Math. Ann.
DOI 10.1007/s00208-013-0952-z

Mathematische Annalen

On the third gap for proper holomorphic maps between balls

Xiaojun Huang · Shanyu Ji · Wanke Yin

Received: 16 June 2012 / Revised: 25 June 2013

Abstract Let F be a proper rational map from the complex ball $\mathbb{B}^n$ into $\mathbb{B}^N$ with $n > 7$ and $3n + 1 \leq N \leq 4n - 7$. Then F is equivalent to a map $(G, 0, \ldots, 0)$ where G is a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{3n}$.

1 Introduction

Write $\mathbb{B}^n$ for the unit ball in the complex space $\mathbb{C}^n$. Recall that a holomorphic map F from $\mathbb{B}^n$ into $\mathbb{B}^N$ is called proper if for any compact subset $K \subset \mathbb{B}^N$, $F^{-1}(K)$ is also a compact subset in $\mathbb{B}^n$. A holomorphic map defined over $\mathbb{B}^n$ is said to be rational if it can be written as $\frac{P}{q}$ with P a holomorphic polynomial map and q a

Xiaojun Huang is Supported in part by DMS-1101481.

Wanke Yin is Supported in part by FANEDD-201117, ANR-09-BLAN-0422, RFDP-20090141120010, NSFC-10901123 and NSFC-11271291.

X. Huang · W. Yin
School of Mathematics and Statistics, Wuhan University, Hubei 430072, China

X. Huang (✉)
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
e-mail: huangx@math.rutgers.edu

S. Ji
Department of Mathematics, University of Houston, Houston, TX 77204, USA
e-mail: shanyuji@math.uh.edu

W. Yin
Laboratoire de Mathématiques Raphaël Salem, Université de Rouen,
Saint-Etienne-du-Rouvray 76801, France
e-mail: wankeyin@whu.edu.cn

Published online: 13 July 2013

holomorphic polynomial function. This paper continues the recent work in Hamada [17], Huang–Ji–Xu [22], etc. Our main purpose is to prove the following gap rigidity theorem:

Theorem 1.1 *Let F be a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $n > 7$ and $3n+1 \le N \le 4n-7$. Then there is an automorphism $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F = (G, 0') = (G, 0, 0, \ldots, 0)$, where G is a proper holomorphic rational map from $\mathbb{B}^n$ into $\mathbb{B}^{3n}$.*

Theorem 1.1 roughly says that there is no new proper rational map added for N in the closed interval denoted by $\mathcal{I}_3 := [3n+1, 4n-7]$. The following example shows that Theorem 1.1 is sharp. (See Remark A in Sect. 5 for more discussions on this example.)

Example 1.2 For $n \ge 2,\ \lambda, \mu \in (0,1)$, define the proper monomial map F from $\mathbb{B}^n$ into $\mathbb{B}^{3n}$ as follows:

$$F = \left(z_1, \ldots, z_{n-2}, \lambda z_{n-1}, z_n, \sqrt{1-\lambda^2} z_{n-1}(z_1, \ldots, z_{n-1}, \mu z_n, \sqrt{1-\mu^2} z_n z)\right). \tag{1.1}$$

For such a map F, there is no $\tau \in Aut(\mathbb{B}^{3n})$ such that $\tau \circ F = (G, 0')$. Also, there are proper monomial maps F from $\mathbb{B}^n$ into $\mathbb{B}^{4n-6}$ [24] such that for any $\tau \in Aut(\mathbb{B}^{4n-6}),\ \tau \circ F$ can not be of the form $(G, 0')$.

The rationality theorem proved in [19,23] says that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \le n(n+1)/2$, that is three times differentiable up to the boundary, must be rational. Hence, Theorem 1.1 can be stated in the following more general form:

Theorem 1.3 *Let F be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $n > 7$ and $3n+1 \le N \le 4n-7$. Assume that F is C^3-smooth up to the boundary. Then there is an automorphism $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F = (G, 0')$, where G is a proper rational map from $\mathbb{B}^n$ into $\mathbb{B}^{3n}$.*

Rigidity property is a fundamental property for holomorphic functions with several variables. The study of various rigidity properties for proper holomorphic maps between balls in complex Euclidean spaces goes back to the pioneer paper of Poincaré [31]. Since then, much attention has been paid to such an investigation. When $n > 1$, a result of Alexander [1] states that any proper holomorphic self-map of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ with $n > 1$ is an automorphism. Recall that two proper holomorphic maps f, g from $\mathbb{B}^n$ into $\mathbb{B}^N$ are said to be equivalent if there are $\sigma \in \mathrm{Aut}(\mathbb{B}^n)$ and $\tau \in \mathrm{Aut}(\mathbb{B}^N)$ such that $g = \tau \circ f \circ \sigma$. A proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is said to be linear or totally geodesic if it is equivalent to the standard big circle embedding $L(z) : z \to (z, 0)$. Webster in [35] considered the geometric structure of proper holomorphic maps between balls in complex spaces of different dimensions. He showed that a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{n+1}$ with $n > 2$, which is three times differentiable up to the boundary, is a totally geodesic embedding. Subsequently, Cima–Suffridge [6] reduced the boundary regularity in Webster's theorem to the C^2-regularity. Motivated by a conjecture in [6], Faran in [12] showed that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N < 2n-1$, that is real analytic up to the boundary, is a totally geodesic embedding. Forstneric in [14] proved that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is rational, if the map is C^{N-n+1}-regular up to the boundary, which, in particular, reduces the regularity assumption in Faran's linearity theorem to the C^{N-n+1}-smoothness. In a paper of Mir [29], the theorem of Forstneric was weakened to the case where the source manifold needs only to be assumed to be a real analytic hyper-surface. See also a related paper by Baouendi–Huang–Rothschild [3] and a later generalization in Meylan–Mir–Zaitsve [28]. At this point, we mention that the discovery of inner functions can be used to show that there is a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{n+1}$, which can not be C^2-smooth at any boundary point (see [15,16,27,34], etc).

Write $\mathcal{I}_1 = [n+1, 2n-2]$. The aforementioned theorem of Faran says that there is no new proper rational map added when the target dimension $N \in \mathcal{I}_1$. We call $\mathcal{I}_1$ the first gap interval for proper holomorphic mappings between balls. In [11], Faran showed that there are four different inequivalent proper holomorphic maps from $\mathbb{B}^2$ into $\mathbb{B}^3$, which are C^3-smooth up to the boundary. However, the only embeddings are linear maps.

In [18] and, subsequently, [21], two questions arising from the above mentioned work were considered. In [18], the first author proved that any proper holomorphic map which is only C^2-regular up to the boundary must be linear if $N < 2n-1$, by applying a very different method from the previous work, answering a long standing open question in the field (see [6,15]). While it has been open for many years to answer if the C^1-boundary regularity is still enough for this super-rigidity to hold, the result in [18] gives a first result in which the required regularity is independent of the codimension. In [21, Theorem 1, Theorem 2.3] and [20, Corollary 2.1], it was shown that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = 2n-1$, $n \ge 3$, which is C^2-smooth up to the boundary, is either linear or equivalent to the Whitney map

$$W : z = (z_1, \ldots, z_n) = (z', z_n) \to (z_1, \ldots, z_{n-1}, z_n z) = (z', z_n z). \tag{1.2}$$

Since the Whitney map is not an immersion, together with the aforementioned work of Faran [11], this shows that any proper holomorphic *embedding* from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = 2n-1$, which is twice continuously differentiable up to the boundary, must be a linear map. Earlier, D'Angelo constructed the following family F_θ of mutually inequivalent proper quadratic monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$ (See [8]):

$$\begin{aligned} F_\theta(z', z_n) = (&z', (\cos\theta) z_n, (\sin\theta) z_1 z_n, \ldots, (\sin\theta) z_{n-1} z_n, \\ &(\sin\theta) z_n^2), \quad 0 < \theta \le \pi/2. \end{aligned} \tag{1.3}$$

Notice that by adding $N - 2n$ zero components to the D'Angelo map F_θ, we get a proper monomial embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ for any $N \ge 2n$. The combining effort in [12,21] gives a complete description to the linearity problem for proper holomorphic embeddings from $\mathbb{B}^n$ into $\mathbb{B}^N$, which are C^2-smooth up to the boundary. However, in applications, one still hopes to get the linearity for mappings with a rich geometric structure. For instance, the following difficult problem initiated from the work of Siu, Mok [30] and others has been open for more than thirty years: (See Cao–Mok [4] for the work when $N \le 2n-1$.)

Conjecture 1.4 (Siu, Mok) *Let f be a proper holomorphic mapping from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $1 < n < N$. Write $M = F(\mathbb{B}^n)$. Suppose that there is a subgroup Γ of Aut$(\mathbb{B}^N)$ such that (1). for any $\sigma \in \Gamma$, $\sigma(M) = M$; (2) M/Γ is compact. Then f is a linear embedding.*

In a recent paper of Hamada [17], based on a careful analysis on the Chern–Moser normal form method as developed in [18,21], it was proved that all proper rational maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$ with $n \ge 4$ are either equivalent to the Whitney map W in (1.2) or the D'Angelo map F_θ. After the work of Hamada [17], the first two authors and Xu in [22] proved that a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $4 \le n \le N \le 3n-4$, that is C^3-smooth up to the boundary, is equivalent to either the map $(W, 0')$ or $(F_\theta, 0')$ with $\theta \in [0, \pi/2)$. An immediate consequence of the work in [22] is that there is no new map added when $N \in \mathcal{I}_2$ with $\mathcal{I}_2 := [2n+1, 3n-4]$. Since there are proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ for $3n-3 \le N \le 3n$ or $2n-1 \le N \le 2n$, that are not equivalent to maps of the form $(G, 0')$, we call $\mathcal{I}_2$ the second gap interval for proper holomorphic maps between balls.

