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A NEW GAP PHENOMENON FOR PROPER HOLOMORPHIC MAPPINGS FROM $\mathbf{B}^n$ INTO $\mathbf{B}^N$

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1. Introduction

Write $\mathbf{B}^n = \{z \in \mathbf{C}^n : |z| < 1\}$ and $Prop(\mathbf{B}^n, \mathbf{B}^N)$ for the collection of all proper holomorphic mappings from $\mathbf{B}^n$ into $\mathbf{B}^N$. Recall that f and $g \in Prop(\mathbf{B}^n, \mathbf{B}^N)$ are said to be equivalent if there are automorphisms $\sigma \in Aut(\mathbf{B}^n)$ and $\tau \in Aut(\mathbf{B}^N)$ such that $f = \tau \circ g \circ \sigma$. By a classical result of Poincaré and Alexander [A77], every proper holomorphic self-map of $\mathbf{B}^n$ with $n \geq 2$ is equivalent to the identity map.

For $N > n > 1$, denote by $Rat(\mathbf{B}^n, \mathbf{B}^N)$ the collection of all rational proper holomorphic mappings from $\mathbf{B}^n$ to $\mathbf{B}^N$. In the past thirty years, there has been much work done on the classification of mappings in $Rat(\mathbf{B}^n, \mathbf{B}^N)$. (For extensive references on this and related studies, see the book of Baouendi-Ebenfelt-Rothschild [BER99] and the survey articles [Fo92] [Hu01]). In [W79], Webster proved that $Rat(\mathbf{B}^n, \mathbf{B}^{n+1})$ has only one equivalence class for $n > 2$. This result was proved to be true by Faran [Fa86] for $N \leq 2n - 2$. For $N = 2n - 1$, when $n = 2$, Faran [Fa82] proved that $Rat(\mathbf{B}^2, \mathbf{B}^3)$ has exactly four equivalence classes. For $n > 2$, the first two authors [HJ01] proved that there are only two equivalence classes in $Rat(\mathbf{B}^n, \mathbf{B}^{2n-1})$.

The situation for $Rat(\mathbf{B}^n, \mathbf{B}^N)$ with $N \geq 2n$ is quite different. First, equivalence classes may form a continuous family. For instance, among other things, D'Angelo constructed in [DA88] the following continuous family of mutually inequivalent proper polynomial embeddings from $\mathbf{B}^n$ into $\mathbf{B}^{2n}$:

$$(1) \quad F_\theta(z', w) = (z', (\cos\theta)w, (\sin\theta)z_1 w, \cdots, (\sin\theta)z_{n-1}w, (\sin\theta)w^2),\ 0 < \theta \leq \pi/2,$$

where $z = (z', w) \in \mathbf{C}^{n-1} \times \mathbf{C}$. More recently, using the same argument that the first two authors developed in [HJ01], Hamada [Ha05] showed that when $n \geq 4$, any map in $Rat(\mathbf{B}^n, \mathbf{B}^{2n})$ is equivalent to F_θ for a certain θ with $0 \leq \theta \leq \pi/2$. In this paper, we provide a new gap phenomenon for proper holomorphic mappings from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N \leq 3n - 4$. We prove the following:

Theorem 1.1. *Let F be a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$, that is C^3-smooth up to the boundary. Suppose that $4 \leq n \leq N \leq 3n - 4$. Then F is equivalent to*

$$F'_\theta := (F_\theta(z, w), 0, \cdots, 0) = (z, w\cos\theta, z_1 w \sin\theta, \cdots, z_{n-1}w\sin\theta, w^2\sin\theta, 0, \cdots, 0)$$

for some θ with $(0 \leq \theta \leq \frac{\pi}{2})$.

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An interesting feature of Theorem 1.1, which is somewhat surprising to us, is that there is no new map added when N runs from $2n$ to $3n-4$. When $N = 3n-3$ with $n \geq 3$, the generalized Whitney map $W_{n,3}$ defined on [pp 463, Example 1.3, Hu03] properly sends $\mathbf{B}^n$ into $\mathbf{B}^{3n-3}$. Notice that the just mentioned $W_{n,3}$ is polynomial but can not be equivalent to F'_θ, for $W_{n,3}$ has geometric rank 2 while F'_θ has geometric rank 1. (See Definition 2.2 of §2 for the definition of the geometric rank). Hence the gap phenomenon in Theorem 1.1 breaks down when $N \geq 3n-3$.

Our proof of Theorem 1.1 is based on a careful analysis of proper holomorphic maps between balls with the so-called degenerate geometric rank, along the lines of researches first carried out in [Hu03]. To state our main theorem, which includes Theorem 1.1 as an application, we next give more definitions and notation:

Denote by $Prop_k(\mathbf{B}^n, \mathbf{B}^N)$ the collection of all proper holomorphic mappings from $\mathbf{B}^n$ into $\mathbf{B}^N$ that are C^k smooth up to the boundary ($k \geq 2$). By the work in [Hu03], each map $F \in Prop_2(\mathbf{B}^n, \mathbf{B}^N)$ can be associated with an invariant integer $\kappa_0 \in \{0, 1, \cdots, n-1\}$, called its geometric rank (see Definition 2.2 in §2 for the precise definition of κ_0). An early result of the first author ([Hu99, Theorem 4.2]) states that F has geometric rank $\kappa_0 = 0$ if and only if F is equivalent to a linear fractional map. For $F \in Prop_3(\mathbf{B}^n, \mathbf{B}^N)$ ($n \geq 3$) with $\kappa_0 \leq n-2$, we say F has degenerate geometric rank. By the results of [Hu03] and [HJX05], such a map must be rational and $(n-\kappa_0)$-linear. Here $F \in Prop(\mathbf{B}^n, \mathbf{B}^N)$ *is said to be k-linear*, if for any point $p \in \mathbf{B}^n$, there is an affine complex subspace S^a_p containing p and of complex dimension k such that the restriction of F to S^a_p is a linear fractional map. At this point, we should mention a theorem of Forstneric [Fo89] which states that $Prop_{N-n+1}(\mathbf{B}^n, \mathbf{B}^N) = Rat(\mathbf{B}^n, \mathbf{B}^N)$ for $N \geq n > 1$. (See also a very interesting paper of Mir [Mir03] later on a more general situation).

Our main purpose of this paper is to study the normalization problem for maps in $Rat(\mathbf{B}^n, \mathbf{B}^N)$ with degenerate geometric rank. For $n \geq 3$ and for maps with geometric rank one, we will also describe precisely the hyperplanes along which the maps are linear fractional, which will, in particular, give Theorem 1.1 as one of the immediate applications. For the case with general degenerate geometric rank, the normal form will be derived in the same manner. However, we have to leave open the question to determine precisely the linearity directions. For a non-linear map $F \in Rat(\mathbf{B}^2, \mathbf{B}^N)$ with $N \geq 3$, it has geometric rank 1 that is also $n-1$. The structure of F could be very complicated, as indicated by the work in [Fa82] [CD96]. Indeed, by a result of Catlin-D'Angelo [CD96], any holomorphic rational map from $\overline{\mathbf{B}^2}$ into $\mathbf{B}^N$ can be extended to an element in $Rat(\mathbf{B}^2, \mathbf{B}^{N^*})$, by suitably choosing the last (N^*-N)-components, when N^* is sufficiently large. In the last section of the paper, we give a normal form for mappings in $Rat(\mathbf{B}^2, \mathbf{B}^N)$ with degree 2 under the action of the isotropic automorphism group of the Heisenberg hypersurfaces.

