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ON THE MAPPING PROBLEM FOR ALGEBRAIC REAL HYPERSURFACES IN THE COMPLEX SPACES OF DIFFERENT DIMENSIONS

by Xiaojun HUANG

0. Introduction.

In this paper, we study algebraicity for mappings between algebraic hypersurfaces. Our main purpose is to prove the following theorem for general codimension k, which was obtained by Webster [We1] in the case $k = 0$ (see §1.1 for relevant notation) :

MAIN THEOREM . — *Let $M_1 \subset \mathbf{C}^{\mathbf{m}}$ and $M_2 \subset \mathbf{C}^{\mathbf{m+k}}$ be strongly pseudoconvex real algebraic hypersurfaces with $m > 1$ and $k \geq 0$. Suppose that f is a holomorphic mapping from a neigborhood of M_1 to $\mathbf{C}^{\mathbf{m+k}}$ so that $f(M_1) \subset M_2$. Then f is algebraic.*

The proof of the theorem is based on a rather careful analysis on the Segre surfaces. The argument, after some technical modifications, can also be used to prove the following version of the reflection principle (Theorem 1), which, in turn, allows our main Theorem to be formulated in the form of Theorem 2.

THEOREM 1. — *Let $M_1 \subset \mathbf{C}^{\mathbf{m}}$ and $M_2 \subset \mathbf{C}^{\mathbf{m+k}}$ ($m > 1$, $k \geq 0$) be strongly pseudoconvex real analytic hypersufaces. Let Ω be a domain which contains M_1 in its boundary and is pseudoconvex along M_1. Suppose that f is a mapping from Ω to $\mathbf{C}^{\mathbf{m+k}}$ that is holomorphic on Ω, C^{1+k}-smooth up to M_1, and sends M_1 into M_2. Then f extends holomorphically to a neighborhood of an open dense subset of M_1.*

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THEOREM 2. — *Let $M_1 \subset \mathbf{C}^{\mathbf{m}}$ and $M_2 \subset \mathbf{C}^{\mathbf{m+k}}$ $(m > 1,\ k \geq 0)$ be strongly pseudoconvex algebraic real hypersufaces. Let Ω be a domain which contains M_1 in its boundary and is pseudoconvex along M_1. Suppose that f is a mapping from Ω to $\mathbf{C}^{\mathbf{m+k}}$ that is holomorphic on Ω, C^{1+k}-smooth up to M_1, and sends M_1 into M_2. Then f is algebraic.*

In the last twenty years, much related work has been done which, in some sense, gave partial results in the direction of the Main Theorem and Theorem 2. In the setting of our main result, when M_1 and M_2 are open subsets of the sphere in $\mathbf{C}^{\mathbf{m}}$, the theorem was proved by Poincaré [Po] $(m = 2)$ and Tanaka [Ta] $(m \geq 2)$. For general algebraic strongly pseudoconvex hypersurfaces of the same dimension, Webster [We1] obtained the analogous result with $k = 0$. In the form of Theorem 2, in case M_1 and M_2 are open subset of spheres, the result is due to Alexander [Al], Pinchuk [Pi] for $k = 0$; and due to Webster [We2], Faran [Fa], Cima-Suffridge [CS], and Forstneric [Fr1] for $k > 0$. In the spherical case, the map f is actually rational, i.e, algebraic of degree 1.

Research along the lines of generalizing the classical Schwartz reflection principle to higher dimensions started with the work of Fefferman [Fe], Lewy [Le], and Pinchuk [Pi]. Since then, considerable attention has been paid to this subject. We mention here the work in [BBR], [BR], and [FD1], to name a few. For extensive surveys on this and related topics, see the papers by Bedford [Be], Forstneric [Fr3], and Bell-Narasimhan [BN]. In the context of Theorem 1, some special cases were investigated by several authors. In [We2], Webster obtained the result when $f \in C^3(\Omega \cup M_1)$, $m > 2$, $k = 1$, and M_2 is the sphere. In [CKS] and [Fa1], Theorem 1 was proved when $k = 1$ and f is three times differentiable on the boundary. In case $f \in C^\infty(\Omega \cup M_1)$ or if M_2 is the sphere, Theorem 1 is one of the main themes of [Fr1].

Remarks. — (a) By the Lewy extension phenomenon and Hopf's lemma, it is easy to see that Theorem 2 implies immediately the following result, which can be viewed as a sort of the real version of Chow's Theorem :

THEOREM 2′. — *Let M_1 and M_2 be two strongly pseudoconvex real algebraic hypersurfaces in (possibly different) complex spaces of dimension at least two. Then every smooth CR mapping from M_1 to M_2 is algebraic.*

(b) The following example shows that the above mentioned results are false when $m = 1$ and $k \geq 0$:

Example. — Let $\Delta_\epsilon = \{\tau \in \mathbf{C}^1 : |\tau|^2 + \epsilon^2|1-e^\tau|^2 < 1\}$. Obviously, when $\epsilon \approx 0$, the domain Δ_ϵ is a strongly convex domain with analytic boundary. Let ϕ_ϵ be a conformal mapping from the unit disk Δ to Δ_ϵ, which is analytic on $\partial\Delta$ by the classical Schwartz reflection principle. Define $f : \Delta \to B_2$(the unit two ball) by $f(\tau)= (\phi_\epsilon(\tau), \epsilon(1-e^{\phi_\epsilon(\tau)}))$. Then f is proper and holomorphic on $\overline{\Delta}$, but is not algebraic.

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1. Preliminaries.

The purpose of this section is to make some necessary preparations. In §1.1, we recall some definitions. In §1.2, we reformulate an immersion result of Pinchuk so that it can be easily applied to our situation (especially, to the proof of Theorem 1).

1.1. Notation and an algebraic lemma.

Let $C(z)$ be the field of rational functions in the variable $z \in \mathbf{C}^\mathbf{n}$. We recall that a function $\chi(z)$ on $U \subset \mathbf{C}^\mathbf{n}$ is called *algebraic* if there is a non-zero polynomial P with coefficients in $C(z)$ so that $P(\chi) = 0$, i.e., the field generated by adding χ to $C(z)$ is of finite extension. A mapping is called *algebraic* if each of its components is. For convenience, we collect here some facts on algebraic functions which will be used frequently in the later discussion :

LEMMA 1. — *Let χ be algebraic in $z \in U \subset \mathbf{C}^\mathbf{n}$. Then the following holds :*

(1) *For any fixed $(z_{k+1}^0, \cdots, z_n^0)$, the function $\chi(z_1, \cdots, z_k, z_{k+1}^0, \cdots, z_n^0)$ is algebraic in $(z_1, \cdots, z_k)$.*

(2) *Let $z_j = g(z_1, \cdots, \hat{z}_j, \cdots, z_n)$ be a solution of $\chi = 0$. Then $z_j = g(z_1, \cdots, \hat{z}_j, \cdots, z_n)$ is algebraic in $(z_1, \cdots, \hat{z}_j, \cdots, z_n) \in \mathbf{C}^{\mathbf{n-1}}$.*

(3) $\dfrac{\partial}{\partial z_j}\chi(z)$ *is algebraic in $(z_1, \cdots, z_n)$ for each j.*

(4) *If $g(z)$ is also algebraic on $U \subset \mathbf{C^n}$, then so are $\chi \pm g$, χg, χ/g.*

(5) *Let $z_j = g_j(s)$ be algebraic in $s \in \mathbf{C^{n'}}$ for $j = 1, \cdots, n$. Then so is $\chi(g_1(s), \cdots, g_n(s))$.*

(6) *Let $N_1 \subset \mathbf{C^{n_1}}$ and $N_2 \subset \mathbf{C^{n_2}}$ be two open subsets. Suppose that $g(z^{(1)}, z^{(2)})$ is a function on $N_1 \times N_2$. Let g be holomorphic and algebraic in $z^{(1)} \in N_1$ (respectively, in $z^{(2)} \in N_2$) when holding $z^{(2)}$ fixed (respectively, when holding $z^{(1)}$ fixed). Then g is algebraic.*

(7) *Let $g(z_1, \cdots, z_n)$ be a holomorphic algebraic function at 0 with $g(0) = 0$ and $g(0, \ldots, 0, z_n) \not\equiv 0$. Then the Weierstrass polynomial g^* of g, with respect to z_n, is also algebraic.*

Proof. — The proofs of (1)-(5) are trivial. The argument for (6) can be found, for example, in [BM] (p. 199-205). So we just say a few words about (7) : For any $z'(= (z_1, \ldots, z_{n-1})) \approx 0$, by a standard argument (see [Kr], for example), we obtain exactly (counting multiplicity) n' solutions of the equation $g(z', z_n) = 0 : \{a_1(z'), \cdots, a_{n'}(z')\}$ (with n' fixed). The Weierstrass polynomial g^* of g is then expressed as $g^* = \prod_{j=0}^{n'} (z_n - a_j(z')) = z_n + \sum_{j=0}^{n'-1} s_j(z') z_n^j$, where $s_j = \sum (-1)^j a_{l_1} \cdots a_{l_j}$. By (6), to check that g^* is algebraic we have only to show that the s_j's are. But this follows easily from (2), (4), and the fact that for a generic point $z'_0 \approx 0$, the $a_j(z')$'s are holomorphic on $z' \approx z'_0$. □

Now let $M \subset \mathbf{C^n}$ be a real analytic hypersurface with $r(z, \overline{z}) = 0$ as its defining function. We call M *algebraic* if the complexification of r, i.e, $r(z, \overline{\omega})$ is algebraic in $(z, \overline{\omega})$ for $(z, \overline{\omega}) \approx M \times \mathrm{Conj}(M)$, where we write $\mathrm{Conj}(M) = \{\overline{z} : z \in M\}$. Fix $p \in M$ and a small open subset $\Omega \subset \mathbf{C^n}$ of p, when $\omega \approx p$ then the Segre surface Q_ω restricted to Ω is a complex manifold of dimension $n-1$. Here we recall that $Q_\omega = \{z \in \Omega : r(z, \overline{\omega}) = 0\}$ and the complexification of M is defined as $M_c = \{(z, \omega) \in \Omega \times \Omega : r(z, \omega) = 0\}$, a complex manifold of dimension $2n - 1$.

1.2. Reformulation of a lemma of Pinchuk.

We now let M_1, M_2 and f as in the main theorem (or in Theorem 1). Without loss of generality, we also let f be non-constant. For a given point $p \in M_1$, after making use of a suitable polynomial holomorphic change of variables (see [Fe]), we can assume that $p = 0, f(0) = 0$, and M_1, M_2 are locally defined by ρ_1 and ρ_2, respectively :

$$\rho_1(z,\overline{z}) = z_m + \overline{z_m} + \sum_{j=1}^{m-1} |z_j|^2 + h_0(z,\overline{z}); \tag{1.1}$$

$$\rho_2(w,\overline{w}) = w_{m+k} + \overline{w_{m+k}} + \sum_{j=1}^{m+k-1} |w_j|^2 + h(w,\overline{w}). \tag{1.2}$$

Here $h_0(z,\overline{z}) = O(|z|^4)$ and $h(w,\overline{w}) = O(|w|^4)$ (when M_2 is the sphere, then $h \equiv 0$).