By [24], for any N with $3n-3 \le N \le 3n$ or $4n-6 \le N \le 4n$, there are many proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$, that are not equivalent to maps of the form $(G, 0')$. Theorem 1.1 in the present paper thus provides a third gap interval $\mathcal{I}_3 := [3n+1, 4n-7]$ for proper holomorphic maps between balls.

More generally, for any $n \ge 3$, write $K(n)$ for the largest positive integer m such that $m(m+1)/2 < n$. Then $K(n) = [\frac{-1+\sqrt{1+8n}}{2}]$ if $\frac{-1+\sqrt{1+8n}}{2}$ is not an integer; and $K(n) = \frac{-1+\sqrt{1+8n}}{2} - 1$, otherwise. For each $1 \le k \le K(n)$, define $\mathcal{I}_k := [kn+1, (k+1)n - \frac{k(k+1)}{2} - 1]$. Then $\mathcal{I}_k$ is a closed interval containing positive integers if $n \ge 2 + \frac{k(k+1)}{2}$. Apparently, $\mathcal{I}_k \cap \mathcal{I}_{k'} = \emptyset$ for $k \ne k'$; and $\mathcal{I}_k$ for $k = 1, 2, 3$ are exactly the same intervals defined above. Write $\mathcal{I} = \cup_{k=1}^{K(n)} \mathcal{I}_k$. Then, for

$$\max_{N \in \mathcal{I}} N = (K(n)+1)n - \frac{K(n)(K(n)+1)}{2}$$
$$-1 \approx \frac{-1+\sqrt{1+8n}}{2} n - n - 1 \approx \sqrt{2} n^{\frac{3}{2}} - n - 1.$$

For any $N \notin \mathcal{I}$ (which certainly is the case when $N \ge 1.42 n^{\frac{3}{2}}$), by not a complicated construction, the authors obtained in [24] many monomial proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$, that can not be equivalent to maps of the form $(G, 0')$. ([24, See Theorem 2.8]). Earlier in [9], for $N \ge n^2 - 2n + 2$, D'Angelo and Lebl, by a different method, constructed a proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$, that is not equivalent to a map of the form $(G, 0')$. However, we have not been able to find a map, not equivalent to a map of the form $(G, 0')$, for $N \in \mathcal{I}$. Indeed, the first, the second and the third gap intervals mentioned above suggest the following conjecture:

Conjecture 1.5 (Huang–Ji–Yin [24]) *Let $n \ge 3$ be a positive integer, and let $\mathcal{I}_k$ $(1 \le k \le K(n))$ be defined above. Then any proper holomorphic rational map F*

from $\mathbb{B}^n$ *into* $\mathbb{B}^N$ *is equivalent to a map of the form* $(G, 0')$ *if and only if* $N \in \mathcal{I}_k$ *for some* $1 \le k \le K(n)$.

As mentioned above, the "$\Longrightarrow$" part follows from Theorem 2.8 of [24]; also the conjecture holds for $k = 1, 2, 3$. An affirmative solution to this gap conjecture would tells exactly for what pair (n, N) there are no new proper rational maps added.

Next, we describe briefly the idea for the proof of Theorem 1.1. The proofs for the first and the second gaps are immediate applications of the much more precise classification results. When $N \in \mathcal{I}_3$, making a precise classification for all maps seems to be hard. We need a different approach from the work in Huang–Ji [21], Hamada [17] and Huang–Ji–Xu [22]. Consider the setting in the Heisenberg hypersurface case. Let F be a holomorphic map defined near 0 with $F(0) = 0$ into $\mathbb{C}^N$. Then the Taylor formula says that $F(z) = \sum_\alpha \frac{D^\alpha F}{\alpha!}(0)z^\alpha$. Hence the image of F stays in the linear subspace spanned by $\{D^\alpha F(0)\}_\alpha$. If $spann\{D^\alpha F(0)\}_\alpha \neq \mathbb{C}^N$, we get a gap from F. The crucial point in our argument is to find, for our map, a basis of $spann\{D^\alpha F(0)\}_\alpha$. The way to achieve is to get a good normal form for F. However, this is a highly non-linear normalization problem, for the maps need to satisfy the fundamental non-linear equation. While it is easy to get linear independent set from the first and the second jets, finding more linearly independent elements to form a basis from the higher order jets is very involved. The basic tool at our disposal for this approach is a lemma of the first author proved in [18, Lemma 3.2]. For $N \in \mathcal{I}_3$, it turns out that there is only one more linearly independent element for the map from the higher order jets. For the study of general but very rough jet determination problems for holomorphic maps, there has been much work done in the past. We refer the reader to the book by Baouendi–Ebenfelt–Rothschild [2] and a paper by Lamel–Mir [25]. However, what we need here is a very precise jet determination, which is only doable due to the extra geometric structure for the maps in our setting.

It appears to us that a fundamental fact which dominates the gap rigidity for holomorphic maps between balls is [18, Lemma 3.2]. In the course of the proof our main theorem, one finds that the assumption $N \in \mathcal{I}_3$ is exactly what is needed, in several induction steps, for applying [18, Lemma 3.2]. We hope that the method of the present paper may motivate the general study of Conjecture 1.5.

Our discussion above only touches the linearity and the gap rigidity part from a vast amount of work for mappings between balls. We would like to mention that there has been a lot of interesting work done in the past on the study of proper monomial maps between balls by D'Angelo and his coauthors. (See the book of D'Angelo [8] for many references therein.) Here, we mention, in particular, two papers on the degree estimates for proper monomial maps by D'Angelo–Kos–Riehl [10] and Lebl–Peters [26]. The study for mappings between balls is also related to the problem of decomposing a positive Hermitian form into the sum square of holomorphic functions, for which we refer the reader to a recent survey article by Putinar [32] as well as many references therein. Here, we just mention a result obtained by Quillen–Catlin–D'Angelo in [5, 33], which states that for any positive bi-homogenous polynomial $H(z, z)$, there is a sufficiently large integer N such that $|z|^{2N} H(z, z) = \sum_{j=1}^{N'} |h_j(z)|^2$ with $h_j(z)$ holomorphic polynomials. This has an immediate consequence (see [5]) that for any homogenous polynomial map $q(z)$ into $\mathbb{C}^N$ with $|q(z)| < 1$ on the sphere , there exists a vector valued polynomial $p(z)$ with $N(q)$-components such that $(q(z), p(z))$ properly holomorphically maps $\mathbb{B}^n$ into $\mathbb{B}^{N+N(q)}$, where $N(q)$ depends on q and the value $1 - |q(z)|^2$ and could be very large.

2 Notations and preliminaries

In this section, we set up notation and recall a result established in Huang–Ji–Xu [22] and a lemma from [18] which will be crucial for our proof of Theorem 1.1.

Write $\mathbb{H}_n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Similarly, we can define the notion of proper rational maps from $\mathbb{H}_n$ into $\mathbb{H}_N$.

Since the Cayley transformation

$$\rho_n : \mathbb{H}_n \to \mathbb{B}^n, \quad \rho_n(z, w) = \left(\frac{2z}{1 - iw}, \frac{1 + iw}{1 - iw} \right) \tag{2.1}$$

is a biholomorphic mapping between $\mathbb{H}_n$ and $\mathbb{B}^n$, we can identify a proper rational map F from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $\rho_N^{-1} \circ F \circ \rho_n$, which is a proper rational map from $\mathbb{H}_n$ into $\mathbb{H}_N$. By a well-known result of Cima–Suffridge [7], F extends holomorphically across the boundary $\partial\mathbb{B}^n$.

Parameterize $\partial\mathbb{H}_n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial\mathbb{H}_n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2u)}{|t|^m} \to 0$ uniformly for (z, u) on any compact subset of U as $t (\in \mathbb{R}) \to 0$. We use the notation $h^{(k)}$ to denote a polynomial h which has weighted degree k. Occasionally, for a holomorphic function (or map) $H(z, w)$, we write $H(z, w) = \sum_{k,l=0}^{\infty} H^{(k,l)}(z) w^l$ with $H^{(k,l)}(z)$ a polynomial of degree k in z.

Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g)$ be a non-constant C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ with $F(0) = 0$. For each $p = (z_0, w_0) \in M$ close to 0, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}_N)$ by defining

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle).$$

Then F is equivalent to

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p). \tag{2.2}$$

Notice that $F_0 = F$ and $F_p(0) = 0$. The following is fundamentally important for the understanding of the geometric properties of F.

Lemma 2.1 ([18, §2, Lemma 5.3]) *Let F be a C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$, $2 \le n \le N$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}_N)$*

such that $F_p^{**} := \tau_p^{**} \circ F_p$ *satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4),\ \textit{with}$$
$$\langle \overline{z}, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Definition 2.2 [19] Write $\mathcal{A}(p) = -2i \left(\frac{\partial^2 (f_p^{**})_l}{\partial z_j \partial w} |_0 \right)_{1 \le j, l \le (n-1)}$ in the above lemma. We call the rank of the $(n-1) \times (n-1)$ matrix $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the *geometric rank* of F at p.

