\zeta_2}\left(B_{10}\overline{\zeta_1} + B_{20}\overline{\zeta_1}^2 + B_{11}\overline{\zeta_1\zeta_2} + B_{21}\overline{\zeta_1}^2\overline{\zeta_2} + B_{12}\overline{\zeta_1\zeta_2}^2\right) \\
&\quad + 2\mu_2\overline{\zeta_2}^2\left(C_{10}\overline{\zeta_1} + C_{20}\overline{\zeta_1}^2 + C_{30}\overline{\zeta_1}^3 + C_{21}\overline{\zeta_1}^2\overline{\zeta_2}\right),
\end{aligned} \tag{4.8}$$

$$
(4.9)\quad
\begin{aligned}
\Phi_{12}(\overline{\zeta_1},\overline{\zeta_2}) &= -2\overline{\zeta_1}^2 b_{21}^{-1} + (1+\mu_2)\overline{\zeta_1\zeta_2} b_{22}^{-1} - 2\mu_2\overline{\zeta_2}^2 b_{23}^{-1} \\
&= -2\overline{\zeta_1}^2\Big(D_{10}\overline{\zeta_1} + D_{01}\overline{\zeta_2} + D_{11}\overline{\zeta_1\zeta_2} + D_{02}\overline{\zeta_2}^2 + D_{12}\overline{\zeta_1\zeta_2}^2 + D_{03}\overline{\zeta_2}^3\Big) \\
&+ (1+\mu_2)\overline{\zeta_1\zeta_2}\Big(E_{00} + E_{10}\overline{\zeta_1} + E_{01}\overline{\zeta_2} + E_{20}\overline{\zeta_1}^2 \\
&\quad + E_{11}\overline{\zeta_1\zeta_2} + E_{02}\overline{\zeta_2}^2 + E_{21}\overline{\zeta_1}^2\overline{\zeta_2} + E_{12}\overline{\zeta_1\zeta_2}^2\Big) \\
&- 2\mu_2\overline{\zeta_2}^2\Big(F_{10}\overline{\zeta_1} + F_{01}\overline{\zeta_2} + F_{20}\overline{\zeta_1}^2 + F_{11}\overline{\zeta_1\zeta_2} + F_{30}\overline{\zeta_1}^3 + F_{21}\overline{\zeta_1}^2\overline{\zeta_2}\Big)
\end{aligned}
$$

and

$$
(4.10)\quad
\begin{aligned}
\Phi_{22}(\overline{\zeta_1},\overline{\zeta_2}) &= 2\overline{\zeta_1}^2 b_{31}^{-1} - (1+\mu_2)\overline{\zeta_1\zeta_2} b_{32}^{-1} + 2\mu_2\overline{\zeta_2}^2 b_{33}^{-1} \\
&= 2\overline{\zeta_1}^2\Big(G_{01}\overline{\zeta_2} + G_{02}\overline{\zeta_2}^2 + G_{12}\overline{\zeta_1\zeta_2}^2 + G_{03}\overline{\zeta_2}^3\Big) \\
&- (1+\mu_2)\overline{\zeta_1\zeta_2}\Big(H_{01}\overline{\zeta_2} + H_{11}\overline{\zeta_1\zeta_2} + H_{02}\overline{\zeta_2}^2 + H_{21}\overline{\zeta_1}^2\overline{\zeta_2} + H_{12}\overline{\zeta_1\zeta_2}^2\Big) \\
&+ 2\mu_2\overline{\zeta_2}^2\Big(I_{00} + I_{10}\overline{\zeta_1} + I_{01}\overline{\zeta_2} + I_{20}\overline{\zeta_1}^2 + I_{11}\overline{\zeta_1\zeta_2} + I_{30}\overline{\zeta_1}^3 + I_{21}\overline{\zeta_1}^2\overline{\zeta_2}\Big).
\end{aligned}
$$