From a result of Pinchuk ([Pi]), it follows that $\frac{\partial f_{m+k}}{\partial z_m}(0) \neq 0$ and $df : T_0^{(1,0)} M_1 \to T_0^{(1,0)} M_2$ is injective. We write $\tilde{L}_j = \frac{\partial \rho_1}{\partial z_m}\frac{\partial}{\partial z_j} - \frac{\partial \rho_1}{\partial z_j}\frac{\partial}{\partial z_m}$ for $j = 1, \ldots, m-1$. Since $f(M_1) \subset M_2$ and $\tilde{L}_j \in T^{(1,0)} M_1$, we see that

(1.3)

$$f_{m+k}(z) + \overline{f_{m+k}(z)} + \sum_{j=1}^{m+k-1} |f_j(z)|^2 + h(f(z), \overline{f(z)}) = 0, \quad \text{for } z \in U \subset M_1.$$

Applying each $\tilde{L}_j$ to (1.3), we then obtain

$$\tilde{L}_l f_{m+k}(z) + \sum_{j=1}^{m+k-1} \tilde{L}_l f_j(z)\overline{f_j(z)} + \sum_{j=1}^{m+k} \frac{\partial h}{\partial w_j}\tilde{L}_l f_j(z) = 0, \quad \text{for } z \in U. \tag{1.4}$$

Now, by letting $z = 0$ in the formula (1.4), we see that $\tilde{L}_j f_{m+k}(0) = 0$ for each j. On the other hand, since $\{\tilde{L}_1, \ldots, \tilde{L}_{m-1}\}$ consists of a local basis of $T^{(1,0)}M$ near 0, we thus conclude that the rank of the matrix $(\tilde{L}_j f_l)_{\substack{1\le j\le m-1 \\ 1\le l\le m+k-1}}$ is $m-1$. Let S be the vector space spanned by

$$\{\tilde{L}_1 f(0), \ldots, \tilde{L}_{m-1} f(0)\}$$

and let $\{T_1, \ldots T_{m-1}\}$ be an orthonormal basis of S. Extend it to an orthonormal basis of $\mathbf{C^{m+k-1}}$: $\{T_1, \ldots, T_{m+k-1}\}$ and set

$$(\tilde{f}_1, \ldots, \tilde{f}_{m+k-1})^t = (\overline{T_1}, \ldots, \overline{T_{m+k-1}})^t (f_1, \ldots, f_{m+k-1})^t.$$

It then follows easily that $\tilde{L}_j \tilde{f}_l(0) = 0$ for $l = m, \ldots, m+k-1$ and that $(\tilde{f}, f_{m+k})$ still satisfies the equation (1.4) (up to a 4^{th} order small term).

Now, by choosing $(L_1,\ldots,L_{m-1})^t = (\tilde{L}_j\tilde{f}_l(0))^{-1}(\tilde{L}_1,\ldots,\tilde{L}_{m-1})^t$ and by making use of the identity (1.4) with $z=0$, we obtain

$$L_j\tilde{f}_l = \delta_j^l = \begin{cases} 0, & \text{if } j \neq l; \\ 1, & \text{if } j = l. \end{cases} \tag{1.5}$$

Consequently, to simplify the notation, we assume in what follows that M_1, M_2, f, and $\{L_1,\ldots L_{m-1}\}$ already have the properties in (1.1), (1.2), and (1.5).

2. Proof of Main Theorem.

In this section, we present the proof of our main result. Our idea is to show that each component of the mapping f stays in the field generated by adding some algebraic elements (which are obtained from suitable operations on the defining functions of the hypersurfaces) to rational functions field. For this purpose, we start by complexifying the identity : $\rho_2(f(z),\overline{f(z)}) = \lambda(z,\overline{z})\rho_1(z,\overline{z})$ and differentiate it along each Segre surface. Then we will obtain the algebraicity by a very careful case-by-case argument according to how degenerate the map is. Since the proof is long, we shall, for clarity, split it into 4 subsections and many small lemmas.

2.1. In this subsection, we concentrate on two major cases which we will study in detail in §2.3 and §2.4.

Let M_1 and M_2 be as in the main theorem. As we have discussed in the above section, we may let M_1, M_2, and f have the properties (1.1), (1.2), and (1.5) mentioned in §1.2. We first choose a small neighborhood $\Omega \subset \mathbf{C^m}$ of 0 so that f is holomorphic on this open subset and the Segre surfaces Q_ω of M_1 restricted to Ω are connected for any $\omega \approx 0$.

Now since $f(M_1) \subset M_2$, we have the equation $\rho_2(f(z),\overline{f(z)}) = \lambda(z,\overline{z})\rho_1(z,\overline{z})$ with $\lambda(z,\overline{z})$ real analytic. By the standard complexification, we then see, for each $\omega \approx 0$, that $\rho_2(f(z),\overline{f(\omega)}) = \lambda(z,\overline{\omega})\rho_1(z,\overline{\omega})$. Thus $f(Q_\omega) \subset Q_{f(\omega)}$ for $\omega \approx 0$. Therefore we obtain the following identity :

$$f_{m+k}(z) + \overline{f_{m+k}(\omega)} + \sum_{j=1}^{m+k-1} f_j(z)\overline{f_j(\omega)} + h(f(z),\overline{f(\omega)}) = 0 \tag{2.1}$$

$$\text{for } z \in Q_\omega, \text{or, } (z,\overline{\omega}) \in M_{1c},$$

where, as before, M_{1c} denotes the complexification of M_1. By (2) of Lemma 1 and the implicit function theorem, the above h can be changed to an algebraic function not involving the $\overline{f_{m+k}(\omega)}$ term. Therefore, we can assume that $h(f,\overline{f}) = h(f,\overline{f_1(\omega)},\ldots,\overline{f_{m+k-1}(\omega)})$, where $h(w,y_1,\ldots,y_{m+k-1})$ is an algebraic holomorphic function on $O_w(0) \times O_{y_1}(0) \times \ldots \times O_{y_{m+k-1}}(0)$. Here and in what follows, we use the symbol $O_*(**)$ to denote a small neighborhood of $**$ in the $*$ variable, which may be different in different contexts.

Now we let L_j (for $j = 1,\ldots,m-1$) be the polarization of the previously defined operator L_j, i.e, L_j is a linear combination of the following operators

$$\left\{ \frac{\partial\rho_1(z,\overline{\omega})}{\partial z_m}\frac{\partial}{\partial z_j} - \frac{\partial\rho_1(z,\overline{\omega})}{\partial z_j}\frac{\partial}{\partial z_m} \right\}_{j=1}^{m-1}.$$

Then for any ω fixed, $\{L_j(z,\omega)\}_{j=1}^{m-1}$ consists of a basis for the holomorphic vector fields of Q_ω.

Applying each L_l to (2.1), $l = 1\ldots,m-1$, we obtain

$$(2.2)\quad L_l f_{m+k}(z) + \sum_{j=1}^{m+k-1} L_l f_j(z)\overline{f_j(\omega)} + \sum_{j=1}^{m+k} \frac{\partial h}{\partial w_j} L_l f_j(z) = 0, \quad \text{for } z \in Q_\omega.$$

Let $V(z,\omega) = (v_{ij}(z,\omega))_{1\le i,j\le m-1}$ with $v_{ij}(z,\omega) = L_i f_j$. Moreover, define

$$\xi(z,\omega) \equiv V^{-1}(z,\omega)(L_1 f_{m+k},\ldots,L_{m-1}f_{m+k})^t,$$

and

$$\eta(z,\omega) \equiv V^{-1}\begin{pmatrix} L_1 f_m \cdots & L_1 f_{m+k-1} \\ \cdots & \\ \cdots & \\ L_{m-1} f_m \cdots & L_{m-1} f_{m+k-1} \end{pmatrix}.$$

Equation (2.2) can then be written in the following matrix form :

$$\xi(z,\omega) + \overline{F_0(\omega)} + \eta(z,\omega)\overline{F}(\omega) + (\mathrm{id},\eta(z,\omega),\xi(z,\omega))Dh(z,\omega) = 0 \text{ for } z \in Q_\omega,$$

where $Dh(z,\omega) = \left(\frac{\partial h}{\partial w_1},\ldots,\frac{\partial h}{\partial w_{m+k}}\right)^t (f(z),\overline{f(\omega)}) = O(|z|^3 + |\omega|^3)$ as $(z,\omega) \to (0,0)$, $F_0 = (f_1,\ldots,f_{m-1})^t$, and $F = (f_m,\ldots,f_{m+k-1})^t$.

Again, by making use of the implicit function theorem and by shrinking Ω, we have, for some holomorphic vector function g, that

(2.3)
$$\xi(z,\omega)+\overline{F}_0(\omega)+\eta(z,\omega)\overline{F}(\omega)+g(f(z),\xi(z,\omega),\eta(z,\omega),\overline{F}(\omega))=0 \quad \text{on } Q_\omega.$$

Since the algebraic function field is closed under the application of the implicit function theorem (see (2) of Lemma 7), it follows that the function $g(w,a,b,y)$ in (2.3) is also algebraic and holomorphic on

$$O_w(0)\times O_a(\xi(0,0))\times O_b(\eta(0,0))\times O_{y_m}(0)\times\ldots O_{y_{m+k-1}}(0),$$

where we identify the variables ξ,η,F with $a,b,Y=(y_m,\ldots,y_{m+k-1})$, respectively. In fact, it is easy to see that g does not depend on f and is identically 0 when M_2 is the sphere. Set

$$\begin{gathered}H_0(f,\xi,\eta)=\xi+g(f,\xi,\eta,0),\\ H_\alpha(f,\xi,\eta)=\eta_{\alpha^*}+\frac{\partial g}{\partial y_{\alpha^*}}(f,\xi,\eta,0),\quad \text{with } |\alpha|=1, \text{ and}\\ H^*(f,\xi,\eta,\overline{F})=g(f,\xi,\eta,\overline{F})-(g(f,\xi,\eta,0)+\sum_{|\alpha|=1}\frac{\partial g}{\partial y_{\alpha^*}}(f,\xi,\eta,0)\overline{f_{\alpha^*}})\\ \text{for } (z,\overline{\omega})\in M_{1c}.\end{gathered}$$

Here and also in what follows, for a multi-index α with the j^{th} element 1 and all other components 0, we let $\alpha^*=m+j-1$, and we let η_{α^*} denote the j^{th} column of the matrix η. Then (2.2) can be written as

(2.4)
$$H_0(f(z),\xi,\eta)+\overline{F}_0(\omega)+\sum_{|\alpha|=1}\overline{f_{\alpha^*}}(\omega)H_\alpha(f(z),\xi,\eta)+H^*(f(z),\xi,\eta,\overline{F}(\omega))=0,$$

for $(z,\overline{\omega})\in M_{1c}$. Let $H^*(f,\xi,\eta,\overline{F})=\sum_{|\alpha|=2}^{\infty}H_\alpha(f,\xi,\eta)\overline{F^\alpha}$. We will continue our discussions according to the following two possibilities :

(AA) $L_{l_0}H_{\alpha_0}(z_0,\overline{z}_0)\neq 0$ for some l_0, α_0, and $(z_0,\overline{z_0})\in M_{1c}$ with $z_0\approx 0$.