Define the *geometric rank* of F to be $\kappa_0(F) = max_{p \in \partial \mathbb{H}_n} Rk_F(p)$. Define the geometric rank of a proper holomorphic map $\mathbb{B}^n$ into $\mathbb{B}^N$, that is C^2-smooth up to the boundary, to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n$. By [19], $\kappa_0(F)$ depends only on the equivalence class of F and when $N < \frac{n(n+1)}{2}$, $\kappa_0(F) \le n-2$. In [22], the authors proved the following normalization theorem for maps with geometric rank bounded by $n-2$, though only part of it is needed later:

Theorem 2.3 [22] *Suppose that* F *is a rational proper holomorphic map from* $\mathbb{H}_n$ *into* $\mathbb{H}_N$*, which has geometric rank* $1 \le \kappa_0 \le n-2$ *with* $F(0) = 0$*. Then there are* $\sigma \in Aut(\mathbb{H}_n)$ *and* $\tau \in Aut(\mathbb{H}_N)$ *such that* $\tau \circ F \circ \sigma$ *takes the following form, which is still denoted by* $F = (f, \phi, g)$ *for convenience of notation:*

$$\begin{cases} f_l = \sum_{j=1}^{\kappa_0} z_j f_{lj}^*(z,w), \quad l \le \kappa_0, \\ f_j = z_j, \quad \kappa_0 + 1 \le j \le n-1, \\ \phi_{lk} = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^*, \quad (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^* = O_{wt}(3), \quad (l,k) \in \mathcal{S}_1, \\ g = w, \\ f_{lj}^*(z,w) = \delta_l^j + \frac{i \delta_l^j \mu_l}{2} w + b_{lj}^{(1)}(z) w + O_{wt}(4), \quad 1 \le l \le \kappa_0,\ \mu_l > 0, \\ \phi_{lkj}^*(z,w) = O_{wt}(2), \quad (l,k) \in \mathcal{S}_1. \end{cases} \tag{2.3}$$

Here, for $1 \le \kappa_0 \le n-2$*, we write* $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$*, the index set for all components of* ϕ*, where* $\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le n-1, j \le l\}$ *and* $\mathcal{S}_1 = \{(j,l) : j = \kappa_0 + 1, \kappa_0 + 1 \le l \le N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\}$*. Also,* $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ *for* $j < l \le \kappa_0$*; and* $\mu_{jl} = \sqrt{\mu_j}$ *if* $j \le \kappa_0 < l$ *or if* $j = l \le \kappa_0$*.*

Finally, we recall the following lemma of the first author in [18], which will play a fundamental role in our proof:

Lemma 2.4 (Huang, Lemma 3.2 [18]) *Let* k *be a positive integer such that* $1 \le k \le n-2$*. Assume that* $a_1, \ldots, a_k,\ b_1, \ldots, b_k$ *are germs at* $0 \in \mathbb{C}^{n-1}$ *of holomorphic functions such that* $a_j(0) = 0,\ b_j(0) = 0$ *and*

$$\sum_{i=1}^{k} a_i(z) \overline{b_i(z)} = A(z, \overline{z}) |z|^2, \tag{2.4}$$

where $A(z,\bar z)$ *is a germ at* $0 \in \mathbb{C}^{n-1}$ *of a real analytic function. Then* $A(z,\bar z) = \sum_{i=1}^{k} a_i(z)\overline{b_i(z)} \equiv 0$.

3 Analysis on the Chern–Moser equation

Suppose now that $F = (f, \phi, g)$ is a proper rational map from $\mathbb{H}_n$ into $\mathbb{H}_N$, and satisfies the normalization as in Theorem 2.3 with $1 \le \kappa_0 \le n-2$. Write the codimension part ϕ of the map F as $\phi := (\Phi_0, \Phi_1)$ with $\Phi_0 = (\phi_{\ell k})_{(\ell,k)\in\mathcal{S}_0}$ and $\Phi_1 = (\phi_{\ell k})_{(\ell,k)\in\mathcal{S}_1}$.

Write

$$\Phi_0^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_j z_j, \quad \Phi_1^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} \hat e_j z_j,$$

with $e_j \in \mathbb{C}^{\#(\mathcal{S}_0)} = \mathbb{C}^{\kappa_0 n - \frac{\kappa_0(\kappa_0+1)}{2}}$, $\hat e_j \in \mathbb{C}^{\#(\mathcal{S}_1)}$, $\xi_j(z) = \overline{e_j}\cdot \Phi_0^{(2,0)}(z)$, and $\xi = (\xi_1,\ldots,\xi_{\kappa_0})$. We also write in the following:

$$\begin{aligned}\phi^{(1,1)}(z)w &= \sum e_j^* z_j w, \text{ with } e_j^* = (e_j, \hat e_j),\\ H &= \sum_{(i_1,\ldots,i_{n-1},i_n)} H^{(i_1,\ldots,i_n)} z_1^{i_1}\cdots z_{n-1}^{i_{n-1}} w^{i_n}\\ &= \sum_{k,j=0}^{\infty} H^{(k,j)}(z) w^j \text{ for } H = f \text{ or } \phi.\end{aligned}$$

Here $H^{(k,j)}(z)$ is a homogeneous polynomial of degree k in z.

In this section, we demonstrate our basic idea of the proof through an easier case. We proceed with the following lemma, that will be used later:

Lemma 3.1 *Let* $(\Gamma_j^{[h]}(z))_{1\le j\le\kappa_0, h=1,2}$ *be some holomorphic functions of* z. *Let* μ_{jl} *and* μ_j *be as in Theorem 2.3. Suppose that for* $h = 1, 2$, $(\Lambda_{j\ell}^{[h]})_{(j,\ell)\in\mathcal{S}_0}$ *are defined as follows:*

1. $\mu_{j\ell}\Lambda_{j\ell}^{[h]}(z) = 2i(z_j\Gamma_\ell^{[h]} + z_\ell\Gamma_j^{[h]}), \quad j < \ell \le \kappa_0,$
2. $\mu_{jj}\Lambda_{jj}^{[h]}(z) = 2iz_j\Gamma_j^{[h]}(z), \quad j \le \kappa_0,$
3. $\mu_{j\ell}\Lambda_{j\ell}^{[h]} = 2iz_\ell\Gamma_j^{[h]}(z), \quad j \le \kappa_0 < \ell.$

Then we have

$$\begin{aligned}\sum_{(j,\ell)\in\mathcal{S}_0} \Lambda_{j\ell}^{[1]}\Lambda_{j\ell}^{[2]} = 4|z|^2\Big(\sum_{j\le\kappa_0}\frac{1}{\mu_j}\Gamma_j^{[1]}\Gamma_j^{[2]}\Big)\\ - \sum_{j<\ell\le\kappa_0}\frac{4}{\mu_j\mu_\ell(\mu_j+\mu_\ell)}\big(\mu_j z_j\Gamma_\ell^{[1]} - \mu_\ell z_\ell\Gamma_j^{[1]}\big)\\ \cdot\big(\mu_j z_j\Gamma_\ell^{[2]} - \mu_\ell z_\ell\Gamma_j^{[2]}\big). \qquad (3.1)\end{aligned}$$

Proof Making use of the formulas between $\mu_{j\ell}$ and μ_j, μ_ℓ in Theorem 2.3, we get, from a straightforward computation, the following:

$$\begin{aligned}\frac{1}{4}\sum_{(j,\ell)\in\mathcal{S}_0}\Lambda^{[1]}_{j\ell}\Lambda^{[2]}_{j\ell} &= \sum_{1\le j\le\kappa_0}\frac{|z_j|^2}{\mu_j}\Gamma^{[1]}_j\Gamma^{[2]}_j + \sum_{j\le\kappa_0<\ell}\frac{|z_\ell|^2}{\mu_j}\Gamma^{[1]}_j\Gamma^{[2]}_j\\ &\quad+\sum_{j<\ell\le\kappa_0}\frac{1}{\mu_j+\mu_\ell}(z_j\Gamma^{[1]}_\ell+z_\ell\Gamma^{[1]}_j)\cdot(z_j\Gamma^{[2]}_\ell+z_\ell\Gamma^{[2]}_j)\\ &=\Big(\sum_{j\le\kappa_0}\frac{1}{\mu_j}\Gamma^{[1]}_j\Gamma^{[2]}_j\Big)|z|^2-\sum_{\ell\le\kappa_0,\ell\ne j\le\kappa_0}\frac{1}{\mu_j}|z_\ell|^2\Gamma^{[1]}_j\Gamma^{[2]}_j\\ &\quad+\sum_{j<\ell\le\kappa_0}\frac{1}{\mu_j+\mu_\ell}(z_j\Gamma^{[1]}_\ell+z_\ell\Gamma^{[1]}_j)\cdot(z_j\Gamma^{[2]}_\ell+z_\ell\Gamma^{[2]}_j)).\end{aligned}$$

Now the lemma follows from the following elementary identity:

$$\begin{aligned}&\frac{\mu_j}{\mu_\ell}|z_j|^2\Gamma^{[1]}_\ell\Gamma^{[2]}_\ell+\frac{\mu_\ell}{\mu_j}|z_\ell|^2\Gamma^{[1]}_j\Gamma^{[2]}_j - z_j\Gamma^{[2]}_\ell\overline{\Gamma^{[1]}_j}\overline{z_\ell}-\overline{z_j}\overline{\Gamma^{[1]}_\ell}\Gamma^{[2]}_j z_\ell\\ &=\frac{1}{\mu_j\mu_\ell}\big(\mu_j z_j\Gamma^{[1]}_\ell-\mu_\ell z_\ell\Gamma^{[1]}_j\big)\cdot\big(\mu_j z_j\Gamma^{[2]}_\ell-\mu_\ell z_\ell\Gamma^{[2]}_j\big).\end{aligned}$$