We now state our main result:

Theorem 1.2. *Let F be a non-linear proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N \geq n \geq 3$. Assume that F is C^3-smooth up to the boundary and has geometric rank $\kappa_0 \leq n-2$. Then F is equivalent to a proper holomorphic map of the form*

$$H := (z_1, \cdots, z_{k^0}, H_1, \cdots, H_{N-k^0}),$$

where $k^0 = n - \kappa_0$ *and* $H_j = \sum_{l=k^0+1}^{n} z_l H_{j,l}$ *with* $H_{j,l}$ *holomorphic over* $\overline{\mathbf{B}^n}$. *Moreover, when* $\kappa_0 = 1$, $(H_1, \cdots, H_{N-n+1}) = z_n \cdot h$ *with* h *a rational proper holomorphic map from* $\mathbf{B}^n$ *into* $\mathbf{B}^{N-n+1}$. *Both* H *and* h *are affine linear maps along each hyperplane defined by* $z_n = constant$.

When $N \leq 3n-3$, then $N-n+1 \leq 2n-2$. Assume that F is given as in Theorem 1.2 with geometric rank one. Then the corresponding map h must be linear fractional by the linearity theorem in [Fa86] [Hu99]. Therefore, one sees that H reduces to a proper map from $\mathbf{B}^n$ into $\mathbf{B}^{2n}$. Now, combining this with [Hu03, Theorem 1.1] [HJX05, Corollary 1.3], [§ 6, HJ01] and [Theorem 1.1, Ha05], we can obtain the result stated in Theorem 1.1. Theorem 1.2 can also be immediately used to derive the following degree estimate:

Corollary 1.3. *Let* $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ *have geometric rank* 1. *Assume that* $n \geq 3$. *The degree of* F *is then bounded by* $\frac{N-1}{n-1}$.

The degree estimate in Corollary 1.3 is optimal. Indeed, the Whitney map has degree 2. By letting h in Theorem 1.2 be the Whitney map, we get a proper polynomial map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = 3n-2$ of degree 3. Inductively, one can construct a proper polynomial map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = kn-(k-1)$ of degree k. At this point, we mention a conjecture of D'Angelo which states that the degree of a rational proper map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $n \geq 3$ is bounded by $\frac{N-1}{n-1}$. Hence, Corollary 1.3 partially provides an affirmative solution to the aforementioned conjecture of D'Angelo.

The organization of the paper is as follows: We set up some notations in §2. We prove Theorem 1.2 and Theorem 1.1 in §3. We also give the map $W_{n,k}$ (first defined in [Example 1.3, Hu3]) in Example 3.3, showing that Theorem 1.1 is optimal. In §4, we give a normal form for rational maps of degree 2 from $\mathbf{B}^2$ into $\mathbf{B}^N$ with $N \geq 4$ under the action of the isotropic automorphisms of the Heisenberg hypersurfaces.

2. Notation and preliminaries

•Maps between balls Write $\mathbf{H}_n := \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \ \mathrm{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Similarly, we can define the space $\mathrm{Rat}(\mathbf{H}_n, \mathbf{H}_N)$, $\mathrm{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$ and $\mathrm{Prop}(\mathbf{H}_n, \mathbf{H}_N)$. Since the Cayley transformation

$$\rho_n : \mathbf{H}_n \to \mathbf{B}^n, \quad \rho_n(z,w) = \Big(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\Big) \tag{2}$$

is a biholomorphic mapping between $\mathbf{H}_n$ and $\mathbf{B}^n$, we can identify a map $F \in \mathrm{Prop}_k(\mathbf{B}^n, \mathbf{B}^N)$ or $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $\mathrm{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$ or $\mathrm{Rat}(\mathbf{H}_n, \mathbf{H}_N)$, respectively. Notice that

$$\rho_n^{-1} : \mathbf{B}^n \to \mathbf{H}_n, \quad \rho_n^{-1}(z,w) = \Big(\frac{z}{1+w}, \frac{i-iw}{1+w}\Big) \tag{3}$$

Parameterize $\partial \mathbf{H}_n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial \mathbf{H}_n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2 u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t(\in \mathbf{R}) \to 0$. We use the notation $h^{(k)}$ to denote a polynomial h which has weighted degree k.

• Partial normalization of F

Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a non-constant C^2-smooth CR map from $\partial \mathbf{H}_n$ into $\partial \mathbf{H}_N$ with $F(0) = 0$. For each $p \in M$ close to 0, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbf{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and $\tau_p^F \in \mathrm{Aut}(\mathbf{H}_N)$ by defining

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle).$$

Then F is equivalent to

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p). \tag{4}$$

Notice that $F_0 = F$ and $F_p(0) = 0$. The following is basic for the understanding of the geometric properties of F.

Lemma 2.1 ([§2, Lemma 5.3, Hu99], [Lemma 2.0, Hu03]): Let F be a C^2-smooth CR map from $\partial \mathbf{H}_n$ into $\partial \mathbf{H}_N$, $2 \le n \le N$. For each $p \in \partial \mathbf{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbf{H}_N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4), \text{ with}$$

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Definition 2.2. Let $\mathcal{A}(p) = -2i(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0)_{1\le j,l\le (n-1)}$. We call the rank of $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the *geometric rank* of F at p.

$Rk_F(p)$ depends only on p and F, and is a lower semi-continuous function on p. Define the *geometric rank* of F to be $\kappa_0(F) = max_{p \in \partial \mathbf{H_n}} Rk_F(p)$. Notice that it always holds that $0 \le \kappa_0 \le n - 1$. Define the geometric rank of $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbf{H}_n, \mathbf{H}_N)$. By [Hu03], $\kappa_0(F)$ depends only on the equivalence class of F. It is known that F is linear fractional if and only if the geometric rank of F is 0 ([Theorem 4.2, Hu99]). Hence, in what follows, we will always assume that $\kappa_0(F) \ge 1$.