(BB) $L_lH_\alpha(z,\overline{z})\equiv 0$ for all l, α, and $(z,\overline{z})\in M_{1c}$ with $z\approx 0$.

2.2. We present in this subsection two lemmas which will be useful in our later discussions.

LEMMA 2. — *Let $\{H_\alpha\}$ as above. If for some open subset $U\subset M_1$ of p, it holds that $L_lH_\alpha(z,\overline{z})\equiv 0$ for all $z\in U$ and l,α, then there is an algebraic holomorphic function Ψ so that $F_0(z)=\Psi(z,F(z))$ for $z\approx p$.*

We first observe that $U \times \mathrm{Conj}(U) \subset M_{1c}$ is a totally real submanifold of maximal dimension, where the notation $\mathrm{Conj}(U)$ is the same as that at the end of §1.1. So $U \times \mathrm{Conj}(U)$ is a set of uniqueness for the holomorphic functions on M_{1c}. Thus, under the hypothesis of the lemma, it follows that $L_l H_\alpha(z,\overline{w}) \equiv 0$ for all l, α and $(z,\overline{w})(\subset M_{1c}) \approx (p,p)$.

Proof of Lemma 2. — Since $\{L_1,\ldots,L_{m-1}\}$ is a basis for the collection of holomorphic vector fields on Q_ω and since $H_\alpha(f(z),\xi(z,\omega),\eta(z,\omega))$ is holomorphic for any fixed ω, it follows, from the just mentioned observation, that $H_\alpha(f(z),\xi(z,\omega),\eta(z,\omega))$ is constant along any Q_ω.

Define

(2.5)
$$\Psi^*(z,\omega,Y) = -\overline{H_0(f(z),\xi,\eta)} - \sum_{|\alpha|=1} y_{\alpha^*}\overline{H_\alpha(f(z),\xi,\eta)} - \overline{H^*(f,\xi,\eta,\overline{Y})},$$

for $(z,\overline{w}) \in M_{1c}$ and $Y \approx 0$. We then can conclude that, for any fixed

$$Y = (y_m,\ldots,y_{m+k-1}) \in \mathbf{C}^{\mathbf{k}},$$

the function Ψ^* is constant on each Q_ω $(\omega \approx p)$. Moreover, it can be seen that for any fixed (z,ω), Ψ^* is algebraic in Y since $\overline{H^*(f,\cdots,\overline{Y})}$ is. For any given $z(\in \Omega) \approx p$ and $Y \approx 0$, we define $\Psi(z,Y) = \Psi^*(\omega,z,Y)$ with $\omega \in Q_z$. This definition makes sense because Ψ^* is independent of the choice of $\omega \in Q_z$. By (2.4), it obviously holds that $F_0(z) = \Psi(z,F(z))$ for $z \approx 0$. We are now going to complete the proof of the lemma by showing that Ψ is algebraic in (z,Y).

First, we notice that, for any given z, Ψ is algebraic in Y by the above discussion. Thus, by (6) of Lemma 1, we have only to prove that Ψ is algebraic in z when holding Y fixed.

Fix $z_0 \approx 0$ and let $z \in Q_{z_0}$. Since z_0 is also contained in Q_z, we see by (2.5) that

$$\Psi(z,Y) = -\overline{H_0(f(z_0),\xi(z_0,z),\eta(z_0,z))}$$

$$- \sum_{|\alpha|=1} y_{\alpha^*}\overline{H_\alpha(f(z_0),\xi(z_0,z),\eta(z_0,z))} - \overline{H^*(f(z_0),\xi(z_0,z),\eta(z_0,z),\overline{Y})}.$$

Therefore it can be seen that $\Psi(z,Y)$ is holomorphic and algebraic along Q_{z_0} for any fixed Y, since H_α and H^* are algebraic in their separate variables and $\xi(z_0,z),\eta(z_0,z)$ are holomorphically algebraic in $\overline{z}$ (by the algebraicity of M_1 and M_2). Now the algebraicity of Ψ follows clearly from the following :

LEMMA 3. — *Let M be a piece of algebraic strongly pseudoconvex hypersurface. If g is a function defined near $p \in M$ which is holomorphic and algebraic on any Segre surface Q_z with $z \approx p$, then $g(z)$ is algebraic in z.*

Proof of Lemma 3. — By an algebraic change of variables, we can assume that $p = 0$ and M is defined by the equation $r(z, \overline{z}) = 0$ with a similar form to (1.1).

Let $\mathrm{e}_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the j^{th} position, and let $\tau(\neq 0) \in \mathbf{C}^1$ but close enough to 0. Obviously, we then have $\tau \mathrm{e}_j \in Q_0$ and thus $0 \in Q_{\tau \mathrm{e}_j}$ for $j = 1, \ldots, m-1$. Write $S_\tau = \bigcap_{j=1}^{m-1} Q_{\tau \mathrm{e}_j}$. We see that $0 \in S_\tau$. Define the map ϕ from $S_\tau \times Q_0$ to $\mathbf{C^m}$ by $\phi(0,0) = 0$ and

$$\phi(s,t) = Q_s \cap \{\cap_{j=1}^{m-1} Q_{\tau \mathrm{e}_j + t}\}.$$

CLAIM 1. — *When $\tau(\neq 0)$ is close enough to 0, then S_τ is a regular algebraic curve near 0 and $\overline{\phi}$ is an algebraic biholomorphism near $(0,0)$.*

Proof of Claim 1. — Notice that S_τ is defined by the equations :

$$z_m + \overline{\tau} z_j + O(|\tau|^4 + |z|^4) = 0 \quad \text{for } j = 1, \ldots, m-1.$$

From the implicit function theorem and Lemma 1, it then follows easily that for $\tau \approx 0$, S_τ is a regular algebraic manifold of complex dimension 1 near (0,0), parametrized by :

$$z_j = a_j(\tau) z_m + h_j(z_m, \tau) \quad j = 1, \ldots, m-1,$$

where $a_j(\tau) = -\frac{1}{\overline{\tau}} + O(\tau)$ and $h_j(0, \tau) = 0$.

Now, let $z = \phi(s,t)$ with $s = (s_1, \ldots, s_m)$ and $t = (t_1, \ldots, t_{m-1}, 0)$. Then, by the above argument and the definition of ϕ, we see that

(i) $$s_j = a_j(\tau) s_m + h_j(s_m, \tau) \quad j = 1, \ldots, m-1;$$

(ii) $$z_m + \overline{s_m} + \sum_{l=1}^{m-1} \overline{s_l} z_l + O(|s|^4 + |z|^4) = 0; \quad \text{and}$$

(iii) $z_m + \sum_{l=1}^{m-1} \overline{t_l} z_l + \overline{\tau} z_j + O(|\tau|^4 + |z|^4 + |t|^4) = 0$ for $j = 1, \ldots, m-1$.

Applying the implicit function to (ii) and (iii), and combining the solution with (i), we can conclude that for any fixed $\tau \approx 0$, when $s, t \approx 0$ there is a unique solution z (with respect to s and t), which takes the form :

$$z = \chi(\overline{s_m}, \overline{t}) \quad \text{with } g(0,0) = 0,$$

where χ is algebraic and holomorphic near $(0,0)$. If we identify S_τ with a small neighborhood of 0 in $\mathbf{C}^1$ through (i), it holds that $\phi = \chi$. Now, to see that χ is a biholomorphism near $(0,0)$, it suffices for us to show that the Jacobian determinant of χ with respect to $\overline{s_m}$ and $\overline{t} \in \mathbf{C^{m-1}}$ does not vanish at $(0,0)$. In fact, substituting (i) into (ii), we have

$$\text{(iv)} \qquad z_m + \overline{s_m} + \overline{s_m} \sum_{l=1}^{m-1} (a_l z_l + z_l \overline{h_l(s_m, \tau)}) + O(|z|^4 + |s|^4) = 0.$$

By combining (iii) with (iv) and a direct computation, we see that the Jacobian matrix of χ at (0,0) has the following form :

$$\begin{pmatrix} \overline{\tau} + O(|\tau|^3) & O(|\tau|^3) & \ldots & O(|\tau|^3) & 1 + O(|\tau|^3) \\ O(|\tau|^3) & \overline{\tau} + O(|\tau|^3) & \ldots & O(|\tau|^3) & 1 + O(|\tau|^3) \\ \ldots & \ldots & & & \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.$$

So, the Jacobian determinant of χ at 0 is $\overline{\tau}^{m-1} + O(|\tau|^m)$. Thus, it cannot vanish when $\tau (\neq 0) \approx 0$. This completes the proof of Claim 1. □

Now, by the way that ϕ was constructed and by the hypotheses of Lemma 2, we see that $g \circ \overline{\phi(s,t)}$ is holomorphic and algebraic on s (respectively, t) when holding t (resp., s) fixed. From (6) of Lemma 1, it thus follows the algebraicity of g. This completes the proof of Lemma 3. □

2.3. We now suppose that (AA) occurs. Then we will obtain the algebraicity of f when $k = 1$, or reduce the situation to the lower codimensional case when $k > 1$.

We first fix some notation. In what follows, we use the symbol $\hat{*}_l$ to denote the tuple obtained by deleting the element with index l from the vector $*$. For example, according to this convention, $\hat{F}_m$ means the vector function $(f_{m+1}, \ldots, f_{m+k-1})^t$, since $F = (f_m, \ldots, f_{m+k-1})^t$; and $\hat{F}_{m,m+1}$ is the vector function $(f_{m+2}, \ldots, f_{m+k-1})^t$.