□

Next we derive the following formula:

Lemma 3.2

$$\begin{aligned}\frac{1}{4}|\Phi^{(3,0)}_0(z)|^2 &= \Big(\sum_{j\le\kappa_0}\frac{1}{\mu_j}|\xi_j(z)|^2\Big)|z|^2\\ &\quad-\sum_{j<\ell\le\kappa_0}\frac{1}{\mu_j+\mu_\ell}\Big|\sqrt{\frac{\mu_j}{\mu_\ell}}z_j\xi_\ell-\sqrt{\frac{\mu_\ell}{\mu_j}}z_\ell\xi_j\Big|^2. \end{aligned}\tag{3.2}$$

Proof Since

$$-\operatorname{Im}\big(g(z,w)\big)+\big|f(z,w)\big|^2+\big|\phi(z,w)\big|^2=0 \quad\text{over}\quad \operatorname{Im}(w)=|z|^2, \tag{3.3}$$

we can consider terms of weighted degree 5 to get, over $\operatorname{Im}(w)=|z|^2$, the following

$$\begin{aligned}&z\overline{f^{(4)}(z,w)}+\overline{z}f^{(4)}(z,w)+\Phi^{(2)}_0(z,w)\overline{\Phi^{(3)}_0(z,w)}+\Phi^{(3)}_0(z,w)\overline{\Phi^{(2)}_0(z,w)}=0,\ \text{or}\\ &z\overline{f^{(2,1)}(z)(u+i|z|^2)}+\overline{z}f^{(2,1)}(z)(u+i|z|^2)+\Phi^{(2)}_0(z)\overline{\Big(\Phi^{(3,0)}_0(z)+(\sum e_jz_j)w\Big)}\\ &+\Big(\Phi^{(3,0)}_0(z)+\Big(\sum e_jz_j\Big)w\Big)\overline{\Phi^{(2)}_0(z)}\equiv 0.\end{aligned}\tag{3.4}$$

Here, we know $f^{(4)}(z,w) = f^{(2,1)}(z)w$ by the above mentioned normalization. Collecting terms of the form $\overline{z}^{\alpha} z^{\beta} u$ with $|\alpha| = 1,\ |\beta| = 2$, we get

$$\overline{z} f^{(2,1)}(z) + \Phi_0^{(2)}(z)\overline{\sum e_j z_j} = 0, \quad or,$$
$$\overline{z} f^{(2,1)}(z) = -\overline{(z_1, \ldots, z_{\kappa_0})} \cdot \xi(z). \tag{3.5}$$

Collecting terms of the form $z^{\alpha}\overline{z}^{\beta}$ with $|\alpha| = 3$ and $|\beta| = 2$, we get

$$i\overline{z} f^{(2,1)}(z)|z|^2 + \overline{\Phi_0^{(2)}(z)}\Phi_0^{(3,0)}(z) + \Phi_0^{(2)}(z)\overline{\sum_{j=1}^{\kappa_0} e_j z_j (i|z|^2)} \equiv 0. \tag{3.6}$$

We thus get

$$\overline{\Phi_0^{(2)}(z)}\Phi_0^{(3,0)}(z) = 2i\overline{(z_1, \ldots, z_{\kappa_0})} \cdot \xi(z)|z|^2. \tag{3.7}$$

Equivalently, we have

1. $\mu_{j\ell}\phi_{j\ell}^{(3,0)}(z) = 2i(z_j\xi_\ell(z) + z_\ell\xi_j(z)), \quad j < \ell \le \kappa_0,$
2. $\mu_{jj}\phi_{jj}^{(3,0)}(z) = 2iz_j\xi_j(z), \quad j \le \kappa_0,$
3. $\mu_{j\ell}\phi_{j\ell}^{(3,0)}(z) = 2iz_\ell\xi_j(z), \quad j \le \kappa_0 < \ell.$

Now Lemma 3.2 follows from Lemma 3.1. □

Lemma 3.3 $|\phi^{(3,0)}(z)|^2 = A(z,\overline{z})|z|^2$ *with* $A(z,\overline{z})$ *a real analytic polynomial in* $(z,\overline{z})$.

Proof Collecting terms of weighted degree 6 in (3.3), we get

$$z\overline{f^{(5)}(z,w)} + \overline{z} f^{(5)}(z,w) + \Phi_0^{(2)}(z,w) \cdot \overline{\Phi_0^{(4)}(z,w)} + \overline{\Phi_0^{(2)}(z,w)} \cdot \Phi_0^{(4)}(z,w)$$
$$+|\phi^{(3)}(z,w)|^2 + |f^{(3)}(z,w)|^2 = 0 \ \text{ over } \mathrm{Im}(w) = |z|^2. \tag{3.8}$$

Collecting terms of the form $z^{\alpha}\overline{z}^{\beta}$ with $|\alpha| = |\beta| = 3$ and applying the normalization for F, we easily see the proof (cf., (4.14) below). □

Notice that $|\phi^{(3,0)}(z)|^2 = |\Phi_0^{(3,0)}(z)|^2 + |\Phi_1^{(3,0)}(z)|^2$ and there are $\frac{\kappa_0(\kappa_0+1)}{2} - \kappa_0$ negative terms in the right hand side of (3.2). Also there are $(N - (\kappa_0+1)n + \frac{\kappa_0(\kappa_0+1)}{2})$ components in Φ_1. Applying Lemma 2.4 and [8, Proposition 3 on page 102 of], we immediately get, after applying a unitary transformation to the Φ_1-components, the following:

Corollary 3.4 *Suppose that* $\kappa_0 \ge 2$ *and* $(\kappa_0 + 1)n - \kappa_0 \le N \le (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2$. *Then*

$$\Phi_1^{(3,0)}(z) = \Big(\frac{2}{\sqrt{\mu_j+\mu_l}}\left(\sqrt{\frac{\mu_j}{\mu_\ell}}z_j\xi_\ell - \sqrt{\frac{\mu_\ell}{\mu_j}}z_\ell\xi_j\right), 0'\Big)_{1\le j<l\le\kappa_0},$$

$$|\phi^{(3,0)}(z)|^2 = 4\left(\sum_{j\le\kappa_0}\frac{1}{\mu_j}|\xi_j(z)|^2\right)|z|^2. \tag{3.9}$$

Here we notice that the condition $N \ge (\kappa_0+1)n-\kappa_0$ implies that $N > \#(\mathcal{S}_0)+n$ for $\kappa_0 > 1$.

4 Partial linearity and further applications of the Chern–Moser equation

We assume in this section that the proper rational map $F=(f,\phi,g)$ from $\mathbb{H}_n$ into $\mathbb{H}_N$ satisfies the normalization as in Theorem 2.3 with $\kappa_0=2$. Moreover, by what is done in the last section, we assume that $\Phi_1^{(3,0)}(z)$ has been normalized to take the form as in Corollary 3.4. Namely, the only possible non-zero element in $\Phi_1^{(3,0)}(z)$ is $\phi_{33}^{(3,0)}(z)$.

In this section, we prove the following result, which will be crucial for our proof of Theorem 1.1:

Theorem 4.1 *Assume that F is as in Theorem 2.3 with $\kappa_0=2$, $n\ge 7$ and $3n-2\le N\le 4n-6$. Also, assume that $\Phi_1^{(3,0)}(z)$ is normalized as in Corollary 3.4. Then the following holds:*

(1): $\Phi_1^{(4,0)}(z) = (\phi_{33}^{(4,0)}(z), 0, \ldots, 0)$, *where*

$$\phi_{33}^{(4,0)}(z) = \frac{2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^*\right),$$
$$\eta_1^* = \phi^{(3,0)}(z)\cdot\overline{e_1^*},\quad \eta_2^* = \phi^{(3,0)}(z)\cdot\overline{e_2^*}.$$

(2): $D_z^\alpha\Phi_1^{(2,1)}(z) \in span\{(1,0,\ldots,0),\hat{e}_1,\hat{e}_2\}$ *for* $|\alpha|=2$.
(3): $D_z^\alpha\Phi_1^{(1,2)}(z) \in span\{\hat{e}_1,\hat{e}_2\}$ *for* $|\alpha|=1$.

Here $\hat{e}_1, \hat{e}_2, e_1^*, e_2^*$, are defined as at the beginning of the last section, and D is the regular differential operator.

This section is devoted to the proof of Theorem 4.1.