From (4.2), it follows that $f_j(z,0) = z_j$, $j = 1,2$. Also $g(z,0) = 0$ always holds. In fact, after complexifying $\mathrm{Im}(g) = |\widetilde{f}|^2$, we have

$$\frac{g(z,w) - \overline{g(\zeta,\eta)}}{2i} = f(z,w)\overline{f(\zeta,\eta)} + \phi(z,w)\overline{\phi(\zeta,\eta)}, \quad \forall \frac{w-\bar\eta}{2i} = \langle z,\bar\zeta\rangle.$$

By putting $z = w = \eta = 0$ in the above equation, we get $\overline{g(\zeta,0)} = 0$ and this means $g(z,0) = 0$. Therefore, in order to prove Theorem 4.1, by Lemma 2.3. it suffices to prove

$$(4.11)\qquad deg(\Phi_{11}(\bar\zeta)) \le 4,\ deg(\Phi_{12}(\bar\zeta)) \le 4,\ deg(\Phi_{22}(\bar\zeta)) \le 4,\ deg(det(B)(\bar\zeta)) \le 3.$$

To prove the first inequality in (4.11), by (4.8), one needs to show

$$2A_{12} - (1+\mu_2)B_{21} + 2\mu_2 C_{30} = 0,\ 2A_{03} - (1+\mu_2)B_{12} + 2\mu_2 C_{21} = 0.$$

This can be verified by the formulas in Lemma 4.2. The second and the third inequalities in (4.11) can be similarly obtained.

To prove the last inequality in (4.11), we write the $3 \times 3$ matrix $B$ in (4.4) as $(\beta_{jl})$. Then

$$det(B) = -\beta_{21} det \begin{pmatrix} \beta_{12} & \beta_{13} \\ \beta_{32} & \beta_{33} \end{pmatrix} + \beta_{22} det \begin{pmatrix} \beta_{11} & \beta_{13} \\ \beta_{31} & \beta_{33} \end{pmatrix} - \beta_{23} det \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{31} & \beta_{32} \end{pmatrix} \tag{4.12}$$

and from (4.5) we obtain

$$\begin{aligned} det(B) = & -\left(2i\overline{\zeta_1}b^{(11)}_{011} + 2i\overline{\zeta_2}b^{(11)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(11)}_{002}\right) \\ & \cdot \Big[(4i\overline{\zeta_1}b^{(12)}_{101} - 8\overline{\zeta_1}^2 b^{(12)}_{002})(2\sqrt{\mu_2} + 4i\overline{\zeta_2}b^{(22)}_{011} - 8\overline{\zeta_2}^2 b^{(22)}_{002}) \\ & - (4i\overline{\zeta_2}b^{(12)}_{011} - 8\overline{\zeta_2}^2 b^{(12)}_{002})(4i\overline{\zeta_1}b^{(22)}_{101} - 8\overline{\zeta_1}^2 b^{(22)}_{002})\Big] \\ & + \left(\sqrt{1+\mu_2} + 2i\overline{\zeta_1}b^{(12)}_{011} + 2i\overline{\zeta_2}b^{(12)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(12)}_{002}\right) \\ & \cdot \Big[(2 + 4i\overline{\zeta_1}b^{(11)}_{101} - 8\overline{\zeta_1}^2 b^{(11)}_{002})(2\sqrt{\mu_2} + 4i\overline{\zeta_2}b^{(22)}_{011} - 8\overline{\zeta_2}^2 b^{(22)}_{002}) \\ & - (4i\overline{\zeta_2}b^{(11)}_{011} - 8\overline{\zeta_2}^2 b^{(11)}_{002})(4i\overline{\zeta_1}b^{(22)}_{101} - 8\overline{\zeta_1}^2 b^{(22)}_{002})\Big] \\ & - \left(2i\overline{\zeta_1}b^{(22)}_{011} + 2i\overline{\zeta_2}b^{(22)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(22)}_{002}\right) \\ & \cdot \Big[(2 + 4i\overline{\zeta_1}b^{(11)}_{101} - 8\overline{\zeta_1}^2 b^{(11)}_{002})(4i\overline{\zeta_2}b^{(12)}_{011} - 8\overline{\zeta_2}^2 b^{(12)}_{002}) \\ & - (4i\overline{\zeta_2}b^{(11)}_{011} - 8\overline{\zeta_2}^2 b^{(11)}_{002})(4i\overline{\zeta_1}b^{(12)}_{101} - 8\overline{\zeta_1}^2 b^{(12)}_{002})\Big]. \end{aligned}$$