Let us choose an integer n in the following way (the existence of such an integer can be seen by the condition in (AA)) :

(i) If for some $p_0 \approx 0$, j_0, and l_0, it holds that $L_{l_0} g_{j_0}(p_0, p_0) \neq 0$, then we let $n = m$. Here $g_0(f(z), \xi(z,\omega), \eta(z,\omega), \hat{\overline{F}}_m(\omega)) = H_0 + \sum_{|\alpha|=1, \alpha^* \neq m} \overline{f_{\alpha^*}} H_\alpha$, $g_1(f, \xi, \eta, \hat{\overline{F}}_m) = H_{(1,0,\cdots,0)}$, and g_j for $j > 1$ is determined by

$$H^*(f, \xi, \eta, \overline{F}) = \sum_{j=2}^{\infty} g_j(f, \xi, \eta, \hat{\overline{F}}_m) \overline{f_m}^j.$$

(ii) If (i) does not hold, we then let n be the smallest integer such that for each j, in the following expansion with respect to $\overline{f}_m, \ldots, \overline{f}_{n-1}$:

$$g_j = \sum_{\alpha} \phi_{j,\alpha}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n-1}) \overline{f_m^{\alpha_1}} \ldots \overline{f_{n-1}^{\alpha_{n-m-1}}},$$

it holds that $L_l \phi_{j,\alpha}(z, \overline{z}) \equiv 0$ on $U' \subset M_1$ for all l and $(n-m-1)$-degree multi-index α. But, for the expansion of some ϕ_{j_0,α_0} with respect to $\overline{f_n}$, there exist some l_0 and i_0 so that $L_{l_0} \phi_{j_0, i_0, \alpha_0}(z, \overline{z}) \not\equiv 0$ on any small neighborhood of 0, where

$$\phi_{j_0,\alpha_0}(f, \xi, \eta, \hat{\overline{F}}_{m,\ldots,n}) = \sum_i \phi_{j_0,i,\alpha_0}(f, \xi, \eta, \hat{\overline{F}}_{m,\ldots,n}) \overline{f}_n^i.$$

LEMMA 4. — *Let n as above. There is an algebraic function Φ and an open neighborhood Ω^* of $p \in M_1 \cap \Omega$ in $\mathbf{C^m}$ such that for any $(z, \overline{\omega}) \in (\Omega^* \times \mathrm{Conj}(\Omega^*)) \cap M_{1c}$ it holds that*

$$\overline{f_n}(\omega) = \Phi(f(z), f^{(1)}(z,\omega), f^{(2)}(z,\omega), \hat{\overline{F}}_n(\omega)). \tag{2.6}$$

Here the $f^{(j)}$'s are certain type of derivatives of f. This notation will be explained below.

Proof of Lemma 4. — We first assume that $n = m$. Then for some j_0, l_0, $p_0 \approx 0$, and the e^{th} element $g_{j_0}^{\mathrm{e}}$ of the vector function g_{j_0}, it holds that

$$L_{l_0} g_{j_0}^{\mathrm{e}}(f(p_0), \xi(p_0, p_0), \eta(p_0, p_0), \hat{\overline{F}}_m(p_0)) \neq 0.$$

We write

$$f^{(1)} = (L_1 f_1, L_1 f_2, \ldots, L_1 f_{m+k}, \cdots, L_{m-1} f_{m+k}),$$

$$f^{(2)} = (L_1 f^{(1)}, L_2 f^{(1)}, \dots, L_{m-1} f^{(1)}),$$

$$f^{(3)} = (L_1 f^{(2)}, L_2 f^{(2)}, \dots, L_{m-1} f^{(2)}),$$

$$\dots.$$

By the way that ξ and η were constructed, we see that they are rational functions of $f^{(1)}$. Hence, it follows that H_α and H^* are algebraic in $(f, f^{(1)}, \hat{\overline{F}}_m)$ for each α. Define

$$A(f, f^{(1)}, \overline{F}) = H_0^e(f, \xi, \eta) + \sum_{|\alpha|=1} \overline{f_{\alpha^*}} H_\alpha^e(f, \xi, \eta, \overline{F}) + H^{*e}(f, \xi, \eta, \overline{F}),$$

where H_α^e and H^{*e} are the e^{th} elements of H_α and H^*, respectively. By (3) and (4) of Lemma 1, we obtain the algebraicity of $L_{l_0} A(f, f^{(1)}, \hat{\overline{F}}_m)$(for simplicity, we denote it by $A^{(1)}(f, f^{(1)}, f^{(2)}, \hat{\overline{F}})$) in $(f, f^{(1)}, f^{(2)}, \hat{\overline{F}})$. This is so because

$$A^{(1)}(f, f^{(1)}, f^{(2)}, \hat{\overline{F}}) = \sum_j \frac{\partial A}{\partial w_j} L_{j_0} f_j + \sum_{i,j} \frac{\partial A}{\partial w_{i,j}^{(1)}} L_{j_0}(L_i f_j),$$

where we identify the variable $w_{i,j}^{(1)}$ with $L_i f_j$ in $f^{(1)}$. Meanwhile, by (2.4) and the definition of g_j, we notice that

$$A^{(1)}(f, f^{(1)}, f^{(2)}, \hat{\overline{F}}) = \sum_j L_{l_0} g_j^e \overline{f_m}^j.$$

Thus, by the choice of p_0, we see that

$$A^{(1)}(f(p_0), f^{(1)}(p_0, p_0), f^{(2)}(p_0, p_0), \overline{\hat{F}_m}(p_0), y_m) \not\equiv 0$$

for $y_m \approx \overline{f}_m(p_0)$.

The proof of Lemma 4 in this case therefore follows from :

Claim 2. — *Let $p \in U$ and let $I(\overline{z}, w, w^{(1)}, \cdots, w^{(i)}, \hat{Y}_j, y_j)$ be holomorphically algebraic on $O = O_{\overline{z}}(\overline{p}) \times O_w(f(p)) \times \cdots \times O_{y_j}(\overline{f_j}(p))$. Suppose that $(\overline{z}, f(z), \cdots, \hat{\overline{F}}_j(z), \overline{f_j}(z))$ is a zero point of I for every $z \in \Omega \cap U$ and suppose that $I(\overline{p}, f(p), \cdots, \hat{\overline{F}_j}(p), y_j) \not\equiv 0$ for $y_j \approx \overline{f_j}(p)$. Then there exists an open subset U' of U so that, for $(z, \overline{\omega}) (\approx U') \in M_{1c}$, it holds that*

$$\overline{f_j}(\omega) = \Phi(\overline{z}, f(z), f^{(1)}(z, \omega), \dots, f^{(i)}(z, \omega), \hat{\overline{F}}_j(\omega))$$

for some algebraic holomorphic function Φ.

Proof of Claim 2. — From the given hypothesis, the Weierstrass preparation theorem, and (7) of Lemma 1, it follows that the equation $I(\overline{z}, w, w^{(1)}, \cdots, y_j) = 0$ is locally equivalent to the following algebraic equation :

$$(2.7) \qquad (y_j - y_0)^{n^*} + \sum_{j=0}^{n^*-1} \lambda_j(\overline{z}, w, w^{(1)}, \cdots, \hat{Y}_j)(y_j - y_0)^j = 0,$$

with $y_0 = \overline{f_j}(p)$. Let D_1 be the variety associated with (2.7), defined by

$$(2.8) \qquad n^*(y_j - y_0)^{n^*-1} + \sum_{j=1}^{n^*-1} j\lambda_j(\overline{z}, w, w^{(1)}, \cdots, \hat{Y}_j)(y_j - y_0)^{j-1} = 0.$$

If for $z \approx p$, the vector $(\overline{z}, f(z), f^{(1)}(z, z), \cdots, \hat{\overline{F}}_j(z), \overline{f_j}(z))$ also satisfies (2.8), whose degree, with respect to y_j, is smaller than that of (2.7), we then pass to the study of the variety associated with D_1. Otherwise, by the implicit function theorem, (2.7) tells us, for $z \approx p'(\approx p)$, that

$$(2.9) \qquad \overline{f_j(z)} = \Phi(\overline{z}, f(z), f^{(1)}, \cdots, \hat{\overline{F}}_j(z)),$$

for some algebraic, holomorphic function $\Phi(\overline{z}, w, w^{(1)}, \cdots, \hat{Y}_j)$ on $O_{\overline{z}}(\overline{p'}) \times O_w(f(p')) \times \cdots \times O_{\hat{Y}_j}(\hat{\overline{F_j}}(p'))$ (here, we need to apply (6) of Lemma 1 to obtain the algebraicity of Φ). Complexifying (2.9) and noting again the maximal total reality of $U' \times \mathrm{Conj}(U')$ in M_{1c}, we thus obtain

$$\overline{f_j(\omega)} = \Phi(\overline{z}, f(z), f^{(1)}(z, \omega), \cdots, \hat{\overline{F}}_j(\omega)),$$

for $(z, \omega)(\approx (p', p')) \in M_{1c}$. We now use an induction argument with respect to n^* and notice that (2.8) will eventually reduce to the equation : $n^*(y_j - y_0) + \lambda_{n^*-1} = 0$. We then conclude the existence of the Φ in the claim. This completes the proof of Claim 1. □

Now let $n > m$. We then have, for each l and α, that

$$L_l \phi_{j,\alpha}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n-1}) \equiv 0 \quad \text{on some } U_1 \subset U,$$

where $\phi_{j,\alpha}$ is defined as before and

$$(2.10) \qquad \phi_{j,\alpha} = \sum_{i=0}^{\infty} \phi_{j,i,\alpha}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n})\overline{f_n^i}.$$

But for some $p_0^* \approx 0$, α_0, i_0 j_0, and l_0, it holds that

$$L_{l_0}\phi_{j_0,i_0,\alpha_0}(p_0^*) \neq 0.$$

Let $L_{l_0}\phi_{j_0,\alpha_0} = \psi_{j_0,\alpha_0}(f, f^{(1)}, f^{(2)}, \overline{f}_{n+1}, \cdots, \overline{f_{m+k-1}}, \overline{f_n})$. We claim that ψ_{j_0,α_0} is algebraic in $(f, f^{(1)}, f^{(2)}, \cdots, \overline{f_{m+k-1}})$. In fact, since ϕ_{j_0,α_0} is the Taylor coefficient of the algebraic function g_{j_0}, we can thus see the algebraicity of ϕ_{j_0,α_0}, by Taylor's formula and by inductively using (3) of Lemma 1. Again from Lemma 1, we determine the algebraicity of ψ_{j_0,α_0} (see the argument for the algebraicity of $A^{(1)}$). Now it is easy to check that Claim 2 can be applied to the equation $\psi_{j_0,\alpha_0}(w, w^{(1)}, w^{(2)}, y_{n+1}, \cdots, y_{m+k-1}, y_n) = 0$ for solving $\overline{f}_n$. So the proof of Lemma 4 is complete. □

An immediate consequence of Lemma 4 is that in case the codimension $k = 1$, then f_m is algebraic. The reason for this is similar to the proof of Lemma 2. In fact, let $z_0 \approx U'$ and $z \in Q_{z_0}$. Since $z_0 \in Q_z$, we see by Lemma 4 that

$$f_m(z) = \overline{\Phi(z_0, f(z_0), f^{(1)}(z_0, z), f^{(2)}(z_0, z))}.$$

Notice that $f^{(j)}(z_0, z)$ is algebraic in $\overline{z}$ and Φ is algebraic in its variables. We thus conclude the algebraicity of f_m along each Q_{z_0}. From Lemma 3, we may then conclude the global algebraicity of f_m.