Let $F = (f, \phi, g) \in Rat(\mathbf{H}_n, \mathbf{H}_N)$, which satisfies the normalization in Lemma 2.1. Define $L_j = 2i\bar{z}_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$, $1 \le j \le n-1$, which form a basis for the sections of the complex tangent bundle $T^{(1,0)}\partial \mathbf{H}_n$. Their complexifications are $\mathcal{L}_j = 2i\bar{\xi}_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$. Consider the basic equation:

$$\frac{g(z, w) - \overline{g(\xi, \eta)}}{2i} = \tilde{f}(z, w)\overline{\tilde{f}(\xi, \eta)},\ w - \bar{\eta} = 2iz \cdot \bar{\xi}. \tag{5}$$

Letting $z = w = \eta = 0$ in (5), we get $g(z, 0) \equiv 0$. Applying $\mathcal{T}^j$ $(j = 1, \cdots, k)$ with $\mathcal{T} = \frac{1}{2}(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{\eta}})$ to (5) and then letting $z = w = \eta = 0$, we get that $g(z, w) = w + o(|(z, w)|^k)$ if $g(0, w) - w, \widetilde{f}(0, w) = o(w^k)$. Finally applying $\mathcal{L}_j$ to (5) for each

j and then letting $\xi = w = \eta = 0$, we have $f(z,0) = z$. In summary, we derived the following:

Lemma 2.3: Let $F = (f, \phi, g) \in Rat(\mathbf{H}_n, \mathbf{H}_N)$, which satisfies the normalization in Lemma 2.1. Then
(A) $g(z,0) = 0$, $\frac{\partial g}{\partial w}(z,0) \equiv 1$, and $f(z,0) = z$.
(B) If we further assume that $(\widetilde{f}(0,w), g(0,w)) = (0,w)$, then $g \equiv w$.

Statements in (A) are well-known in the literature (see, for instance [BER99] [HJ01], etc). (B) was first obtained in [HJ01] for $N = 2n-1$ and was later generalized to the case of $N = 2n$ in [Ha05]. The argument in [Ha05] can also be used to get the statement in (B) for any $N \geq n$.

• **Degree of a rational map** For a rational holomorphic map $H = \frac{(P_1,\dots,P_m)}{Q}$ over $\mathbf{C}^n$, where P_j, Q are holomorphic polynomials and $(P_1, ..., P_m, Q) = 1$, we define

$$deg(H) = max\{deg(P_j),\ 1 \leq j \leq m,\ deg(Q)\}.$$

For a rational map H and a complex affine subspace S of dimension k, we say that H is linear fractional along S, if S is not contained in the singular set of H and for any linear parameterization $z_j = z_j^0 + \sum_{l=1}^k a_{jl}t_l$ with $j = 1, \cdots, n$, $H^*(t_1, \cdots, t_k) := H(z_1^0 + \sum_{l=1}^k a_{1l}t_l, \cdots, z_n^0 + \sum_{l=1}^k a_{jn}t_j)$ has degree 1 in $(t_1, \cdots, t_k)$.

3. Mappings with geometric rank bounded by $n-2$

Let $F \in Prop_3(\mathbf{H}_n, \mathbf{H}_N)$ have geometric rank κ_0 with $1 \leq \kappa_0 \leq n-2$. By [Theorem 2.3; Hu03] and [Corollary 1.3; HJX05], we know that F is a rational map. Making use of [Lemmas 3.2, 3.3, 4.1, 4.3, Corollaries 4.2, 5.2, (3.6.4), Claim 4.4; Hu03] and making use of Lemma 2.3, this map is equivalent to a new map with the following normalization:

$$(6) \quad \begin{cases} f_l = z_l + \frac{\sqrt{-1}}{2}\mu_l z_l w + a_l^{(2)}(z)w + o_{wt}(4);\ \mu_l > 0,\ l \leq \kappa_0 \\ f_j = z_j + o_{wt}(4),\ \text{for } \kappa_0 + 1 \leq j \leq n-1; \\ \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2),\ \text{for } \ (j,l) \in \mathcal{S}_0; \phi_{jl} = o_{wt}(2),\ \text{for } (j,l) \notin \mathcal{S}_0; \\ g = w; \\ F(0, \cdots, 0, z_{\kappa_0+1}, \cdots, z_{n-1}, w) = (0, \cdots, 0, z_{\kappa_0+1}, \cdots, z_{n-1}, 0, \cdots, 0, w). \end{cases}$$

Here, for $1 \leq \kappa_0 \leq n-2$, we write $\mathcal{S}_0 = \{(j,l) : 1 \leq j \leq \kappa_0, 1 \leq l \leq (n-1), j \leq l\}$.

Also, $\mu_{jl} = \sqrt{\mu_j + \mu_l}\ for\ j < l \leq \kappa_0; and\ \mu_{jl} = \sqrt{\mu_j}$ if $j \leq \kappa_0$ and $l > \kappa_0$ or if $j = l \leq \kappa_0$.

Let E_0 be the (proper) complex analytic variety consisting of the poles and the non-immerse points of F in (6). As in [Hu03], we define

$$\mathcal{V}_F := \{(Z, S_Z) \in (\mathbf{C}^n \setminus E_0) \times G_{n,k^0}(\mathbf{C}), F \text{ is linear fractional when restricted to } S_Z + Z\}.$$

Here $G_{n,k^0}(\mathbf{C})$ is the Grassmannian manifold consisting of all k^0-dimensional complex subspaces in $\mathbf{C}^n$ with $k^0 = n - \kappa_0$. Then, as in [Hu03, Lemma 5.1], one can similarly verify that $\mathcal{V}_F$ is a complex analytic variety with $\pi : \mathcal{V}_F \to \mathbf{C}^n \setminus E_0$ as its proper holomorphic projection. Also, making use of a similar argument as in [Lemma 5.3, Hu03], we see, away from a certain proper complex analytic subvariety E_1 of $X := \mathbf{C}^n \setminus E_0$, that any point $Z \in X \setminus E_1$ has a unique preimage in $\mathcal{V}_F$. Namely, for any $Z \in X \setminus E_1$, there is a unique complex subspace S_Z of dimension k^0 such that F is linear fractional when restricted to $S_Z + Z$. Indeed, write $\cup_j \mathcal{V}^{(j)}$ for the irreducible decomposition of $\mathcal{V}_F$. Then there is only one irreducible component, say $\mathcal{V}^{(1)}$, whose projection to $\mathbf{C}^n \setminus E_0$ contains a sufficiently small domain that lies inside $\mathbf{H}_n$ and has a small piece of $\partial \mathbf{H}_n$ containing 0 as part of its boundary. (See the proof of [Lemma 5.3, Hu03]). Let $(\mathcal{K}, \rho, \mathcal{V}^{(1)})$ be the desingularization of $\mathcal{V}^{(1)}$. Let A be the singular set of $\pi \circ \rho$, namely, the set where $\pi \circ \rho$ fails to be biholomorphic. Write $B = \pi \circ \rho(A) \cup \pi(sing(\mathcal{V}^{(1)}))$. Then E_1 can be taken as the locally finite union of B with the proper projections of the other irreducible components. Hence, by moving to a nearby point, if necessary, we can assume that $0 \notin E_1$ and thus π is biholomorphic near $(0, S_0) \in \mathcal{V}_F$. Then for any $\epsilon = (\epsilon_1, \cdots, \epsilon_{\kappa_0}) \approx 0$, $S_{(\epsilon,0)} + (\epsilon, 0)$ can be defined by an equation of the form:

$$z_l = \sum_{j=\kappa_0+1}^{n} a_{jl}(\epsilon) z_j + \epsilon_l, \quad l = 1, \cdots, \kappa_0,$$

with a_{jl} holomorphic in ϵ. Here, we identify z_n with w. Notice that $a_{jl}(0) = 0$. Consider the equations:

$$z_l = \sum_{j=\kappa_0+1}^{n} a_{jl}(z'_1, \cdots, z'_{\kappa_0}) z_j + z'_l, \ l = 1, \cdots, \kappa_0.$$