Notice that $g=w$. By the partial linearity theorem of the first author proved in [19], we can assume that for any $\epsilon=(\epsilon_1,\epsilon_2)(\in\mathbb{C}^2)\approx 0$, there is a unique affine subspace L_ϵ of codimension two defined by equations of the form:

$$z_1 = \sum_{i=3}^{n-1}a_i(\epsilon)z_i + a_n(\epsilon)w + \epsilon_1,$$
$$z_2 = \sum_{i=3}^{n-1}b_i(\epsilon)z_i + b_n(\epsilon)w + \epsilon_2,\ a_i(0)=b_i(0)=0 \tag{4.1}$$

such that F is a linear map on L_ϵ. Here a_j, b_j are holomorphic functions in ϵ near 0. Hence we have

$$\frac{\partial^2 H}{\partial w^2}\Big|_{L_\epsilon} = 0 \text{ for } H = f \text{ or } \phi.$$

Namely, for $H(L_\epsilon) = H\big(\sum_{i=3}^{n-1} a_i(\epsilon)z_i + a_n(\epsilon)w + \epsilon_1,\ \sum_{i=3}^{n-1} b_i(\epsilon)z_i + b_n(\epsilon)w + \epsilon_2, z_3, \ldots, z_{n-1}, w\big)$, we have

$$\begin{aligned} 0 = \frac{\partial^2 H(L_\epsilon)}{\partial w^2}\Big|_{(\epsilon_1,\epsilon_2)} &= \Big(\frac{\partial^2 H}{\partial z_1^2}a_n^2 + \frac{\partial^2 H}{\partial z_2^2}b_n^2 + 2\frac{\partial^2 H}{\partial z_1 \partial z_2}a_n b_n + 2\frac{\partial^2 H}{\partial z_1 \partial w}a_n \\ &\quad + 2\frac{\partial^2 H}{\partial z_2 \partial w}b_n + \frac{\partial^2 H}{\partial w^2}\Big)\Big|_{(\epsilon_1,\epsilon_2,0,\ldots,0)}. \end{aligned} \tag{4.2}$$

Let $a_n^{(1)}(\epsilon)$ and $b_n^{(1)}(\epsilon)$ be the linear parts in a_n and b_n, respectively. Set $H = f_1, f_2$ and ϕ in (4.2), respectively. We then get

$$\begin{aligned} \frac{i}{2}\mu_1 a_n^{(1)}(\epsilon) + f_1^{(1,2)}(\epsilon, 0, \ldots, 0) = 0, \ \frac{i}{2}\mu_2 b_n^{(1)}(\epsilon) + f_2^{(1,2)}(\epsilon, 0, \ldots, 0) &= 0, \\ \phi^{(1,2)}(\epsilon, 0, \ldots, 0) + e_1^* a_n^{(1)}(\epsilon) + e_2^* b_n^{(1)}(\epsilon) &= 0. \end{aligned} \tag{4.3}$$

Notice that by Theorem 2.3, $F^{(1,m)}(z)$ depends only on (z_1, z_2) for any m. It then follows:

$$\begin{aligned} \phi^{(1,2)}(\epsilon, 0, \ldots, 0) &= -e_1^* a_n^{(1)}(\epsilon) - e_2^* b_n^{(1)}(\epsilon) \\ &= -\frac{2i}{\mu_1} f_1^{(1,2)}(\epsilon, 0, \ldots, 0)e_1^* - \frac{2i}{\mu_2} f_2^{(1,2)}(\epsilon, 0, \ldots, 0)e_2^*. \end{aligned} \tag{4.4}$$

This proves Theorem 4.1 (3). Moreover, we obtain

$$\begin{aligned} \Phi_0^{(1,2)}(z) \cdot \Phi_0^{(2,0)}(z) &= \frac{2i}{\mu_1} f_1^{(1,2)}(z)e_1 \cdot \Phi_0^{(2,0)}(z) + \frac{2i}{\mu_2} f_2^{(1,2)}(z)e_2 \cdot \Phi_0^{(2,0)}(z) \\ &= \frac{2i}{\mu_1}(f_1^{(I_1+2I_n)} z_1 + f_1^{(I_2+2I_n)} z_2)\xi_1 \\ &\quad + \frac{2i}{\mu_2}(f_2^{(I_1+2I_n)} z_1 + f_2^{(I_2+2I_n)} z_2)\xi_2. \end{aligned} \tag{4.5}$$

Here and in what follows, write $I_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$, where the non-zero element 1 is in the jth-position. From (4.5), we also have

$$\Phi_0^{(I_1+2I_n)} \cdot \Phi_0^{(2,0)}(z) = 2i\left(\frac{\xi_1}{\mu_1} f_1^{(I_1+2I_n)} + \frac{\xi_2}{\mu_2} f_2^{(I_1+2I_n)}\right)$$
$$\Phi_0^{(I_2+2I_n)} \cdot \Phi_0^{(2,0)}(z) = 2i\left(\frac{\xi_1}{\mu_1} f_1^{(I_2+2I_n)} + \frac{\xi_2}{\mu_2} f_2^{(I_2+2I_n)}\right), \tag{4.6}$$

and the following:

$$\begin{aligned} &2i\left(\frac{\xi_1}{\mu_1}\Phi_0^{(I_1+2I_n)} \cdot \Phi_0^{(2,0)}(z) + \frac{\xi_2}{\mu_2}\Phi_0^{(I_2+2I_n)} \cdot \Phi_0^{(2,0)}(z)\right) \\ &= \frac{-4\xi_1}{\mu_1} \cdot \left(\frac{\xi_1}{\mu_1} f_1^{(I_1+2I_n)} + \frac{\xi_2}{\mu_2} f_2^{(I_1+2I_n)}\right) + \frac{-4\xi_2}{\mu_2}\left(\frac{\xi_1}{\mu_1} f_1^{(I_2+2I_n)} + \frac{\xi_2}{\mu_2} f_2^{(I_2+2I_n)}\right) \\ &= -4\frac{\xi_1}{\mu_1}\left(f_1^{(I_1+2I_n)}\frac{\xi_1}{\mu_1} + f_1^{(I_2+2I_n)}\frac{\xi_2}{\mu_2}\right) - 4\frac{\xi_2}{\mu_2}\left(f_2^{(I_1+2I_n)}\frac{\xi_1}{\mu_1} + f_2^{(I_2+2I_n)}\frac{\xi_2}{\mu_2}\right). \end{aligned} \tag{4.7}$$

Considering terms of weighted degree 6 in the basic Eq. (3.3), we get

$$\begin{aligned} &2\mathrm{Re}\Big\{z f^{(5)}(z,w) + \Phi_0^{(2)}(z,w) \cdot \Phi_0^{(4)}(z,w)\Big\} \\ &\quad + \big|f^{(3)}(z,w)\big|^2 + \big|\phi^{(3)}(z,w)\big|^2 = 0 \end{aligned} \tag{4.8}$$

over $\mathrm{Im}(w) = |z|^2$. Namely, we have

$$\begin{aligned} &2\mathrm{Re}\Big\{z\Big(f^{(3,1)}(z)(u+i|z|^2) + f^{(1,2)}(z)(u+i|z|^2)^2\Big) \\ &\quad + \Phi_0^{(2,0)}(z)\Big(\Phi_0^{(4,0)}(z) + \Phi_0^{(2,1)}(z)(u+i|z|^2)\Big)\Big\} + \big|f^{(1,1)}(z)(u+i|z|^2)\big|^2 \\ &\quad + \big|\phi^{(3,0)}(z) + \phi^{(1,1)}(z)(u+i|z|^2)\big|^2 = 0. \end{aligned} \tag{4.9}$$

Here we notice that the $f^{(5,0)}(z)$ term is not involved (cf. [22, Lemma 2.3(A)]). Collecting terms of the form $z^\alpha z^\beta u^2$ with $|\alpha| = 1$, $|\beta| = 1$, we get

$$2\mathrm{Re}\big(z f^{(1,2)}(z)\big) + |f^{(1,1)}(z)|^2 + |\phi^{(1,1)}(z)|^2 = 0. \tag{4.10}$$

Collecting terms of the form $z^\alpha z^\beta u$ with $|\alpha| = 3$, $|\beta| = 1$, we get

$$z f^{(3,1)}(z) + \phi^{(3,0)}(z) \cdot \phi^{(1,1)}(z) = 0. \tag{4.11}$$

Collecting terms of the form $z^\alpha z^\beta u$ with $|\alpha| = 2$, $|\beta| = 2$, we get

$$2\mathrm{Re}\Big(2iz f^{(1,2)}(z)|z|^2 + \Phi_0^{(2,0)}(z) \cdot \Phi_0^{(2,1)}(z)\Big) = 0. \tag{4.12}$$

Collecting terms of the form $z^\alpha z^\beta$ with $|\alpha| = 4$, $|\beta| = 2$, we get

$$i|z|^2 z f^{(3,1)}(z) + \Phi_0^{(2,0)}(z) \cdot \Phi_0^{(4,0)}(z) - i|z|^2\phi^{(1,1)}(z) \cdot \phi^{(3,0)}(z) = 0. \tag{4.13}$$

Collecting terms of the form $z^\alpha z^\beta$ with $|\alpha| = 3$, $|\beta| = 3$, we get

$$2\mathrm{Re}\Big(- zf^{(1,2)}(z)|z|^4 + i|z|^2\Phi_0^{(2,0)}(z)\cdot\Phi_0^{(2,1)}(z)\Big) + |z|^4\cdot|f^{(1,1)}(z)|^2 + |\phi^{(3,0)}(z)|^2 + |z|^4\cdot|\phi^{(1,1)}(z)|^2 = 0. \tag{4.14}$$

Combining (4.11) with (4.13), we get

$$\Phi_0^{(2,0)}(z)\cdot\Phi_0^{(4,0)}(z) = 2i|z|^2\phi^{(1,1)}(z)\cdot\phi^{(3,0)}(z). \tag{4.15}$$

Substituting (4.10) into (4.14), we get

$$2\mathrm{Re}\Big(- 2zf^{(1,2)}(z)|z|^2 + i\Phi_0^{(2,0)}(z)\cdot\Phi_0^{(2,1)}(z)\Big)|z|^2 + |\phi^{(3,0)}(z)|^2 = 0. \tag{4.16}$$