Now, returning to (2.4) with $\alpha = 1$, we get

$$\begin{aligned} F_0(\omega) = &-\overline{H_0(f(z), \xi(z,\omega), \eta(z,\omega))} \\ &-f_m(\omega)\overline{H_1(f(z), \xi(z,\omega), \eta(z,\omega))} - \overline{H^*(f(z), \xi, \eta, \overline{f_m(\omega)})}, \end{aligned}$$

where $z \in Q_\omega$. Notice the algebraicity of $f_m(\omega)$, $\overline{\xi(z,\omega)}$, and $\overline{\eta(z,\omega)}$ with respect to the variables ω and the algebraicity of H_0, H_1, and H^* with respect to their own variables. From the above argument, we therefore also see the algebraicity of $F_0(z)$ along each Q_{z_0} $(z_0 \approx U)$. Thus $F_0(z) = (f_1(z), \ldots, f_{m-1}(z))$ is algebraic in z. By the same token, we can prove the algebraicty of f_{m+1} by using the equality (2.1) and the just obtained results. So, we have

LEMMA 5. — *When $k = 1$, then f is algebraic in case (AA).*

For the general codimension $k(>1)$, we have

LEMMA 6. — *Under the assumptions in Lemma 4, there exist a small open subset U'' of U' and an algebraic holomorphic function Ψ so that*

$$f_n(z) = \Psi(z, \hat{F}_n(z)) \text{ for } z \in U'', \tag{2.11}$$

where n is as in Lemma 4.

The proof of Lemma 6 follows easily from the following slightly more general assertion :

CLAIM 3. — *Let $p \in U \subset M_1$ and Ψ^* an algebraic holomorphic function on $O_{\overline{z}}(\overline{p}) \times O_w(f(p)) \times \ldots \times O_{w^{(k')}}(f^{(k')}(p,p)) \times O_{\hat{\overline{Y}}_j}(\hat{\overline{F}}_j(p))$ $(j > m-1)$ so that for some $i \geq 1$, it holds that*

$$\overline{f_i(z)} = \Psi^*(\overline{z}, f(z), f^{(1)}(z,z), \cdots, f^{(k')}(z,z), \hat{\overline{F}}_j(z)), \quad z(\approx p) \in U. \tag{2.12}$$

Then there is a holomorphically algebraic function Ψ, defined on $O_z(p^*) \times O_{\hat{F}_j(p^*)}$ with $p^*(\approx p) \in U$, such that

$$f_i(z) = \Psi(z, \hat{F}_j(z)), \quad \text{for } z(\approx p^*) \in U.$$

Proof of Claim 3. — We proceed by induction on the number of the variables $\overline{f_l}$'s in the formula of Ψ^*. First, if Ψ^* involves no $\overline{f_l}$ terms $(l \geq m)$, then Claim 3 follows immediately from the argument presented to prove Lemma 5 (in this situation, the complexification of (2.12) is :

$$\overline{f_i(\omega)} = \Psi^*(\overline{\omega}, f(z), f^{(1)}(z,\omega), \cdots, f^{(k')}(z,\omega)), \quad z \in Q_\omega).$$

In the general case, to simplify the notation, we let $j = m$ and expand Ψ^* as follows :

$$\Psi^*(\overline{z}, f(z), f^{(1)}(z,z), \ldots, \hat{\overline{F}}_m(z)) = \phi_0(\overline{z}, f(z), \ldots, f^{(k')}(z,z))$$

$$+ \sum_{|\alpha|=1, \alpha^* \neq m} \phi_\alpha(\overline{z}, f, \ldots, f^{(k')})(\overline{f_{\alpha^*}}(z) - \overline{f_{\alpha^*}}(p)) + \phi^*(\overline{z}, \ldots, \hat{\overline{F}}_m),$$

where

$$\phi^* = \sum_{|\alpha| \geq 2} \phi_\alpha(\overline{z}, f_1, \ldots, f^{(k')})(\hat{\overline{F}}_m(z) - \hat{\overline{F}}_m(p))^\alpha.$$

(1) In case $L_l\phi_\alpha(z) \equiv 0$ for all α, l, and $z(\approx p) \in U$, we may complete the proof of the claim by applying Lemma 2 in the following way :

Since $L_l\phi_\alpha(\overline{\omega}, f(z), f^{(1)}(z,\omega), \ldots, f^{(k')}(z,\omega))$ is holomorphic in $(z,\overline{\omega})$, by the observation which we made before the proof of Lemma 2, we know that the given hypothesis implies that $\phi_\alpha(\overline{\omega}, f(z), f^{(1)}(z,\omega), \ldots, f^{(k')}(z,\omega))$ is constant along each Segre surface Q_ω. Set

$$\widetilde{\Psi}(z,\omega,\hat{Y}_m) = \overline{\phi_0(\overline{\omega}, f(z), f^{(1)}(z,\omega), \ldots, f^{(k')}(z,\omega))}$$

$$+ \overline{\sum_{|\alpha|=1,\alpha^*\neq m} \phi_\alpha(\overline{\omega}, f(z), \ldots, f^{(k')}(z,\omega))}(y_{\alpha^*} - f_{\alpha^*}(p)) + \overline{\phi^*(\overline{\omega}, \ldots, \hat{Y}_m)},$$

where $z \in Q_\omega$. Thus we similarly see that $\Psi(z,\hat{Y}_m) = \widetilde{\Psi}(z^*, z, \hat{Y}_m)$ with $z^* \in Q_z$ is well defined. Moreover, the same argument as in Lemma 2 shows that Ψ is holomorphically algebraic in its variables. So, the proof of Claim 3 in this case is complete; for it obviously holds that $f_i(z) = \Psi(z, \hat{F}_m(z))$.

(2) Now, we assume that (1) does not occur. We then define a nature number n' in a similar way as we did for n (the existence of such an n' can also be seen from the hypotheses) :

(a) If for some $p'(\approx p)$, α_0, and l_0, it holds that $L_{l_0}(\psi_{\alpha_0}(p')) \neq 0$, we then let $n' = m+1$. Here

$$\psi_0 = \phi_0 + \sum_{|\alpha|=1,\alpha^*\neq m,m+1} \phi_\alpha \left(\overline{f_{\alpha^*}(z)} - \overline{f_{\alpha^*}(p)}\right),$$

$\psi_1 = H_{(0,1,0,\ldots,0)}$, and ψ_j for $j > 1$ are determined by

$$\phi^*(\overline{z}, f, \ldots, \hat{\overline{F}}_m) = \sum_{j\geq 2} \psi_j(\overline{z}, \ldots, f^{(k')}) \left(\overline{f_{m+1}(z)} - \overline{f_{m+1}(p)}\right)^j.$$

(b) When (a) does not hold, we let n' be the smallest integer such that : for each j, in the expansion of ψ_j with respect to $(\overline{f_{m+1}}(z) - \overline{f_{m+1}(p)}), \ldots, (\overline{f_{n'-1}}(z) - \overline{f_{n'-1}(p)})$, all coefficients are annihilated by the operators $\{L_l\}$; but at least for one coefficient of certain ψ_{j_0}, say $b_{j_0}(\overline{z}, f, \ldots, \hat{\overline{F}}_{m,\cdots,n'-1})$, there exist some l_0, i_0 with $L_{l_0}b_{j_0,i_0}(z) \not\equiv 0$ on a small neighborhood p in U. Here

$$b_{j_0}(\overline{z}, f, \ldots, \hat{\overline{F}}_{m,\cdots,n'-1}) = \sum_i b_{j_0,i}(\overline{z}, f, \ldots, \hat{\overline{F}}_{m,\cdots,n'})(\overline{f_{n'}}(z) - \overline{f_{n'}(p)})^i.$$

We now apply the argument in Lemma 4 with g_j's there being replaced by the ϕ_j 's here (in case (a)), or with ϕ_{j_0,α_0} and ϕ_{j_0,i,α_0} being replaced by b_{j_0} and $b_{j_0,i}$, respectively (in case (b)). We then obtain an algebraic holomorphic function Ψ^{**} so that

$$(2.12)' \qquad \overline{f_{n'}(z)} = \Psi^{**}(\overline{z}, f(z), \cdots, f^{(k')}(z,z), \overline{\hat{F}}_j(z)),$$
$$\text{for } z \text{ on a small open subset of } U \text{ near } p.$$

Substitute (2.12)′ into the $\overline{f_{n'}}$ variable in the formula of Ψ^*. Since, the number of $\overline{f_l}$'s is now decreased by 1, we can thus conclude the proof of Claim 3 by the induction hypothesis. □

Replace $\overline{f_n}(z)$ in (2.4) by $\overline{\Psi(z, \hat{F}_n(z))}$ obtained in Lemma 6. Then we have, for each $i \leq m-1$, that

$$(2.13) \qquad \overline{f}_i(z) = g_i^*(\overline{z}, f, f^{(1)}, \overline{\hat{F}}_n) \text{ on } U^{(2)} \subset U'',$$

where g_i^* is holomorphic and algebraic on $O_{\overline{z}}(\overline{p''}) \times O_w(f(p'')) \times \cdots \times O_{\hat{Y}_j}(\overline{\hat{F}_j}(p''))$ with p'' being some point in U''. From Claim 3, it follows that on some $U^{(3)} \subset U''$, there exist algebraic holomorphic functions $\{\Psi_1, \cdots, \Psi_{m-1}\}$ so that it holds for each $i \leq m-1$ that

$$(2.14) \qquad f_i(z) = \Psi_i(z, \hat{F}_n(z)) \quad \text{for } z \in U^{(3)}.$$

Similarly, by substituting (2.11) and (2.14) to (2.1), we obtain

$$(2.15) \qquad \overline{f}_{m+k}(z) = g_{m+k}^*(\overline{z}, f(z), \overline{\hat{F}}_n(z)) \quad \text{on } U^{(4)} \subset U^{(3)},$$

with g_{m+k}^* holomorphic and algebraic in $(f, \overline{z}, \overline{\hat{F}}_n)$. Thus it can be seen, after shrinking $U^{(4)}$, that we have

$$(2.16) \qquad f_{m+k}(z) = \Psi_{m+k}(z, \hat{F}_n(z)) \quad \text{on } U^{(4)},$$

for some algebraic holomorphic function Ψ_{m+k}. Combining all these formulas, we now come up with

LEMMA 7. — *There are a small neighborhood $\Omega^* \subset \mathbf{C}^{\mathbf{m}}$ of some $p \in U^{(4)}$ and a nonsingular algebraic complex variety $M^* \subset \mathbf{C}^{\mathbf{m+k}}$, which contains $f(p)$, so that $f(\Omega^*) \subset M^*$.*

Proof of Lemma 7. — Let Ψ and n as in Lemma 6. Consider the equation $w_n = \Psi(z, \hat{w_n^*})$, where $w = (w_*, w^*, w_{m+k})$ with $w_* =$