We see that

$$z'_l = \psi_l(z_1, \cdots, z_n) = z_l + \sum_{j=\kappa_0+1}^{n} O(|z|) z_j, \quad l = 1, \cdots, \kappa_0.$$

We will next show that $f_j \equiv z_j$ for $j \geq \kappa_0 + 1$. Fix a $j \geq \kappa_0 + 1$. Indeed, since $(\widetilde{f}(z), w)$ has degree one along each complex affine subspace defined by $z'_l = \epsilon_l$ $(l = 1, \cdots, \kappa_0)$, we can write

$$f_j(z, w) = b_0(z'_1, \cdots, z'_{\kappa_0}) + \sum_{l=\kappa_0+1}^{n-1} b_l(z'_1, \cdots, z'_{\kappa_0}) z_l + b_n(z'_1, \cdots, z'_{\kappa_0}) w.$$

Since $f_j(z_1, \cdots, z_{n-1}, 0) = z_j$, we have $0 = f_j(z_1, \cdots, z_{\kappa_0}, 0') = b_0(z'_1, \cdots, z'_{\kappa_0})$. We conclude that $b_0 \equiv 0$.

Write

$$b_l(z'_1, \cdots, z'_{\kappa_0}) - \delta_l^j = b_l^{(k_l)}(z'_1, \cdots, z'_{\kappa_0}) + o(|(z'_1, \cdots, z'_{\kappa_0})|^{k_l})$$

with $b_l^{(k_l)}$ a certain homogeneous polynomial in $(z'_1, \cdots, z'_{\kappa_0})$ of degree k_l. Here, we assume that $b_l^{(k_l)} \not\equiv 0$ when $b_l \not\equiv \delta_l^j$. Then by the formulas for ψ_l, we conclude that

$$b_l = \delta_l^j + b_l^{(k_l)}(z_1, \cdots, z_{\kappa_0}) + \sum_{l=\kappa_0+1}^{n} O(|(z_1, \cdots, z_n)|^{k_l}) z_j.$$

Since $f_j(z_1, z_2, \cdots, z_{n-1}, 0) = z_j$, we get

$$\sum_{l=\kappa_0+1}^{n-1} b_l^{(k_l)}(z_1, \cdots, z_{\kappa_0}) z_l + \sum_{l=\kappa_0+1}^{n-1} o(|(z_1, \cdots, z_{n-1})|^{k_l}) z_l = 0.$$

We claim that the above equation implies that $b_l \equiv 0$ for $l \neq n$. Suppose not. Assume then $b_{l_0}^{(k_{l_0})} \not\equiv 0$ has a minimum degree for a certain $l_0 \leq n-1$. Comparing the coefficients of the homogeneous polynomials in $(z_1, \cdots, z_{n-1})$ of degree $k_{l_0}+1$, we immediately see that $b_{l_0}^{(k_{l_0})} \equiv 0$. This is a contradiction. Hence $b_l \equiv 0$ for $l < n$.

Complexifying the first equation in [(3.4.5), Hu03]:

$$\overline{T(\widetilde{f})} \cdot L_j(\widetilde{f})^t = \frac{1}{2i} L_j T(g) - (L_j T(\widetilde{f})) \cdot \overline{\widetilde{f}}, \tag{7}$$

and letting $z, w, \eta = 0$, we conclude, from the normalization in (6) and the property that $\frac{\partial^2 \phi_{jl}}{\partial z_j \partial w}|_0 = 0$ for any $j \geq \kappa_0 + 1$ ([Lemma 3.3(C), Hu03]), that

$$b_n(\psi_1(z_1, \cdots, z_{n-1}, 0), \cdots, \psi_{\kappa_0}(z_1, \cdots, z_{n-1}, 0)) \equiv 0,$$

which then forces $b_n \equiv 0$ as argued above. This proves that $f_j \equiv z_j$.

Since $\phi, f_l(0, \cdots, 0, z_{\kappa_0+1}, \cdots, z_{n-1}, w) \equiv 0$ for $l \leq \kappa_0$, we see

$$f_l = \sum_{\tau=1}^{\kappa_0} z_\tau f_{l\tau}^*, \ \phi = \sum_{\tau=1}^{\kappa_0} z_\tau \phi_{l\tau}^*.$$

Now, assume $\kappa_0 = 1$ and write the defining equation of $S_{(\epsilon,0)} + (\epsilon, 0)$ for $\epsilon \approx 0$ as $z_1 = \sum_{j=2}^n a_j(\epsilon) z_j + \epsilon$. (Here we identify z_n with w). There are three cases to be considered:

Case (11). $a_j(\epsilon) = \epsilon a_j^0(\epsilon)$ with $a_{j_0}^0(\epsilon) \neq$ *constant* for a certain $j_0 \leq n$.

Case (22). $a_j(\epsilon) = \epsilon a_j^0$ with $a_j^0 =$ *constant* and $\Im a_n^0 = -\sum_{j=2}^{n-1} |\frac{a_j^0}{2}|^2$.

Case (33). $a_j(\epsilon) = \epsilon a_j^0$ with $a_j^0 =$ *constant* but not in the case of (22).

In the case of (11), S_0 intersects $S_{(\epsilon,0)} + (\epsilon, 0)$ for a generic ϵ. As ϵ varies, we conclude that the union of all such intersections contains an open piece of S_0. Notice that $S_0 \setminus E_0$ is an irreducible complex analytic variety of $\mathbf{C}^n \setminus E_0$. By the uniqueness of complex analytic varieties, we see that $S_0 \subset E_1 \cup E_0$. This contradicts the initial assumption that $0 \notin E_1 \cup E_0$. (One can also argue as follows: By the continuity of $a_j(\epsilon)$ on ϵ, we also see for any $0 < |\epsilon_0| << 1$, that $E_1 \cup E_0$ contains an open piece of $S_{(\epsilon_0,0)}$ and thus $S_{(\epsilon_0,0)} \subset E_1 \cup E_0$. That is a contradiction.) Therefore, Case (11) cannot occur.