Combining (4.12) with (4.16), we get

$$2\Big(- 2zf^{(1,2)}(z)|z|^2 + i\Phi_0^{(2,0)}(z)\cdot\Phi_0^{(2,1)}(z)\Big)|z|^2 + |\phi^{(3,0)}(z)|^2 = 0. \tag{4.17}$$

Recall that in Corollary 3.4, we have obtained

$$|\phi^{(3,0)}(z)|^2 = 4|z|^2\Big(\frac{|\xi_1|^2}{\mu_1} + \frac{|\xi_2|^2}{\mu_2}\Big). \tag{4.18}$$

Hence we have

$$2\Big(- 2zf^{(1,2)}(z)|z|^2 + i\Phi_0^{(2,0)}(z)\cdot\Phi_0^{(2,1)}(z)\Big) + 4\Big(\frac{|\xi_1|^2}{\mu_1} + \frac{|\xi_2|^2}{\mu_2}\Big) = 0. \tag{4.19}$$

Notice that $|\xi_i|^2 = \xi_i\overline{\xi_i} = \xi_i e_i\cdot\Phi_0^{(2,0)}(z)$. Set

$$\widetilde{\phi}^{(2,1)}(z) = \phi^{(2,1)}(z) - 2i\sum_{j=1}^{2}\frac{\xi_j}{\mu_j}e_j^*,\ \widetilde{\Phi}_0^{(2,1)}(z) = \Phi_0^{(2,1)}(z) - 2i\sum_{j=1}^{2}\frac{\xi_j}{\mu_j}e_j. \tag{4.20}$$

Then we have

$$\Phi_0^{(2,0)}(z)\cdot\widetilde{\Phi}_0^{(2,1)}(z) = -2i|z|^2 z\cdot f^{(1,2)}(z). \tag{4.21}$$

Recall $f_j = z_j$ for $3 \le j \le n-1$ so that $\overline{z}\cdot f^{(1,2)}(z) = (\overline{z}_1, \overline{z}_2)\cdot(f_1^{(1,2)}(z), f_2^{(1,2)}(z))$ and thus we get

$$\widetilde{\Phi}_{11}^{(2,1)}(z)=\frac{-2i}{\sqrt{\mu_1}}z_1 f_1^{(1,2)}(z),\quad \widetilde{\Phi}_{12}^{(2,1)}(z)=\frac{-2i}{\sqrt{\mu_1+\mu_2}}\big(z_1 f_2^{(1,2)}(z)+z_2 f_1^{(1,2)}(z)\big),$$
$$\widetilde{\Phi}_{22}^{(2,1)}(z)=\frac{-2i}{\sqrt{\mu_2}}z_2 f_2^{(1,2)}(z), \tag{4.22}$$
$$\widetilde{\Phi}_{1j}^{(2,1)}(z)=\frac{-2i}{\sqrt{\mu_1}}z_j f_1^{(1,2)}(z),\quad \widetilde{\Phi}_{2j}^{(2,1)}(z)=\frac{-2i}{\sqrt{\mu_2}}z_j f_2^{(1,2)}(z) \text{ with } j\geq 3.$$

Making use of Lemma 3.1, we get

$$|\widetilde{\Phi}_0^{(2,1)}(z)|^2=4|z|^2\sum_{j=1}^{2}\frac{1}{\mu_j}|f_j^{(1,2)}(z)|^2 -\frac{4}{\mu_1\mu_2(\mu_1+\mu_2)}\big|\mu_1 z_1 f_2^{(1,2)}(z)-\mu_2 z_2 f_1^{(1,2)}(z)\big|^2, \tag{4.23}$$

and

$$\widetilde{\Phi}_0^{(2,1)}(z)\Phi_0^{(3,0)}(z)=-4|z|^2\sum_{j=1}^{2}\frac{1}{\mu_j}f_j^{(1,2)}(z)\xi_j +\frac{4}{\mu_1\mu_2(\mu_1+\mu_2)}\Big(\mu_1 z_1 f_2^{(1,2)}(z)-\mu_2 z_2 f_1^{(1,2)}(z)\Big) \cdot\Big(\mu_1 z_1\xi_2-\mu_2 z_2\xi_1\Big). \tag{4.24}$$

Notice that if we replace z_1, z_2 by $\frac{\xi_1}{\mu_1}, \frac{\xi_2}{\mu_2}$, respectively, in (4.10), we get

$$2\mathrm{Re}\Big\{\frac{\xi_1}{\mu_1}\Big(f_1^{(I_1+2I_n)}\frac{\xi_1}{\mu_1}+f_1^{(I_2+2I_n)}\frac{\xi_2}{\mu_2}\Big)+\frac{\xi_2}{\mu_2}\Big(f_2^{(I_1+2I_n)}\frac{\xi_1}{\mu_1}+f_2^{(I_2+2I_n)}\frac{\xi_2}{\mu_2}\Big)\Big\} +\frac{1}{4}(|\xi_1|^2+|\xi_2|^2)+|\frac{\xi_1}{\mu_1}e_1^*+\frac{\xi_2}{\mu_2}e_2^*|^2=0. \tag{4.25}$$

Here we have used $f_j^{(1,1)}(z)=\frac{i}{2}\mu_j z_j$ for $j=1,2$. Combining this with (4.7), we get

$$-2\mathrm{Re}\Big\{2i\Big(\frac{\xi_1}{\mu_1}\Phi_0^{(I_1+2I_n)}+\frac{\xi_2}{\mu_2}\Phi_0^{(I_2+2I_n)}\Big)\cdot\Phi_0^{(2,0)}(z)\Big\} +(|\xi_1|^2+|\xi_2|^2)+4\Big|\frac{\xi_1}{\mu_1}e_1^*+\frac{\xi_2}{\mu_2}e_2^*\Big|^2=0. \tag{4.26}$$

Considering terms of weighted degree 7 in the basic Eq. (3.3), we get

$$2\mathrm{Re}\Big\{zf^{(6)}(z,w)+f^{(3)}(z,w)f^{(4)}(z,w)+\Phi_0^{(2)}(z,w)\Phi_0^{(5)}(z,w) +\phi^{(3)}(z,w)\phi^{(4)}(z,w)\Big\}=0 \tag{4.27}$$

over $\mathrm{Im}(w) = |z|^2$. Namely, we have

$$2\mathrm{Re}\Big\{z\Big(f^{(4,1)}(z)(u+i|z|^2) + f^{(2,2)}(z)(u+i|z|^2)^2\Big) + f^{(1,1)}(z)(u+i|z|^2)\cdot f^{(2,1)}(z) \cdot(u+i|z|^2) + \Phi_0^{(2,0)}(z)\Big(\Phi_0^{(5,0)}(z) + \Phi_0^{(3,1)}(z)(u+i|z|^2) + \Phi_0^{(1,2)}(z)(u+i|z|^2)^2\Big) + \big(\phi^{(3,0)}(z) + \phi^{(1,1)}(z)(u+i|z|^2)\big)\cdot\big(\phi^{(4,0)}(z) + \phi^{(2,1)}(z)(u+i|z|^2)\big)\Big\} = 0. \quad (4.28)$$

Here we notice that the $f^{(6,0)}(z)$ term is not involved (cf. [22, Lemma 2.3(A)]). Collecting terms of the form $z^\alpha z^\beta u^2$ with $|\alpha| = 2$, $|\beta| = 1$, we get

$$zf^{(2,2)}(z) + f^{(1,1)}(z)\cdot f^{(2,1)}(z) + \Phi_0^{(1,2)}(z)\cdot\Phi_0^{(2,0)}(z) + \phi^{(1,1)}(z)\cdot\phi^{(2,1)}(z) = 0. \quad (4.29)$$

Collecting terms of the form $z^\alpha z^\beta u$ with $|\alpha| = 3$, $|\beta| = 2$, we get

$$2iz|z|^2 f^{(2,2)}(z) + \Phi_0^{(2,0)}(z)\cdot\Phi_0^{(3,1)}(z) - 2i|z|^2\Phi_0^{(1,2)}(z)\cdot \Phi_0^{(2,0)}(z) + \phi^{(2,1)}(z)\cdot\phi^{(3,0)}(z) = 0. \quad (4.30)$$

Collecting terms of the form $z^\alpha z^\beta$ with $|\alpha| = 4$, $|\beta| = 3$, we get

$$-zf^{(2,2)}(z)|z|^4 + f^{(1,1)}(z)\cdot f^{(2,1)}(z)|z|^4 + i|z|^2\Phi_0^{(2,0)}(z)\cdot\Phi_0^{(3,1)}(z) - |z|^4 \Phi_0^{(1,2)}(z)\cdot\Phi_0^{(2,0)}(z) + \phi^{(3,0)}(z)\cdot\phi^{(4,0)}(z) - i|z|^2\phi^{(2,1)}(z)\cdot\phi^{(3,0)}(z) + |z|^4\phi^{(1,1)}(z)\cdot\phi^{(2,1)}(z) = 0. \quad (4.31)$$

By calculating (4.31)$-|z|^4\cdot$(4.29), we get

$$-2zf^{(2,2)}(z)|z|^4 + i|z|^2\Phi_0^{(2,0)}(z)\Phi_0^{(3,1)}(z) - 2|z|^4\Phi_0^{(1,2)}(z)\cdot\Phi_0^{(2,0)}(z) + \phi^{(3,0)}(z)\cdot\phi^{(4,0)}(z) - i|z|^2\phi^{(2,1)}(z)\cdot\phi^{(3,0)}(z) = 0. \quad (4.32)$$