$(w_1,\ldots,w_{m-1})$ and $w^* = (w_m,\ldots,w_{m+k-1})$. Set $\chi_1(z,w) = w_n - \Psi(z,\hat{w_n^*})$. If χ_1 does not involve any z terms, we then define $M^* \subset \mathbf{C}^{\mathbf{m}+\mathbf{k}}$ by the equation $w_n = \Psi(p,\hat{w_n^*})$. Obviously, M^* is a regular algebraic manifold near $f(p)$ and $f(U^{(5)}) \subset M_2$, where $U^{(5)} \subset U^{(4)} \subset U''$ is a small neighborhood of p in M_1 (see Lemma 6). Since $U^{(5)}$ is a set of uniqueness for the holomorphic function f, it follows that $f(\Omega^*) \subset M^*$ for certain small neighborhood of $U^{(5)}$ in $\mathbf{C}^{\mathbf{m}}$. So, without loss of generality, we assume that the Taylor expansion of Ψ at $(p,\hat{F}_n(p))$ does have z terms. After a rotation around p in $\mathbf{C}^{\mathbf{m}}$ (if necessary), we may assume $\dfrac{\partial^j\Psi}{\partial z_1^j}(p) \neq 0$ for some $j \geq 1$. Notice that $\chi_1(p,f(p)) = 0$. By the Weierstrass preparation theorem, the equation $\chi_1 = 0$ is therefore equivalent to

$$(2.17) \qquad (z_1 - p^1)^{n^*} + \sum_{j=0}^{n^*-1} a_j(w^*,z_2,\cdots,z_m)(z_1-p^1)^j = 0$$

with $n^* \geq 1$, where $p = (p^1,\cdots,p^m)$ and a_j's are algebraic. Arguing as in Claim 2, we can conclude that $z_1 = \chi_1^*(z_2,\ldots,z_m,F(z))$ for $z(\in U^{(5)}) \approx p^*$ (here we may have to shrink $U^{(5)}$). Now substituting this into (2.14), we obtain, for $i < m$:

$$(2.17)' \qquad f_i(z) = \Psi_i^{(1)}(z_2,\cdots,z_m,F(z)) \ \text{ for } z \in U^{(5)},$$

where $\Psi_i^{(1)} = \Psi_i(\chi_1^*(z_2,\cdots,z_m,w^*),z_2,\cdots,z_n,\hat{w^*}_n)$. Consider especially the equation :

$$\chi_2(z_2,\ldots,z_m,w) = w_1 - \Psi_1^{(1)}(z_2,\cdots,z_m,w^*) = 0.$$

By the same token, if the above equation is independent of $(z_2,\ldots,z_m)$, the M^* in the lemma can be defined by $w_1 = \Psi_1^{(1)}(p_2^*,\cdots,p_m^*,w^*)$, where $(p_1^*,\cdots,p_m^*)$ is a fixed point in $U^{(5)}$. Otherwise, after a rotation at $(p_2^*,\cdots,p_m^*)$ with respect to the variables $(z_2,\cdots,z_m)$, we can also assume that $\dfrac{\partial^j\Psi_1^{(1)}}{\partial z_2^j}(p^*) \neq 0$ for some $j \geq 1$. Then it follows similarly that there exists an algebraic holomorphic function $\chi_2^*(z_3,\cdots,z_m,w_1,w^*)$ with $z_2 = \chi_2^*(z_3,\cdots,z_m,f_1(z),F(z))$ for z in a small open subset of $U^{(5)}$. Now, substitute this again into (2.17)$'$ and consider the equation :

$$(2.17)'' \qquad w_2 - \Psi_2^{(2)}(z_3,\cdots,z_m,w_1,w^*) = 0,$$

where $\Psi_2^{(2)} = \Psi_2^{(1)}(\chi_2^*(z_3,\cdots,z_m,w_1,w^*),z_3,\cdots,w^*)$. Repeating what we just did, we see that either we complete the proof of the lemma, or we can solve from $(2.17)''$ that $z_3 = \chi_3^*(z_4, \cdots, z_m, w_1, w_2, w^*)$ with $f_3(z) = \chi_3^*(z_4, \cdots, z_m, f_1(z), f_2(z), F(z))$ for z in a small subset of $U^{(5)}$. Arguing inductively in this way, we then either come up with the proof of Lemma 7, or we obtain algebraic holomorphic functions

$$\chi_j^*(z_{j+1}, \cdots, z_m, w_1, w_2, \cdots, w_{j-1}, w^*) \ (j = 1, \ldots, m)$$

so that $z_j = \chi_j^*(z_{j+1}, \cdots, f_1(z), \cdots, f_{j-1}(z), F(z))$ for z on a small open subset of $U^{(5)}$. Here we understand w_0 and z_{m+1} as 0. In the latter case, we can easily obtain an algrebraic vector function $\chi(w_*, w^*)$ with $z = \chi(F_0(z), F(z))$ for z in a small open subset of $U^{(5)}$. Meanwhile, combining this equality with (2.16), we also see that the algebraic manifold M^*, defined by the equation $w_{m+k} = \Psi_{m+k}(\chi(w_*, w^*), \hat{w}_n^*)$, does our job in the lemma. □

LEMMA 8. — *Let M^* be as in Lemma 7. Then $M^* \cap M_2$ is an algebraic strongly pseudoconvex hypersurface of M_2.*

Proof of Lemma 8. — Through a linear change of variables, we may assume that $p = 0$ and that the complex tangent space of M_2 is defined by $w_{m+k} = 0$. Since $f(\Omega^*) \subset M^*$ and since $f(\Omega^*)$ is transversal to $T_0^{(1,0)} M_2$ (see §1.2), it follows that $T_0^{(1,0)} M^* \neq \{w_{m+k} = 0\}$. Thus, by the implicit function theorem, we see that M^* can be locally expressed by the equation : $w_l = \phi(\hat{w}_l)$ for some $l \neq m + k$. Now it is easy to see that $\rho_2^* = \rho_2(w_1, \cdots, \phi(\hat{w}_l), \cdots)$ is a non degenerate real algebraic defining function of $M^* \cap M_1$, which is obviously strongly plurisubharmonic at 0. □

2.4. We now are in a position to study the main theorem in case (BB). We will either reduce to the situation (AA) or obtain the algebraicity of f.

By Lemma 2, we have an algebraic function Φ so that

$$F_0 = \Phi(z, F) \tag{2.18}$$

for $z \approx 0$. Substituting this to (2.1) (here we assume that the h in (2.1) does not contain $\overline{f_{m+k}}(\omega)$ term), we obtain

$$f_{m+k}(z) + \overline{f_{m+k}(\omega)} + \sum_{j=1}^{m+k-1} f_j(z)\overline{f_j(\omega)} + h^*(\overline{\omega}, f(z), \overline{F(\omega)}) = 0 \tag{2.19}$$
$$\text{for } z \in Q_\omega \text{ or } \ (z, \overline{\omega}) \in M_{1c}$$

with $h^*(\overline{\omega}, f(z), \overline{F(\omega)}) = h(f(z), \overline{\Phi(\omega, F(\omega))}, \overline{F(\omega)})$. We now repeat what we did at the beginning of this section. Then we obtain also the equation with a form similar to (2.2) :

$$(2.20) \qquad \xi + \overline{F}_0 + \eta\overline{F} + g(\overline{\omega}, f, \xi, \eta, \overline{F}) = 0, \quad \text{for } (z, \overline{\omega}) \in M_{1c},$$

where $g(\overline{\omega}, f, \xi, \eta, \overline{F}) = (\mathrm{id}, \eta, \xi)Dh^*$. Define similarly the new H_α functions in terms of (2.20). We also consider the conditions (AA) and (BB), respectively. By §2.3, in case (AA) occurs, then either we obtain the algebraicity ($k = 1$) or we can transform the problem to the case of codimension $k-1$. Assume that we are still in case (BB). So the new H_α's are also constant along each Segre surface. We note that

$$(2.21) \qquad H_0 = \xi + (\mathrm{id}, \eta, \xi)Dh^*|_{Y=0}$$

and

$$(2.22) \qquad H_\alpha = \eta_{\alpha^*} + (\mathrm{id}, \eta, \xi)\frac{\partial}{\partial y_{\alpha^*}}Dh^*|_{Y=0} \text{ for } |\alpha| = 1.$$

From (2.21) and the definition of η and ξ, it follows, for any l, that

$$L_l\left(\sum_{j=1}^{m-1} H_0^j(z,\omega)f_j\right) = L_l(f_{m+k}) + L_l h^*,$$

where $H_0 = (H_0^1, \cdots, H_0^{m-1})^t$. Thus if we set

$$(2.23) \qquad E_0 = \sum_{j=1}^{m-1} H_0^j(z,\omega)f_j - f_{m+k} - L_l h^*,$$

then $E_0(z,\omega)$ is constant on Q_ω for every $\omega \approx 0$. Similarly, (2.22) tells that

$$(2.24) \qquad E_j = \sum_{i=1}^{m-1} H^i_{(0,\cdots,1(j^{th}),\cdots,0)}(z,\omega)f_i - f_j - \frac{\partial h^*}{\partial y_j},$$

is also constant along Q_ω for $j = m, \cdots, m+k-1$. Notice $H_\alpha(0,0) = 0$. Applying the implicit function theorem to (2.18), (2.23), and (2.24), we then obtain

$$f(z) = G(z, \overline{\omega}, E, H_\alpha), \quad (z,\omega) \in M_{1c},$$

where $E = (E_0, E_m, \cdots, E_{m+k-1})$ and $|\alpha| \leq 1$. From Lemma 1, it follows that G is also algebraic in its variables. By Lemma 3, to show that f is algebraic in z, it suffices for us to prove that $f(z)$ is algebraic along any

Q_ω. However, this follows immediately from the fact that E and H are constant along each Segre surface.

2.5. Summarizing all the above discussions, we can conclude the algebraicity when $k = 1$. In case $k > 1$, we either obtain the algebraicity of f (see §2.4) or we may reduce to a problem with smaller codimension (see §2.3 and §2.4). Thus, by a simple induction argument, the proof of our main theorem follows.

Remark. — From the proof, it is easy to see that when f is a priori assumed to be an immersion from M_1 to M_2, then M_1 and M_2 in the main theorem can be relaxed to the non-degenerate real algebraic hypersurfaces. However, the main theorem is obviously false if M_1 and M_2 are allowed to be Levi flat.

3. Proof of Theorem 1.

The purpose of this section is to prove Theorem 1 by modifying the previous argument. We still start with the equation $\rho_2(f(z), \overline{f(z)}) = \lambda(z, \overline{z})\rho_1(z, \overline{z})$. Since we do not know the existence of the complexification in the present setting, we will differentiate the equation along M_1. Then we will come up with a new equation similar to (2.3), which also enables us to divide the discussions according to how degenerate the map f is : in a sort of the totally degenerate case (analogous to (BB)), we will reduce the analytic extendibility to the hyper-ellipticity of a differential equation by making use of the CR-extension results. In the other situations, we will similarly obtain the analyticity of f (in case k=1) or get a reduction with respect to the codimension.

For the sake of brevity, we retain most of the notation in §2.

3.1. We now let M_1, M_2, and L_l as in §1.2. To prove Theorem 1, we proceed by seeking a point $q \in U$, where U is an arbitrarily fixed small neighborhood of 0 in M_1, so that f has an analytic extension at q.