Consider Case (22). We can write

$$z_1' = \frac{z_1}{1 + \sum_{j=2}^n a_j^0 z_j} = \frac{z_1}{1 - 2i\langle \overline{\alpha}, z\rangle + (r - i|\alpha|^2)w}$$

with a certain $r \in \mathbf{R}$ and $\alpha = (0, \alpha_2, \cdots, \alpha_{n-1}) \in \mathbf{C}^{n-1}$. Define

$$\sigma(z', w') := \frac{((z' - \alpha w'), w')}{q'(z', w')}$$

with $q'(z',w')=1+2i\langle\overline{\alpha},z'\rangle+(-r-i|\alpha|^2)w'$, and define

$$\tau^*(z^*,w^*):=\frac{((z^*+\alpha^*w^*),w^*)}{q^*(z^*,w^*)}$$

with $q^*(z^*,w^*)=1-2i\langle\overline{\alpha^*},z^*\rangle+(r-i|\alpha^*|^2)w^*$, where $\alpha^*=(\alpha,0)$. Then $\sigma\in Aut_0(\mathbf{H}_n)$ and $\tau^*\in Aut_0(\mathbf{H}_N)$. Moreover, as in [Lemma 2.2 A, Hu03], it is straightforward to verify that $\tau^*\circ F\circ\sigma$ still satisfies the normalizations in (6). The new map, still denoted by F, now is linear fractional along each hyperplane defined by $z_1'=constant$. Since $g=w$, F has to be affine linear along the hyperplanes defined by $z_1'=constant$. Write (z,w) for (z',w'). Then $f_1=z_1f_1^*$ and $\phi_j=z_1\phi_j^*$. Also, it is easy to see that $\Psi=:(\phi_j^*,f_1^*)$ properly maps $\mathbf{H}_n$ into $\mathbf{B}^{N-n+1}$ and Ψ is affine linear along $z_1=constant$. In particular, letting $z_1=0$, we get that

$$\Psi(0,z_2,\cdots,w)=(b_1w,z_2+b_2w,\cdots,z_{n-1}+b_{n-1}w,0,\cdots,0,1+\frac{i}{2}w)$$

maps $Im(w)=\sum_{j=2}^{n-1}|z_j|^2$ into the unit sphere. Namely,

$$|b_1w|^2+\sum_{j=2}^{n-1}|b_jw+z_j|^2+|1+\frac{i}{2}w|^2=1 \quad \text{over} \quad w=u+i\sum_{j=2}^{n-1}|z_j|^2.$$

Comparing the coefficients of terms with the u factor, we get a contradiction. Namely, this case cannot occur neither.

In the case of (33), after composing F by unitary transformations on both sides, we can assume that F is linear fractional (in fact, affine linear) along hyperplanes defined by equations: $z_1=constant\cdot(1+bz_2+cw)$ with $b\geq 0$. Next for $\alpha=(0,K,0,\cdots,0)$ with $K\in\mathbf{C}$, we notice that the inverse of the map

$$\sigma(z',w'):=\frac{((z'-\alpha w'),w')}{q(z',w')}$$

with $q(z',w')=1+2i\langle\overline{\alpha},z'\rangle+(-r-i|\alpha|^2)w'$ transforms the hyperplanes defined by $z_1=constant\cdot(1+bz_2+cw)$ into hyperplanes defined by $z_1'=constant\cdot(q(z',w')+bz_2'-bKw'+cw')$. Notice that

$$q(z',w')+bz_2'-bKw'+cw'=1+2i(\overline{K}-\frac{i}{2}b)z_2'+(-r-i|K|^2+c-bK)w'.$$

Choose $K=-\frac{i}{2}b$ and $r=Re(c)$. Hence, we can easily see that in Case (33), we can make $b=0$ and $c(\neq 0)$ purely imaginary, after composing F with this σ on the right and some other suitable τ on the left as discussed in Case (22). Moreover, composing F by suitable dilation of both sides of F, we can further make $c=\pm i$. We will see later that the case $c=i$ can not occur. Assuming this statement, we thus obtain the following normalization:

Theorem 3.1: *Let $F \in Prop_3(\mathbf{H}_n, \mathbf{H}_N)$ have geometric rank $\kappa_0 \le n-2$. Then F is equivalent to a map of the following form, which is still denoted by F:*

$$(8) \qquad \begin{cases} f_l = \sum_{j=1}^{\kappa_0} z_j f_{lj}^*(z); \ l \le \kappa_0 \\ f_j = z_j, \ \textit{for } \kappa_0 + 1 \le j \le n-1; \\ \phi_{lk} = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^* \ \textit{for} \quad (l,k) \in \mathcal{S}_0, \\ g = w; \\ f_{lj}^*(z,w) = \delta_l^j + \frac{i\delta_l^j \mu_l}{2} w + O(|(z,w)|)^2), \\ \phi_{lkj}^*(z,w) = O_{wt}(2), \quad (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^* = O(|(z,w)|^3) \ \ \textit{for } (l,k) \notin \mathcal{S}_0 \end{cases}$$

Here μ_j, μ_{jl} are as in (6). Moreover, when $\kappa_0 = 1$, F is an affine linear map along each hyperplane defined by: $z_1 = constant \cdot (1 - iw)$.

Completion of the proof of Theorem 3.1 and proof of Theorem 1.2: We first notice that when $\kappa_0 = 1$, F in Theorem 3.1 must be affine linear along the hyperplanes defined by: $z_1 = constant \cdot (1 - iw)$ due to the fact that $deg((\widetilde{f}, w)) = 1$ along such hyperplanes.

To complete the proof of Theorem 3.1, it remains only to explain that the case of $kappa_0 = 1$, F can not be affine linear along the hyperplanes defined by $z_1 = constant\cdot (1+iw)$. Let F be normalized as in (8). Following the procedure in [HJ01] and using the part of Theorem 3.1 which we have proved, we can use the Cayley transformation to obtain the corresponding proper holomorphic mapping $H = \rho_N \circ F \circ \rho_n^{-1}$ from $\mathbf{B}^n$ to $\mathbf{B}^N$, which takes the following form:

$$H = (H_1, \cdots, H_{\kappa_0}, z_{\kappa_0+1}, \cdots, z_{n-1}, H_n, \cdots, H_{N-1}, w),$$

where $H_j = \sum_{l=1}^{\kappa_0} z_l H_{j,l}$ with $H_{j,l}$ holomorphic over $\overline{\mathbf{B}}^n$. Notice that

$$\rho^{-1}(z', w') = (\frac{iz'}{i + iw'}, \frac{w' - 1}{i + iw'}).$$

Hence in case $\kappa_0 = 1$, H is affine linear along one of the following two families of the hyperplanes defined by (1): $z_1 = constant$; (2): $z_1 = constant \cdot w$. Suppose that F is affine linear along each hyperplane defined by $z_1 = constant \cdot w$. Then $H^*(z_1, \cdots, z_n) = (H_{1,1}, \cdots, H_{N-1,1})$ must be a constant map along each hyperplane defined by $z_1 = constant \cdot w$. That is impossible, for H^* is proper from $\mathbf{B}^n$ into $\mathbf{B}^{N-n+1}$. This completes the proof of Theorem 3.1.

After re-ordering the coordinates, we can assume that the map H is of the following form:

$$H = (z_1, \cdots, z_{k^0}, H_1, \cdots, H_{N-k^0}),$$

where $k^0 = n - \kappa_0$ and $H_j = \sum_{l=\kappa_0+1}^{n} z_l H_{j,l}$.