By calculating (4.32)$-i|z|^2$(4.30), we get

$$\phi^{(3,0)}(z)\cdot\phi^{(4,0)}(z) = 4|z|^4\Phi_0^{(1,2)}(z)\cdot\Phi_0^{(2,0)}(z) + 2i|z|^2\phi^{(2,1)}(z)\cdot\phi^{(3,0)}(z). \quad (4.33)$$

Combining this with (4.30), we get

$$\phi^{(3,0)}(z)\cdot\phi^{(4,0)}(z) = -2i|z|^2\Big(2i|z|^2 zf^{(2,2)}(z) + \Phi_0^{(2,0)}(z)\cdot\Phi_0^{(3,1)}(z)\Big). \quad (4.34)$$

By (4.15), we have

$$\begin{aligned} \mu_{11}\cdot\Phi^{(4,0)}_{11}(z) &= 2iz_1\phi^{(3,0)}(z)\cdot e_1^*, \\ \mu_{12}\cdot\Phi^{(4,0)}_{12}(z) &= 2iz_1\phi^{(3,0)}(z)\cdot e_2^* + 2iz_2\phi^{(3,0)}(z)\cdot e_1^*, \\ \mu_{22}\cdot\Phi^{(4,0)}_{22}(z) &= 2iz_2\phi^{(3,0)}(z)\cdot e_2^*, \\ \mu_{1j}\cdot\Phi^{(4,0)}_{1j}(z) &= 2iz_j\phi^{(3,0)}(z)\cdot e_1^*, \quad j\geq 3, \\ \mu_{2j}\cdot\Phi^{(4,0)}_{2j}(z) &= 2iz_j\phi^{(3,0)}(z)\cdot e_2^*, \quad j\geq 3. \end{aligned} \tag{4.35}$$

Write

$$\eta_1^* = \phi^{(3,0)}(z)\cdot e_1^*,\ \eta_2^* = \phi^{(3,0)}(z)\cdot e_2^*;\ \eta_1 = \Phi^{(3,0)}_0(z)\cdot e_1,\ \eta_2 = \Phi^{(3,0)}_0(z)\cdot e_2.$$

Making use of Lemma 3.1, we get

$$\begin{aligned} \Phi^{(3,0)}_0(z)\Phi^{(4,0)}_0(z) = 4|z|^2\Big(\frac{\xi_1\eta_1^*}{\mu_1}+\frac{\xi_2\eta_2^*}{\mu_2}\Big) - \frac{4}{\mu_1+\mu_2}\Big(\sqrt{\frac{\mu_2}{\mu_1}}z_2\xi_1 - \sqrt{\frac{\mu_1}{\mu_2}}z_1\xi_2\Big) \\ \cdot\Big(\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^*\Big). \end{aligned} \tag{4.36}$$

Combining (4.34) with (4.36) and making use of Lemma 2.4, we get

$$\Phi^{(3,0)}_1(z)\Phi^{(4,0)}_1(z) = \frac{4}{\mu_1+\mu_2}\Big(\sqrt{\frac{\mu_2}{\mu_1}}z_2\xi_1 - \sqrt{\frac{\mu_1}{\mu_2}}z_1\xi_2\Big)\cdot\Big(\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^*\Big).$$

Now, by Corollary 3.4, we have

$$\phi^{(4,0)}_{33}(z) = \frac{2}{\sqrt{\mu_1+\mu_2}}\Big(\sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^*\Big). \tag{4.37}$$

Moreover

$$2i\left(\frac{\xi_1\eta_1^*}{\mu_1}+\frac{\xi_2\eta_2^*}{\mu_2}\right) = 2i|z|^2zf^{(2,2)}(z) + \Phi^{(2,0)}_0(z)\cdot\Phi^{(3,1)}_0(z). \tag{4.38}$$

Write

$$\begin{aligned} \widetilde{\phi}^{(3,1)}(z) &= \phi^{(3,1)}(z) - 2i\left(\frac{\eta_1^*}{\mu_1}e_1^* + \frac{\eta_2^*}{\mu_2}e_2^*\right), \\ \widetilde{\Phi}^{(3,1)}_0(z) &= \Phi^{(3,1)}_0(z) - 2i\left(\frac{\eta_1^*}{\mu_1}e_1 + \frac{\eta_2^*}{\mu_2}e_2\right). \end{aligned} \tag{4.39}$$

Then we have

$$\Phi^{(2,0)}_0(z)\widetilde{\Phi_0}^{(3,1)}(z) = -2i|z|^2zf^{(2,2)}(z).$$

Hence, we get

$$\begin{aligned}
&\mu_{11}\cdot\widetilde{\Phi}^{(3,1)}_{11}(z)=-2iz_1f^{(2,2)}_1(z),\\
&\mu_{12}\cdot\widetilde{\Phi}^{(3,1)}_{12}(z)=-2i\big(z_1f^{(2,2)}_2(z)+z_2f^{(2,2)}_1(z)\big),\\
&\mu_{22}\cdot\widetilde{\Phi}^{(3,1)}_{22}(z)=-2iz_2f^{(2,2)}_2(z),\\
&\mu_{1j}\cdot\widetilde{\Phi}^{(3,1)}_{1j}(z)=-2iz_jf^{(2,2)}_1(z),\quad j\geq 3,\\
&\mu_{2j}\cdot\widetilde{\Phi}^{(3,1)}_{2j}(z)=-2iz_jf^{(2,2)}_2(z),\quad j\geq 3.
\end{aligned}\tag{4.40}$$

By Lemma 3.1, we have

$$\begin{aligned}
\Phi^{(3,0)}_0(z)\widetilde{\Phi}^{(3,1)}_0(z)=&-4|z|^2\Big(\frac{\xi_1}{\mu_1}f^{(2,2)}_1(z)+\frac{\xi_2}{\mu_2}f^{(2,2)}_2(z)\Big)\\
&+\frac{4}{\mu_1+\mu_2}\Big(\sqrt{\frac{\mu_1}{\mu_2}}z_1\xi_2-\sqrt{\frac{\mu_2}{\mu_1}}z_2\xi_1\Big)\\
&\cdot\Big(\sqrt{\frac{\mu_1}{\mu_2}}z_1f^{(2,2)}_2(z)-\sqrt{\frac{\mu_2}{\mu_1}}z_2f^{(2,2)}_1(z)\Big).
\end{aligned}\tag{4.41}$$

Notice that

$$\Phi^{(3,0)}_0(z)\cdot 2i\left(\frac{\eta^*_1}{\mu_1}e_1+\frac{\eta^*_2}{\mu_2}e_2\right)=2i\left(\frac{\eta^*_1}{\mu_1}\eta_1+\frac{\eta^*_2}{\mu_2}\eta_2\right).\tag{4.42}$$

Hence

$$\begin{aligned}
&\Phi^{(3,0)}_0(z)\Phi^{(3,1)}_0(z)\\
&=2i\Big(\frac{\eta^*_1}{\mu_1}\eta_1+\frac{\eta^*_2}{\mu_2}\eta_2\Big)-4|z|^2\Big(\frac{\xi_1}{\mu_1}f^{(2,2)}_1(z)+\frac{\xi_2}{\mu_2}f^{(2,2)}_2(z)\Big)\\
&+\frac{4}{\mu_1+\mu_2}\Big(\sqrt{\frac{\mu_1}{\mu_2}}z_1\xi_2-\sqrt{\frac{\mu_2}{\mu_1}}z_2\xi_1\Big)\\
&\cdot\Big(\sqrt{\frac{\mu_1}{\mu_2}}z_1f^{(2,2)}_2(z)-\sqrt{\frac{\mu_2}{\mu_1}}z_2f^{(2,2)}_1(z)\Big).
\end{aligned}\tag{4.43}$$

Combining (4.33) with (4.36) and making use of Lemma 2.4 and Corollary 3.4 again, we get

$$4|z|^2\Phi^{(1,2)}_0(z)\cdot\Phi^{(2,0)}_0(z)+2i\phi^{(2,1)}(z)\cdot\phi^{(3,0)}(z)=4\left(\frac{1}{\mu_1}\xi_1\eta^*_1+\frac{1}{\mu_2}\xi_2\eta^*_2\right).\tag{4.44}$$

Namely, we have

$$|z|^2A(z,z)+2i\widetilde{\phi}^{(2,1)}(z)\cdot\phi^{(3,0)}(z)=0.\tag{4.45}$$

Here, as before, we write $A(z, \bar{z})$ for a real analytic function which may be different in different contexts.

Combining (4.24) with (4.45) and making use of Lemma 2.4 and Corollary 3.4, we get

$$\widetilde{\phi}_{33}^{(2,1)}(z) = \frac{-2}{\sqrt{\mu_1 + \mu_2}}\left(\sqrt{\frac{\mu_1}{\mu_2}}\bar{z}_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}}\bar{z}_2 f_1^{(1,2)}(z)\right). \tag{4.46}$$

Next we will prove that $\phi_{3j}^{(4,0)}(z) = 0,\ \widetilde{\phi}_{3j}^{(2,1)}(z) = 0$ for $j = 4, \ldots, K$ with $K = N - n - (n-1) - (n-2)$.