By direct computation, one can verify that the $\overline{\zeta_1}^j\overline{\zeta_2}^l$ terms of degree 4, 5 and 6 above all vanish so that the last inequality in (4.11) holds. ∎

## References

[A77] H. Alexander, Proper holomorphic maps in $\mathbf{C}^n$, Indiana Univ. Math. Journal 26, 137-146 (1977).

[BER99] M. S. Baouendi, P. Ebenfelt and L. Rothschild, Real Submanifolds in Complex Spaces and Their Mappings, Princeton Univ. Mathematics Series 47, Princeton University, New Jersey, 1999.

[CS83] J. Cima and T. J. Suffridge, A reflection principle with applications to proper holomorphic mappings, Math Ann. 265, 489-500 (1983).
[CS90] J.Cima and T. J. Suffridge, Boundary behavior of rational proper maps, Duke Math. J. 60, 135-138 (1990).
[DA93] J. P. D'Angelo, Several Complex Variables and the Geometry of Real Hypersurfaces, CRC Press, Boca Raton, 1993.
[DA88] J. P. D'Angelo, Proper holomorphic mappings between balls of different dimensions, *Mich. Math. J.* 35, 83-90 (1988).
[DA02] J. D'Angelo, Inequalities from Complex Analysis, Carus Mathematical Monograph No. 28, Mathematics Association of America, 2002.
[DC] J. D'Angelo and D. Catlin, A stabilization theorem for Hermitian forms and applications to holomorphic mappings, Math Research Letters 3, 149-166 (1996).
[DL] J. D'Angelo and D. Lichtblau, Spherical space forms, CR mappings, and proper maps between balls, J. Geom. Anal. 2 (1992), no. 5, 391–415.
[Fa82] J. Faran, Maps from the two ball to the three ball, Invent. Math. 68, 441-475 (1982).
[Fa86] J. Faran, On the linearity of proper maps between balls in the lower dimensional case, Jour. Diff. Geom. 24, 15-17 (1986).
[Fo89] F. Forstneric, Extending proper holomorphic mappings of positive codimension, Invent. Math., 95, 31-62 (1989).
[Fo92] F. Forstneric, A survey on proper holomorphic mappings, Proceeding of Year in SCVs at Mittag-Leffler Institute, Math. Notes 38 (1992), Princeton University Press, Princeton, N.J.
[H99] X. Huang, On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions, Jour. of Diff. Geom. Vol (51) No. 1, 13-33 (1999).
[H01] X. Huang, On some problems in several complex variables and CR geometry. First International Congress of Chinese Mathematicians (Beijing, 1998), 383–396, AMS/IP Stud. Adv. Math., 20, Amer. Math. Soc., Providence, RI, 2001.
[H02] X. Huang, On a Semi-Rigidity Property for Holomorphic Maps, preprint, 2002. (To appear in Asian Jour. of Math., a special issue in honor of Professor Y-T Siu's 60th birthday).
[HJ01] X. Huang and S. Ji, Mapping $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$, Invent. Math. 145, 219-250(2001).
[JX03] S. Ji and D. Xu, Rational maps between $\mathbf{B^n}$ and $\mathbf{B^N}$ with geometric rank $\kappa_0 \leq n-2$ and minimal target dimension, preprint, 2003. (To appear in Asian Jour. of Math., a special issue in honor of Professor Y-T Siu's 60th birthday).
[P07] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, Ren. Cire. Mat. Palermo, II. Ser. 23, 185-220 (1907).

[T62] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan 14, 397-429(1962).
[W79] S. Webster, On mappings an (n+1)-ball in the complex space, Pac. J. Math. 81, 267-272 (1979).

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