Apparently, when $\kappa_0 = 1$, write $(H_1, \cdots, H_{N-n+1}) = z_n \cdot h = z_n(h_1, \cdots, h_{N-n+1})$. Then h is a proper holomorphic mapping from $\mathbf{B}^n$ to $\mathbf{B}^{N-n+1}$, and h is affine linear along each hyperplane defined by $z_n = constant$. This completes the proof of Theorem 1.2. ☐

Proof of Corollary 1.3: For each $N \ge n \ge 3$, there is a unique positive integer k such that $k(n-1)+1 \le N \le (k+1)(n-1)$. We apply the induction on k. When $k = 1$, $F \in Rat(\mathbf{B}^n, \mathbf{B}^{2n-2})$ and thus $deg(F) = 1 \le \frac{N-1}{n-1}$. Assume $deg(F) \le \frac{N-1}{n-1}$ for $k = k_0$. For

$k = k_0 + 1$ and $N \le (k_0 + 2)(n-1)$, by Theorem 1.1, F is equivalent to (z, wh) where $h \in Rat(\mathbf{B}^n, \mathbf{B}^{N-n+1})$ with geometric rank 0 or 1. Since $N-(n-1) \le (k_0+1)(n-1)$, by the induction assumption, $deg(F) \le 1 + deg(h) \le 1 + \frac{(N-n+1)-1}{n-1} = \frac{N-1}{n-1}$. $\square$

Proof of Theorem 1.1: Let F be as in Corollary 1.3. We assume that F is not linear. By [Hu03], F must have geometric rank one. By Theorem 1.2, we can assume that F is equivalent to H, that is of the following special normal form: $H = (z_1, \cdots, z_{n-1}, w \cdot h(z,w))$ with $h(z,w)$ a proper holomorphic rational map from $\mathbf{B}^n$ into $\mathbf{B}^{N'}$ with $N' = N - n + 1$. Since we now have $N \le 3n - 3$ and $N - n + 1 \le 2n - 2$, h is equivalent to a linear embedding. Hence, there is a unitary transformation U which maps $h(\mathbf{B}^n)$ into the intersection of the ball with the affine complex subspace of $\mathbf{C}^{N'}$ defined by $z_1 = c_1, \cdots, z_{N'-n} = c_{N'-n}$, where $c_j's$ are non-negative constant. Write $|c|^2 = \sqrt{c_1^2 + \cdots + c_{N'-n}^2}$. Hence composing H by (Id, U) from the left, we see that H is equivalent to a map of the form: (Still denote the new map by H)

$$H = (z_1, \cdots, z_{n-1}, (c_1, \cdots, c_{N'-n})w, \sqrt{1-|c|^2}w\tilde{h}(z,w)).$$

Here $\widetilde{h}(z,w)$ is an automorphism of $\mathbf{B}^n$. After a unitary transformation, the above map is equivalent to the following map:

$$H = (z_1, \cdots, z_{n-1}, |c|w, 0, \cdots, 0, w\sqrt{1-|c|^2}\widetilde{h}).$$

Hence, H is reduced to the following rational map from $\mathbf{B}^n$ into $\mathbf{B}^{2n}$:

$$H' = (z_1, \cdots, z_{n-1}, |c|w, w\sqrt{1-|c|^2}\widetilde{h}), \quad \widetilde{h} \in Aut(\mathbf{B}^n).$$

Now, we can apply a result of Hamada [Ha05, §4] to conclude that H is equivalent to the map $H = (F_\theta, \mathbf{0})$, $\theta \in (0, \pi/2]$. (See already very similar arguments on this matter in §6 of [HJ01]). Indeed, at this stage, we can just apply [Ha05, Theorem 1.1] to make a conclusion. The proof of Theorem 1.1 is complete. $\square$

The above argument apparently also gives the following slightly stronger result than what is stated in Theorem 1.1:

Theorem 3.2 Let $F \in Prop_3(\mathbf{B}^n, \mathbf{B}^N)$ with $4 \le n \le N = 3n - 3$. Suppose that F has geometric rank 1. Then F is equivalent to

$$F'_\theta := (F_\theta(z,w), 0, \cdots, 0) = (z, w\cos\theta, z_1 w \sin\theta, \cdots, z_{n-1} w \sin\theta, w^2 \sin\theta, 0, \cdots, 0)$$

for some θ with $(0 \le \theta \le \frac{\pi}{2})$.

The following example, copied from Example 1.3 of [Hu03], shows that Theorem 1.1 is optimal.

Example 3.3 ([Example 1.3, Hu03]): Let

$$\begin{aligned}
\psi_1 &= (z_1^2, \sqrt{2}z_1z_2, \cdots, \sqrt{2}z_1z_{k-1}, z_1z_k, \cdots, z_1z_n),\\
\psi_2 &= (z_2^2, \sqrt{2}z_2z_3, \cdots, \sqrt{2}z_2z_{k-1}, z_2z_k, \cdots, z_2z_n),\\
&\cdots\\
&\cdots\\
\psi_{k-1} &= (z_{k-1}^2, z_{k-1}z_k, \cdots, z_{k-1}z_n),\\
\psi_k &= (z_k, \cdots, z_n).
\end{aligned}$$

Let $W_{n,k} = (\psi_1, \cdots, \psi_k)$. Then $W_{n,1}$ is the identical map, $W_{n,2}$ is the standard Whitney map. More generally, for any $k \leq n$, $W_{n,k}$ is a proper polynomial map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = n + P(n, k-1)$. Notice that $W_{n,k}$ is not $(n-(k-1)+1)$-linear, where $P(n,k) = \frac{k(2n-k-1)}{2}$. In particular, $W_{n,3}$ (for $n \geq 3$), from $\mathbf{B}^n$ into $\mathbf{B}^{3n-3}$, can not be $(n-1)$-linear. Hence, $W_{n,3}$ is not equivalent to $(F_\theta, 0, \cdots, 0)$. (Notice that first and the third k in the last line of page 463 of [Hu03] are misprint and should be $k-1$.)

4. A normal form for $F \in Rat(\mathbf{B}^2, \mathbf{B}^N)$ with degree 2

In this section, we study the normalization problem for mappings, which have geometric rank one, but may not be partially linear. By the work in [Hu03] and [CD96], such maps have to be from $\mathbf{B}^2$ to $\mathbf{B}^N$ and may take very abstract form. We thus focus on the normalization problem under the action of the isotropic group of the Heisenberg hypersurfaces. Even in this setting, we still have to restrict ourselves to maps which have degree 2 to be able to get some clean results.