By the properness of f (i.e., the fact : $f(M_1) \subset M_2$), we see that

(3.1)
$$f_{m+k}(z) + \overline{f_{m+k}(z)} + \sum_{j=1}^{m+k-1} |f_j(z)|^2 + h(f(z), \overline{f(z)}) = 0 \quad \text{for } z \in U \subset M_1.$$

As did in §2.1, by using the implicit function theorem, we can assume that $h(f,\overline{f}) = h(f,\overline{f_1},\ldots\overline{f_{m+k-1}})$, where $h(w,y_1,\ldots,y_{m+k-1})$ is a holomorphic function on $O_w(0)\times O_{y_1}(0)\times\ldots\times O_{y_{m+k-1}}(0)$.

Applying L_l to (3.1) for each l, we obtain

$$(3.2)\quad L_l f_{m+k}(z) + \sum_{j=1}^{m+k-1} L_l f_j(z)\overline{f_j(z)} + \sum_{j=1}^{m+k} \frac{\partial h}{\partial w_j} L_l f_j(z) = 0, \quad \text{for } z\in U.$$

Let V,ξ, and η as defined in §2.1, except replacing $\overline{\omega}$ by $\overline{z}$. Equation (3.2) can then be written as

$$\xi(z) + \overline{F_0(z)} + \eta(z)\overline{F}(z) + (\mathrm{id},\eta(z),\xi(z))Dh(z) = 0 \ \text{ for } z\in U,$$

where $Dh(z) = \left(\frac{\partial h}{\partial w_1},\ldots,\frac{\partial h}{\partial w_{m+k}}\right)^t(z) = O(|z|^3)$ as $z\to 0$, $F_0 = (f_1,\ldots,f_{m-1})^t$, and $F = (f_m,\ldots,f_{m+k-1})^t$.

Again, by making use of the implicit function theorem and by shrinking U, we have that

$$(3.3)\qquad \xi + \overline{F}_0 + \eta\overline{F} + g(f,\xi,\eta,\overline{F}) = 0 \quad \text{on} \quad U.$$

Here g is holomorphic in its variables and if

$$g(f,\xi,\eta,\overline{F}) = \sum_\alpha g_\alpha(f,\xi,\eta)\overline{F^\alpha},$$

then

$$\frac{\partial g_\alpha}{\partial\xi},\frac{\partial g_\alpha}{\partial\eta}\to 0$$

as $z(\in U)\to 0$ for $|\alpha|\le 1$ (by (1.2) and (1.3)).

We now expand g with respect to f_m :

$$g = \sum_{j=0}^{\infty} g_j(f,\xi,\eta,\hat{\overline{F}}_m)\overline{f_m^j},$$

where $g_j(w,\xi,\eta,y_{m+1},\cdots,y_{m+k-1})$ is holomorphic on $O_1 = O_w(0)\times O_\xi(0)\times O_\eta(0)\times\cdots\times O_{y_{m+k-1}}(0)$ and there exists a number $R >> 1$ so that $|g_j(w,\xi,\eta,\cdots y_{m+k-1})| \le R^j$ for each j and for every $(w,\xi,\eta,\cdots,y_{m+k-1})\in O_1$.

Set

$$H_0 = \xi + \hat{\eta}_m\hat{\overline{F}}_m + g_0(f,\xi,\eta,\hat{\overline{F}}_m),$$

$$H_1 = \eta_m + g_1(f, \xi, \eta, \overline{\hat{F}}_m),$$

and

$$H_j = g_j(f, \xi, \eta, \overline{\hat{F}}_m) \quad \text{for } j = 2, \cdots.$$

Here, as we defined before, we write $\eta = (\eta_m, \cdots, \eta_{m+k-1})$ and $\hat{\eta_m} = (\eta_{m+1}, \cdots, \eta_{m+k-1})$. Now (3.3) reads as

$$H_0 + \overline{F}_0 + \overline{f_m} H_1 + \sum_{j=2}^{\infty} \overline{f_m^j} H_j = 0 \quad \text{on } U_0 \subset U, \tag{3.4}$$

where U_0 is a small neighborhood of $0 \in M_1$.

3.2. In this subsection, we study a situation similar to (BB) in §2. We will obtain the analyticity by using CR-extension and PDE results.

From (3.4), we now define $\mathrm{Ind}(1) = 0$ if, for each k and j, $L_l(H_j) = 0$ on a small neighborhood $U_1 \subset U$ of 0. Otherwise, we define $\mathrm{Ind}(1) = 1$. In case $\mathrm{Ind}(1) = 0$, we then let, for each j_0,

$$H_{j_0}(z) = \sum_{j=0}^{\infty} \phi_{j_0,j}(f, \xi, \eta, \overline{\hat{F}}_{m,m+1}) \overline{f}_{m+1}^j. \tag{3.5}$$

Applying L_l to (3.5) for each l, we see that

$$\sum_j L_l(\phi_{j_0,j}(f, \xi, \eta, \overline{\hat{F}}_{m,m+1})) \overline{f}_{m+1}^j = 0, \quad \text{on } U_1. \tag{3.6}$$

Define $\mathrm{Ind}(2) = 0$ if $L_l(\phi_{j_0,j}) = 0$ for all l, j_0, and j on a small neighborhood $U_2 (\subset U_1)$ of 0; otherwise let $\mathrm{Ind}(2) = 1$.

If it still happens that $\mathrm{Ind}(2) = 0$, we expand ϕ_{j_0,j_1}, for every j_0 and j_1, with respect to $\overline{f}_{m+2}$. Then we can similarly define the value of $\mathrm{Ind}(3) \cdots$. Arguing inductively, if it always happens that $\mathrm{Ind}(j) = 0$ for $j = 1, \cdots, k$, we then easily see, for any index i, that

$$H_i(f, \xi, \eta, \overline{\hat{F}}_m) = \sum_{|\alpha|=0}^{\infty} h_{i,\alpha}(f, \xi, \eta) \overline{\hat{F}}_m^{\alpha}$$
$$\text{for } z \in U_k \text{ (a small neighborhood of 0)},$$

where $L_l h_{i,\alpha}(f, \xi, \eta) \equiv 0$ for all indices i, l, α, and $h_{i,\alpha}(w, a, b)$ is holomorphic on $O_2 = O_w(0) \times O_a \times O_b(0)$ with $|h_{i,\alpha}(w, a, b)| < R^{|\alpha|}$ for each

(k-1)-multi-index α and for some $R >> 1$. Returning to (3.4), we then obtain the expansion :

$$\overline{F_0} + \sum_{|\alpha|=0}^{\infty} H_\alpha^* \overline{F^\alpha} = 0, \tag{3.7}$$

with $L_l H_\alpha^* \equiv 0$, for all l and k-multi-index α, on U' (a neighborhood of 0). By the unique power series expansion property of a holomorphic function and by combining the upper bound (Cauchy) estimates of $h_{i,\alpha}$ with those of g_j, it therefore follows that

$$|H_\alpha^*(w, \xi, \eta)| \le R^{|\alpha|} \tag{3.7$'$}$$

for some $R >> 1$ and that

$$H_0^* = \xi + g_0(f, \xi, \eta, 0),$$

$$H_{(1,0,\ldots,o)}^* = \eta_m + g_1(f, \xi, \eta, 0),$$

and

$$H_\alpha^* = \eta_{\alpha^*} + \frac{\partial g_0}{\partial y_{\alpha^*}}(f, \xi, \eta, 0) \tag{3.8}$$

where the notation α^* is the same as before.

LEMMA 9. — *Under the above circumstances, we have, for each α, a holomorphic extension $A_\alpha(z)$ of $\overline{H}_\alpha^*$ on Ω^* (a small neigborhood of U' in $\mathbf{C^m}$). Moreover, it holds that $\max_{z \in \Omega^*} |A_\alpha(z)| < R^\alpha$.*

Proof of Lemma 9. — First, we note that $\overline{H_\alpha^*}(z)$ is a CR-function on U_k for each α, since $\overline{L_l}(\overline{H_\alpha^*})(z) = 0$ for all l. Consequently, by the Lewy extension theorem, we have, for each $\overline{H_\alpha^*}$, a holomorphic extension $A_\alpha^+(z)$ defined on some open subset $\Omega'(\subset \Omega)$ whose size depends only on U'. Since the analytic discs with their boundaries attached on U_k sweep out an open subset of Ω, so after shrinking Ω', the maximal principle then implies that $\max_{\Omega'} |A_\alpha^+| = \max_{U'} |\overline{H_\alpha^*}| < R^\alpha$ (by (3.7)$'$).

Now, let $\phi : V \subset \mathbf{C^1} \to \mathbf{C^m}$ be an embedding such that $\phi(V \cap \Delta) \subset \Omega'$, $\phi(1) \approx 0$, $\phi(\partial(\Delta \cap V)) \subset M_1$, and $\phi(\Delta^c \cap V) \subset \mathbf{C^m} - \Omega'$ (where Δ denotes the unit disk in $\mathbf{C^1}$). Since M_1 is real analytic, we can extend $f \circ \phi(\tau)$, $\xi \circ \phi(\tau)$, and $\eta \circ \phi(\tau)$ holomorphically into $\Delta \cap V'$ (where $V' \subset V$

is a small neighborhood of $\partial\Delta \cap V$ and symmetric with respect the unit circle). For $\tau \in \Delta^c \cap V'$, we define

$$A_\alpha^-(\tau) = \overline{H_\alpha^*(f \circ \phi(1/\overline{\tau}), \xi \circ \phi(1/\overline{\tau}), \eta \circ \phi(1/\overline{\tau}))}.$$

Then A_α^- is holomorphic on $\Delta^c \cap V'$ and $A_\alpha^-(\tau) = A_\alpha^+(\tau)$ for $\tau \in \partial\Delta \cap V'$. It thus follows from the Hartogs theorem that A_α^+ has a holomorphic extension A_α, on an open subset Ω^* near 0, which does not depend on α. By the construction of A_α and (3.7)$'$, it obviously holds that $\max_{\Omega^*} |A_\alpha| < R^\alpha$ (we may have to shrink Ω^* here). This completes the proof of Lemma 1. □

Now by (3.7), we have that

$$F_0(z) + \sum_\alpha A_\alpha(z) F^\alpha(z) = 0. \tag{3.9}$$

Let $J(z, w^*) = -\sum_{|\alpha|=0}^{\infty} A_\alpha(z) w^{*\alpha}$. For $(z, w^*) \in \Omega^* \times O_{w^*}(0)$, since

$$\max_{\Omega^*} |A_\alpha(z)| < R^\alpha$$

for some $R >> 1$,we see that $J(z, w^*)$ is holomorphic on $\Omega^* \times O_{w^*}(0)$ (where we may have to shrink the domains). On the other hand, by making use of the formulas in (3.8) and the implicit function theorem, we have that

$$\xi = H_0^* + G_0(f, H_0^*, H_\alpha^*)$$

and

$$\eta_{\alpha^*} = H_\alpha^* + G_\alpha(f, H_0^*, H_\alpha^*).$$

Here G_0 and G_α ($|\alpha| = 1$)) are holomorphic in their variables and have no linear terms. Applying $\mathbf{L} = (L_1, \cdots, L_{m-1})$ to (3.9), we obtain

$$V = \mathbf{L}_z J + V \times (H_0^* + G_1, H_\alpha^* + G_\alpha) \times \frac{\partial J}{\partial w},$$

where $|\alpha| \leq 1$ and $\mathbf{L}_z J$ is the partial differential operator $\mathbf{L}$ applied to J while holding w^* fixed. So it follows easily that

$$V = (\mathbf{L}J)(\mathrm{id} - (H_0^* + G_1, H_\alpha^* + G_\alpha))^{-1} = G_1^*(z, f),$$

where $G_1^*(z, w)$ is real analytic on $O_z(0) \times O_w(0)$ for H_0^* and H_α^* are real analytic on U_k by Lemma 9. Combining this with the formulas for ξ and

η, we therefore conclude that $L_i f_j = G_{ij}^{**}(z, f)$ with G_{ij}^{**} real analytic in z and f (when $z \approx 0$).