Let $F = (f, \phi, g)$ be a proper rational map of degree two from $\mathbf{H}_2$ into $\mathbf{H}_N$. Assume that $F(0) = 0$ and 0 is a generic point of F, namely, $\kappa_F(0) = 1$. Without loss of generality, we assume that $N \geq 4$. We then want to give a complete classification of F under the action of $Aut_0(\partial\mathbf{H}_2)$ and $Aut_0(\partial\mathbf{H}_N)$. By [Lemma 3.2, Hu03], we have $\sigma \in Aut_0(\partial\mathbf{H}_2)$ and $\tau \in \partial Aut_0(\mathbf{H}_N)$ such that $\tau \circ F \circ \sigma$, still denoted by $F = (f, \phi, g)$, takes the following form:

$$
\begin{aligned}
f =& z + \frac{i}{2} zw + o_{wt}(3), \quad \frac{\partial^2 f}{\partial w^2}(0) = 0,\\
g =& w + o_{wt}(4),\\
\phi_1 =& z^2 + A_1 zw + B_1 w^2 + E_1 z^3 + \cdots,\\
\phi_j =& o_{wt}(2), \ j \geq 2.
\end{aligned} \tag{9}
$$

Replacing $(\phi_2, \cdots, \phi_{N-2})$ by $(\phi_2, \cdots, \phi_{N-2}) \cdot U$ with U a certain $(N-3) \times (N-3)$ unitary matrix, we can assume that $\phi_j = A_j zw + B_j w^2 + o(|(z,w)|^2)$ for $j \geq 2$ and $A_j = 0$ for $j \geq 3$. In a similar manner, we can assume that $B_j = 0$ for $j \geq 4$ (if $N \geq 6$). Making use of the assumption that F has degree 2, we can thus assume in (9) that

$$
\begin{aligned}
\phi_2 =& A_2 zw + B_2 w^2 + o(|(z,w)|^2),\\
\phi_3 =& B_3 w^2 + o(|(z,w)|^2),\\
\phi_j =& 0, \ j \geq 4.
\end{aligned} \tag{10}
$$

Considering the weighted 5^{th}-order terms of the basic equation

$$
\Im g = |f|^2 + \|\phi\|^2, \quad \Im w = |z|^2 \tag{11}
$$

we get $\Im(g^{(5)} - 2iz\overline{f^{(4)}}) = 2\Re(z^2 \overline{A_1} \overline{zw} + z^2 \overline{E_1 z^3})$. ¿From this, we easily see that $g^{(5)} \equiv 0$ and $f^{(4)} = az^2 w$ with

$$a = -\overline{A_1}, \quad E_1 = -2ia. \tag{12}$$

Also, by considering the coefficients of u^3 in the weighted 6^{th}-order terms of (11), we can see that

$$g = w + \mu w^3 + O(|(z,w)|^4) \cap o_{wt}(5) \quad \text{with } \mu \in \mathbf{R}. \tag{13}$$

Now, write $f = P_0/Q,\ \phi_j = P_j/Q,\ j = 1, \cdots, N-2,\ g = G/Q$. Here $P_0,\ P_j, G$ are polynomials of degree at most 2, and $Q = 1 + L(z,w) + Q_0(z,w)$ with $L,\ Q_0$ linear and quadratic polynomials in (z,w), respectively. Then from (9) and (13), it follows easily that $P_0 = z(1 + L(z,w) + iw/2)$, $G = w(1 + L(z,w))$ and $Q_0 = -\mu w^2$.

Write $L = id_1 z + id_2 w$. Let

$$\sigma = \frac{(z,w)}{1 + rw}, \quad \tau = \frac{(z^*, w^*)}{1 - rw^*}$$

with $r = -\Re(id_2)$. Replacing F by $\tau \circ F \circ \sigma$, we can assume, without loss of generality, that $d_2 \in \mathbf{R}$.

Next, replacing F by $\tau \circ F \circ \sigma$, where $\sigma(z,w) = (e^{i\theta} z, w)$ and $\tau(z_1^*, z_2^*, \cdots, z_{N-1}^*, w^*) = (e^{-i\theta} z_1^*, e^{-2i\theta} z_2^*, e^{i\beta_3} z_3^*, \cdots, e^{i\beta_{N-1}} z_{N-1}^*, w^*)$ with appropriate $\theta,\ \beta_j's$, we can assume that $A_1 \geq 0,\ A_2 \geq 0, B_3 \geq 0$. Also, when $A_2 = 0$, we can make $B_2 \geq 0$.

Rewrite (11) as

$$\begin{aligned} \Im\left(w(1 + L(z,w))(1 + \overline{L(z,w)} - \mu \overline{w}^2)\right) &= |z(1 + L(z,w) + i/2w)|^2 + \\ &+ |z^2 + A_1 zw + B_1 w^2|^2 + |A_2 zw + B_2 w^2|^2 + |B_3 w^2|^2. \end{aligned} \tag{14}$$

Comparing the coefficients of weighted degree 8, we obtain

$$-\Im(id_2 \mu w^2 \overline{w}^2) = \sum_{j=1}^{3} |B_j|^2 |w|^4.$$

From this, we obtain

$$-\mu d_2 = \sum_{j=1}^{3} |B_j|^2 \tag{15}$$

Comparing the coefficients of weighted degree 7 in (14), we get

$$-\Im(id_1 \mu z w \overline{w}^2) = \sum_{j=1}^{2} (A_j \overline{B_j} z w \overline{w}^2 + B_j \overline{A_j} w^2 \overline{zw}),$$

from which, it follows that

$$-\mu d_1 = 2 \sum_{j=1}^{2} A_j \overline{B_j}. \tag{16}$$

Similarly, comparing the coefficients of weighted degree 6 in (14), we get precisely the following:

$$B_1 = 0,\ \mu = 1/4 + d_2 + \sum_{j=1}^{2} |A_j|^2. \tag{17}$$

Since $E_1 = -id_1$ and $a = d_1/2$, by (12), we have $d_1 \in \mathbf{R}$ and $A_1 = -d_1/2$. Thus, $d_1 \le 0$.

Making use of (16), we get that $B_2 \in \mathbf{R}$. When $A_2 \neq 0$, (16) can be also used to show that $B_2 \geq 0$. If $A_2 = 0$, then we can similarly make $B_2 \geq 0$, $B_3 = 0$ by applying a unitary action to the $(\phi_2, \cdots, \phi_{N-2})$-components as we mentioned before.

Notice that when $\mu = 0$, then $g = w$, $f = zf^*$, $\phi = z\phi^*$ with (f^*, ϕ^*) a linear fractional map from $\mathbf{H}_2$ into $\mathbf{B}^{N-1}$. As in the proof of Theorem 1.1, we can see, in a similar way, that the pull back of F to the map from $\mathbf{B}^2$ into $\mathbf{B}^N$ then must be equivalent to the D'Angelo map

$$F_\theta(z,w) = (z, \cos(\theta)w, \sin(\theta)zw, \sin(\theta)w^2, 0, \cdots, 0).$$

In summary, we have the following:

Theorem 4.1: *Let $F \in Rat(\mathbf{H}_2, \mathbf{H}_N)$ have degree 2 with $F(0) = 0$ and $\kappa_F(0) = 1$ ($N \geq 4$). Then there are $\sigma \in Aut_0(\partial \mathbf{H}_2)$ and $\tau \in Aut_0(\partial \mathbf{H}_N)$ such that $\tau \circ F \circ \sigma$, still denoted by (f, ϕ, g), takes the following normal form:*

$$\begin{aligned} &f(z,w) = \tfrac{z - 2id_1z^2 + (i/2 - id_2)zw}{1 - id_2w - \mu w^2 - 2id_1z}; \ \phi_1(z,w) = \tfrac{z^2 + d_1zw}{1 - id_2w - \mu w^2 - 2id_1z};\\ &\phi_2(z,w) = \tfrac{c_1w^2 + \nu zw}{1 - id_2w - \mu w^2 - 2id_1z}; \ \phi_3(z,w) = \tfrac{c_2w^2}{1 - id_2w - \mu w^2 - 2id_1z};\\ &\phi_j \equiv 0, \ j \geq 4, \qquad g(z,w) = \tfrac{w - id_2w^2 - 2id_1zw}{1 - id_2w - \mu w^2 - 2id_1z}. \end{aligned} \tag{18}$$

Here when $N = 4$, ϕ only has two components (ϕ_1, ϕ_2). $\nu, \mu, d_1, d_2, c_1, c_2$ are non-negative real numbers. Also, the following relations hold:

$$\mu d_2 = c_1^2 + c_2^2, \ \mu + d_2 = 1/4 + d_1^2 + \nu^2, \mu d_1 = \nu c_1; c_2 = 0 \text{ if } \nu = 0. \tag{19}$$

Moreover, we have
(I). $\nu, \mu, d_1, d_2, c_1, c_2$ are uniquely determined by F. Conversely, for any non-negative real numbers

$$\nu, \mu, d_1, d_2, c_1, c_2$$

satisfying the relations in (19), the map F defined in (18) is an element in $Rat(\mathbf{H}_2, \mathbf{H}_N)$ of degree 2 with $F(0) = 0$ and $\kappa_F(0) = 1$.
(II). If $\mu = 0$, then $\rho_N^{-1} \circ F \circ \rho_2$, where ρ_n is defined as in (2), is equivalent to $(F_\theta, 0)$ with F_θ as in (1).

Remark 4.2: An immediate consequence of Theorem 4.1 is that any rational proper holomorphic map from $\mathbf{B}^2$ into $\mathbf{B}^N$ with $N \geq 5$ of degree 2 is equivalent to a rational proper holomorphic map from $\mathbf{B}^2$ into $\mathbf{B}^5$. A similar argument can be used to show that for any positive integers k, n, N with $N \geq n \geq 2$, there is an integer N_0, depending only on k, n, N, such that any rational proper map of degree bounded by k from $\mathbf{B}^n$ into $\mathbf{B}^N$ is equivalent to a rational proper holomorphic7 map from $\mathbf{B}^n$ into $\mathbf{B}^{N_0}$.

Proof of Theorem 4.1: It only remains to prove the uniqueness of $\mu, \nu, d_1, \ d_2, c_1, c_2$.

Suppose $F^* = \tau^* \circ F \circ \sigma = (f^*, \phi^*, g^*)$ with $\sigma \in Aut_0(\partial \mathbf{H}_2)$ and $\tau^* \in Aut_0(\partial \mathbf{H}_N)$. And suppose that both F and F^* satisfy the normalization in Theorem 4.1. By [Lemma 2.2(A), Hu03] [(2.4.1), Hu03] and [(2.4.2), Hu03], we have

$$\sigma = \frac{(\lambda(z + aw) \cdot U, \ \lambda^2 w)}{q(z,w)}, \quad \tau^*(z^*, w^*) = \frac{(\lambda^*(z^* + a^*w^*) \cdot U^*, \lambda^{*2}w^*)}{q^*(z^*, w^*)}. \tag{20}$$

Here $q(z,w) = 1 - 2i\langle \overline{a}, z\rangle + (r - i|a|^2)w$, $\lambda > 0$, $r \in \mathbf{R}$, $a \in \mathbf{C}$, $|U| = 1$, $q^*(z^*, w^*) = 1-2i\langle \overline{a^*}, z^*\rangle + (r^* - i|a^*|^2)w^*$, $\lambda^* > 0$, $r^* \in \mathbf{R}$, $a^* \in \mathbf{C}^{N-1}$ and U^* is an $(N-1)\times(N-1)$ unitary matrix. Also the following holds (see [((2.5.1), (2.5.2), Hu03]):

$$\lambda^* = \lambda^{-1},\ a_1^* = -\lambda^{-1}aU,\ a_2^* = 0,\ r^* = -\lambda^{-2}r,\ U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix}, \tag{21}$$

where $a^* = (a_1^*, a_2^*)$, U_{22}^* is an $(N-2)\times(N-2)$ unitary matrix. Write

$$A = -2i\frac{\partial^2 f}{\partial z\partial w}(0), \quad B^i = \frac{\partial^2 \phi_i}{\partial z^2}(0),\ B^{*i} = \frac{\partial^2 \phi_i^*}{\partial z^2}(0), i = 1,\cdots,N-2,$$

$$\mathcal{B} = (\frac{\partial^2\phi_1}{\partial z\partial w},\cdots,\frac{\partial^2\phi_{N-2}}{\partial z\partial w}), \mathcal{B}^* = (\frac{\partial^2\phi_1^*}{\partial z\partial w},\cdots,\frac{\partial^2\phi_{N-2}^*}{\partial z\partial w}).$$

The same computation in [Hu03, Lemma 2.2 (A)] yields the following:

$$\begin{aligned} &\frac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 aU\cdot A\cdot U^{-1} + \lambda^3\frac{\partial^2 f}{\partial w^2}(0)U^{-1}, \\ &[B^{*1},\cdots,\ B^{*N-2}] = \lambda U[B^1,\cdots,B^{N-2}]U^tU_{22}^*, \\ &\mathcal{B}^* = \lambda U[B^1,\cdots,B^{N-2}]U^ta^tU_{22}^* + \lambda^2 U\mathcal{B}U_{22}^*, \\ &\frac{\partial^2\phi^*}{\partial w^2}(0) = \lambda aU[B^1,\cdots\ B^{N-2}]U^ta^tU_{22}^* + 2\lambda^2 aU\mathcal{B}U_{22}^* + \lambda^3\frac{\partial^2\phi}{\partial w^2}(0)U_{22}^*. \end{aligned} \tag{22}$$

Since both F and F^* satisfy the normalization in (18), we have $a = 0$, $\lambda = 1$. Also, since $d_2, d_2^* \in \mathbf{R}$, a direct computation shows that $r = 0$. Write $U_{22} = (\alpha_{kl})$. By the second equality in (22), we see that $\alpha_{11}e^{2i\theta} = 1$ and $\alpha_{1j} = 0$ for $j \geq 2$. Notice that $d_1, d_1^*, \nu, \nu^* \geq 0$. Making use of this and the third equality of (22), we see that $(d_1^*, \nu^*) = (d_1, \nu)$. If $\mu = 0$, then $g^* = g = w$. Thus, we have $c_1, c_2, c_1^*, c_2^*, \mu^* = 0$. Since $f^* = f$, we get $d_1 = d_1^*, d_2 = d_2^*, \nu = \nu^*$. Next, suppose that $\mu, \mu^* \neq 0$. If ν does no vanish neither, then making use of the fact $g^* = g$ and (19), we can also conclude the proof of the theorem. The case that $\mu \neq 0$ and $\nu = 0$ can be treated similarly. □

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