Let $T = [L_1, \overline{L_1}]$. Then $Tf_j = \overline{L_1}(L_1 f_j) = \overline{L_{1z}}(G_{1j}^{**}) + \sum \frac{\partial G_{1j}^{**}}{\partial w_l} \overline{G_{1l}^{**}}$, which is also real analytic on z and f for $j = 1, \cdots, m+k$. Since $\{\mathbf{L}, \overline{\mathbf{L}}, T\}$ consists of a local analytic basis of TM_1, we can conclude that $f \in C^\infty(U_k)$. Now, from Lemma 10 (or [Fri]), we thus have the real analyticity of f on U_k.

Lemma 10. — *Let $f \in C^\infty(B_n, R^m)$ be such that $f(0) = 0$ and*

$$\frac{\partial f}{\partial x} = G(x, f).$$

Here B_n stands for the ball in $\mathbf{R^n}$ and $G(x, f)$ is real analytic in x and f. Then f is real analytic at 0.

Proof of Lemma 10. — Omitted.

3.3. This subsection is very similar to §2.3. We will directly show the analyticity of f in case $k = 1$ and obtain a reduction in case $k > 1$.

By the argument in the above subsection, to complete the proof of Theorem 1 it now suffices for us to assume that there is an $n \geq m$ so that $\mathrm{Ind}(j) = 0$ for $j \leq n - m$, but $\mathrm{Ind}(n - m + 1) = 1$. This similarly implies the following :

Lemma 11. — *There exist an open subset $U' \subset U$ and a holomorphic function Φ so that it holds that $\overline{f_n}(z) = \Phi(f(z), f^{(1)}(z), f^{(2)}(z), \hat{\overline{F}}_n(z))$ for $z \in U'$.*

Proof of Lemma 11. — We first assume that $n = m$. Then for some j_0, l_0,, $p_0^* \approx p_0$, and the e^{th} element $H_{j_0}^e$ of the vector function H_{j_0}, it holds that

$$L_{l_0} H_{j_0}^e(f, \xi, \eta, \hat{\overline{F}}_m)(p_0^*) \neq 0.$$

Then it is easy to see that for each j, $L_l H_j^e = \psi_j(f, f^{(1)}, f^{(2)}, \overline{f_{m+1}}, \cdots, \overline{f_{m+k-1}})$ for some ψ_j that is holomorphic in its variables and satisfies the corresponding Cauchy estimates.

Define

$$\begin{aligned} I_1(w, w^{(1)}, w^{(2)}, y_{m+1}, \cdots, y_{m+k-1}, u) \\ = \sum \psi_j(w, w^{(1)}, w^{(2)}, y_{m+1}, \cdots, y_{m+k-1})(u - u_0)^j \end{aligned}$$

with $u_0 = \overline{f_m}(p_0^*)$. Obviously, by (3.4), I_1 is holomorphic on

$$O_w(f(p_0^*)) \times O_{w^{(1)}}(f^(1)(p_0^*)) \cdots \times O_u(0).$$

We note that $(f(z), f^{(1)}(z), f^{(2)}(z), \overline{f_{m+1}(z)}, \cdots, \cdots, \overline{f_{m+k-1}}(z), \overline{f_m}(z) + u_0)$ satisfies the equation $I_1 = 0$ and $I_1(f(p_0^*), \cdots, \overline{f_{m+k-1}}(p_0^*), u) \not\equiv 0$ for $u \approx 0$. The proof in this case thus follows from Claim 2 of §2.

Now let $n > m$. We then have, for each l and α, that

$$L_l\phi_\alpha(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n-1}) = 0 \text{ on some } U_1 \subset U,$$

where α is a (n-1)-multi-index, ϕ_α is defined as in §3.2, and

$$\phi_\alpha = \sum_{j=0}^{\infty} \phi_{\alpha,j}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n})\overline{f_n^j}. \tag{3.10}$$

But for some $p_0^* \approx 0$, α_0, j_0, and l_0, it holds that

$$L_{l_0}\phi_{\alpha_0,j_0}(p_0^*) \neq 0.$$

Let $L_l\phi_{\alpha_0,j} = \psi_{\alpha_0,j}(f, f^{(1)}, f^{(2)}, \cdots, \overline{f_{m+k-1}})$ and define

$$\begin{aligned} &I_2(w, w^{(1)}, w^{(2)}, y_{m+1}, \cdots, y_{m+k-1}, u) \\ &\qquad = \sum \psi_{n,\alpha_0,j}(w, w^{(1)}, \cdots, y_{m+k-1})(u - u_0)^j \end{aligned}$$

with $u_0 = \overline{f_n}(p_0)$. Then it is easy to see that Claim 2 can be applied to the equation $I_2 = 0$ for solving $\overline{f}_n$. So the proof of Lemma 11 is complete. □

When the codimension $k = 1$, Lemma 11 tells that f_m admits a holomorphic extension on U'; for the formula of Φ involves no conjugate holomorphic terms (see the proof of Lemma 9 or [Pi] for details on this matter). Returning to (3.3) with $k = 1$, we see that

$$\overline{F_0}(z) = A(z, \xi(z), \eta(z), f(z)) \text{ for } z(\approx p_0) \in U',$$

where

$$A(z, \xi, \eta, f) = -\xi - \eta\overline{f_m(z)} - g(f, \xi, \eta, \overline{f_m(z)}).$$

We claim that this also implies the analyticity of $F_0(z) = (f_1(z), \cdots, f_m(z))$ near p_0. In fact, from the analyticity of $f_m(z)$, it follows that A is holomorphic in $(\overline{z}, \xi, \eta, f)$. So, let ϕ and V as constructed in Lemma 9, then $A(\phi(\tau), \xi(\phi(\tau)), \eta(\phi(\tau)), f(\phi(\tau)))$ admits a (uniform) holomorphic extension to Δ near $1 \in \partial\Delta$ (see the proof of Lemma 9 for more details concerning this matter). Notice that $\overline{F_0}(\phi(\tau))$ allows a uniform holomorphic extension to the outside of Δ near 1 and coincides with $A(\phi(\tau), \xi(\phi(\tau)), \eta(\phi(\tau)), f(\phi(\tau)))$ on part of the circle $\partial\Delta$ near 1. So, by using the Hartogs theorem, we can conclude the claim (see the proof of Lemma 9). By the same token, with these results at our disposal and returning to (3.1), we see the analyticity of f_{m+1}. For the general codimension, we have

LEMMA 12. — *Under the assumptions in Lemma 11, there exist a small open subset U'' of U' and a holomorphic function Ψ so that*

$$f_n(z) = \Psi(z, \hat{F}_n(z)) \text{ for } z \in U'', \tag{3.11}$$

where n is as in Lemma 11.

Proof of Lemma 12. — Make use of the assumption that f is of class C^{k+1} and copy the proof for Lemma 6 (Claim 3). □

3.4. Now we replace $\overline{f_n}$ in (3.4) by $\overline{\Psi(z, \hat{F}_n)}$. Then we have, for each $i < m - 1$, that

$$\overline{f}_i = g_i^*(\overline{z}, f, f^{(1)}, \hat{\overline{F}}_n) \text{ on } U^{(2)} \subset U'', \tag{3.12}$$

where g_i^* is holomorphic on $O_z(p'') \times O_w(f(p'')) \times \cdots \times O_{\hat{Y}_j}(\hat{F}_j(p''))$ with p'' being some point in U''. By a slight modification of Lemma 12, it follows that on some $U^{(3)} \subset U^{(2)}$ there exist holomorphic functions $\{\Psi_1, \cdots, \Psi_{m-1}\}$ so that it holds for each i that

$$f_i(z) = \Psi_i(z, \hat{F}_n(z)) \quad \text{for } z \in U^{(3)}. \tag{3.13}$$

Similarly, by substituting (3.12) and (3.13) to (3.1), we obtain

$$\overline{f}_{m+k} = g_{m+k}^*(\overline{z}, f, \hat{\overline{F}}_n) \quad \text{on } U^{(4)} \subset U^{(3)}, \tag{3.14}$$

with g_{m+k}^* holomorphic in $(f, z, \hat{\overline{F}}_n)$. Thus it can be seen, after shrinking $U^{(4)}$, that we have

$$f_{m+k} = \Psi_{m+k}(z, \hat{F}_n) \quad \text{on } U^{(4)}, \tag{3.15}$$

for some holomorphic function Ψ_{m+k}. Combining all these formulas, we similarly have

Lemma 13. — *There are a small neighborhood $\Omega^* \subset \mathbf{C}^{\mathbf{m}}$ of some $p \in U^{(4)}$ and a nonsingular complex variety $M^* \subset \mathbf{C}^{\mathbf{m+k}}$, which contains $f(p)$, so that $f(\Omega^*) \subset M^*$.*

Proof of Lemma 13. — Copy that for Lemma 7. □

Lemma 14. — *f admits a holomorphic extension on some point near p.*

Proof of Lemma 14. — First, summarizing the argument in §3.2 and the argument following Lemma 11, we note that Lemma 14 is true in case $k = 1$. For $k > 1$, we also see that either f has a holomorphic extension at some point on $U^{(4)}$ or f has no analytic extension at any point on $U^{(4)}$ but Lemma 13 holds. In the latter case, similar to Lemma 8, it implies that there is a complex manifold M^* of dimension $m + k - 1$ so that $f(\Omega) \subset M^*$ and $f(U^{(4)})$ is contained in some strongly pseudoconvex real analytic hypersurface of M^* (here we may have to shrink $U^{(4)}$). By making use a local coordinates chart of M^*, we then see that Theorem 1 is false in the case of codimension $k - 1$. Inductively, this would result in a contradiction with the situation of $k = 1$. □

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