Considering terms of weighted degree 8 in the basic Eq. (3.3), we get

$$\begin{aligned} &2\mathrm{Re}\Big\{\bar{z} f^{(7)}(z, w) + \overline{f^{(3)}(z, w)} f^{(5)}(z, w) + \overline{\Phi_0^{(2)}(z, w)}\Phi_0^{(6)}(z, w) \\ &\quad + \overline{\phi^{(3)}(z, w)}\phi^{(5)}(z, w)\Big\} + \big|f^{(4)}(z, w)\big|^2 + \big|\phi^{(4)}(z, w)\big|^2 = 0 \end{aligned} \tag{4.47}$$

over $\mathrm{Im}(w) = |z|^2$. Namely, we have

$$\begin{aligned} &2\mathrm{Re}\Big\{\bar{z}\Big(f^{(5,1)}(z)(u + i|z|^2) + f^{(3,2)}(z)(u + i|z|^2)^2 + f^{(1,3)}(z)(u + i|z|^2)^3\Big) \\ &+ \overline{f^{(1,1)}(z)(u + i|z|^2)} \cdot \Big(f^{(3,1)}(z)(u + i|z|^2) + f^{(1,2)}(z)(u + i|z|^2)^2\Big) \\ &+ \overline{\Phi_0^{(2,0)}(z)} \cdot \Big(\Phi_0^{(6,0)}(z) + \Phi_0^{(4,1)}(z)(u + i|z|^2) + \Phi_0^{(2,2)}(z)(u + i|z|^2)^2\Big) \\ &+ \Big(\overline{\phi^{(3,0)}(z) + \phi^{(1,1)}(z)(u + i|z|^2)}\Big) \\ &\cdot \Big(\phi^{(5,0)}(z) + \phi^{(3,1)}(z)(u + i|z|^2) + \phi^{(1,2)}(z)(u + i|z|^2)^2\Big)\Big\} \\ &+ \Big|f^{(2,1)}(z)(u + i|z|^2)\Big|^2 + \Big|\phi^{(4,0)}(z) + \phi^{(2,1)}(z)(u + i|z|^2)\Big|^2 = 0. \end{aligned} \tag{4.48}$$

Here we notice that the $f^{(7,0)}(z)$ term is not involved (cf. [22, Lemma 2.3(A)]). Collecting terms of the form $z^\alpha \bar{z}^\beta$ with $|\alpha| = 4, |\beta| = 4$, we get

$$\begin{aligned} &2\mathrm{Re}\Big\{ - i\bar{z} f^{(1,3)}(z)|z|^6 - \overline{f^{(1,1)}(z)}(-i|z|^2) f^{(1,2)}(z)|z|^4 + \overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z)(-|z|^4) \\ &+ \overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z) i|z|^2 + \overline{\phi^{(1,1)}(z)}(-i|z|^2)\phi^{(1,2)}(z)(-|z|^4)\Big\} \\ &+ |\phi^{(4,0)}(z)|^2 + \big(\big|f^{(2,1)}(z)\big|^2 + \big|\phi^{(2,1)}(z)\big|^2\big)|z|^4 = 0. \end{aligned} \tag{4.49}$$

Collecting terms of the form $z^\alpha \bar{z}^\beta u^2$ with $|\alpha| = 2, |\beta| = 2$, we get

$$\begin{aligned} &2\mathrm{Re}\Big\{\bar{z} f^{(1,3)}(z) 3i|z|^2 + \overline{f^{(1,1)}(z)} f^{(1,2)}(z) i|z|^2 + \overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z) \\ &\quad + \overline{\phi^{(1,1)}(z)}\phi^{(1,2)}(z) i|z|^2\Big\} + \big(\big|f^{(2,1)}(z)\big|^2 + \big|\phi^{(2,1)}(z)\big|^2\big) = 0. \end{aligned} \tag{4.50}$$

Collecting terms of the form $z^\alpha z^\beta u$ with $|\alpha| = 3$, $|\beta| = 3$, we get

$$2\mathrm{Re}\Big\{zf^{(1,3)}(z)3(-|z|^4) + f^{(1,1)}(z)f^{(1,2)}(z)|z|^4 + \Phi_0^{(2,0)}(z)\Phi_0^{(2,2)}(z)2i|z|^2 \\ +\phi^{(3,0)}(z)\phi^{(3,1)}(z) + \phi^{(1,1)}(z)\phi^{(1,2)}(z)|z|^4\Big\} = 0. \quad (4.51)$$

By (4.49) and (4.50), we get

$$|z|^2 \cdot 2\mathrm{Re}\Big\{-4izf^{(1,3)}(z)|z|^4 - 2\Phi_0^{(2,0)}(z)\Phi_0^{(2,2)}(z)(|z|^2) \\ +i\phi^{(3,0)}(z)\phi^{(3,1)}(z)\Big\} + \big|\phi^{(4,0)}(z)\big|^2 = 0. \quad (4.52)$$

Combining this with (4.51), we get

$$|z|^6 A(z, z) + 2|z|^2 \cdot \big(-2\Phi_0^{(2,0)}(z)\Phi_0^{(2,2)}(z)(|z|^2) + i\phi^{(3,0)}(z)\phi^{(3,1)}(z)\big) \\ +\big|\phi^{(4,0)}(z)\big|^2 = 0. \quad (4.53)$$

By (4.35) and Lemma 3.1, we get

$$\frac{1}{4}\big|\Phi_0^{(4,0)}(z)\big|^2 = |z|^2\left(\frac{1}{\mu_1}|\eta_1^*|^2 + \frac{1}{\mu_2}|\eta_2^*|^2\right) \\ -\frac{1}{\mu_1+\mu_2}\Big|\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^*\Big|^2. \quad (4.54)$$

Combining this with (4.53) and making use of Lemma 2.4, we get

$$|z|^4 A(z, z) - 4|z|^2\Phi_0^{(2,0)}(z)\Phi_0^{(2,2)}(z) + 2i\phi^{(3,0)}(z)\phi^{(3,1)}(z) \\ +\frac{4}{\mu_1}|\eta_1^*|^2 + \frac{4}{\mu_2}|\eta_2^*|^2 = 0, \quad (4.55)$$

and

$$\frac{1}{4}|\Phi_1^{(4,0)}(z)|^2 = \tfrac{1}{\mu_1+\mu_2}\Big|\sqrt{\tfrac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\tfrac{\mu_1}{\mu_2}}z_1\eta_2^*\Big|^2. \quad (4.56)$$

By (4.37) and (4.56), we get

$$\phi_{33}^{(4,0)}(z) = \frac{2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^*\right), \\ \phi_{3j}^{(4,0)}(z) = 0 \quad \text{for } j > 3. \quad (4.57)$$

This proves Theorem 4.1 (1).

Substituting (4.43) into (4.55), we get

$$
\begin{aligned}
&|z|^4 A(z,z) - 4|z|^2 \Phi_0^{(2,0)}(z)\Phi_0^{(2,2)}(z) + 2i\,\Phi_1^{(3,0)}(z)\Phi_1^{(3,1)}(z)\\
&-8i|z|^2\left(\frac{\xi_1}{\mu_1} f_1^{(2,2)}(z) + \frac{\xi_2}{\mu_2} f_2^{(2,2)}(z)\right) + \frac{8i}{\mu_1+\mu_2}\left(\sqrt{\frac{\mu_1}{\mu_2}} z_1\xi_2 - \sqrt{\frac{\mu_2}{\mu_1}} z_2\xi_1\right)\\
&\cdot\left(\sqrt{\frac{\mu_1}{\mu_2}} z_1 f_2^{(2,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}} z_2 f_1^{(2,2)}(z)\right) + \frac{4}{\mu_1}\eta_1^*(\eta_1^* - \eta_1) + \frac{4}{\mu_2}\eta_2^*(\eta_2^* - \eta_2) = 0.
\end{aligned}
\tag{4.58}
$$

Notice that $\Phi_1^{(3,0)}(z) = (\phi_{33}^{(3,0)}(z), 0, \ldots, 0)$ and $n \geq 7$. Making use of Lemma 2.4, we get

$$
\Phi_0^{(2,0)}(z)\Phi_0^{(2,2)}(z) = -2i\big(\frac{\xi_1}{\mu_1} f_1^{(2,2)}(z) + \frac{\xi_2}{\mu_2} f_2^{(2,2)}(z)\big) + |z|^2 A(z,z). \tag{4.59}
$$

By (4.29), we have

$$
\begin{aligned}
f_1^{(2,2)}(z) &= \frac{i}{2}\mu_1 f_1^{(2,1)}(z) - \Phi_0^{(I_1+2I_n)}\Phi_0^{(2,0)}(z) - e_1^*\phi^{(2,1)}(z),\\
f_2^{(2,2)}(z) &= \frac{i}{2}\mu_2 f_2^{(2,1)}(z) - \Phi_0^{(I_2+2I_n)}\Phi_0^{(2,0)}(z) - e_2^*\phi^{(2,1)}(z).
\end{aligned}
\tag{4.60}
$$

Thus we get

$$
2\mathrm{Re}\Big\{-2i\big(\frac{\xi_1}{\mu_1} f_1^{(2,2)}(z) + \frac{\xi_2}{\mu_2} f_2^{(2,2)}(z)\big)\Big\} = I + II + III. \tag{4.61}
$$

Here

$$
\begin{aligned}
I &= 2\mathrm{Re}\Big\{-2i\Big(\frac{\xi_1}{\mu_1}\frac{i}{2}\mu_1(-\xi_1) + \frac{\xi_2}{\mu_2}\frac{i}{2}\mu_2(-\xi_2)\Big)\Big\} = -2(|\xi_1|^2 + |\xi_2|^2).\\
II &= 2\mathrm{Re}\Big(2i\frac{\xi_1}{\mu_1}\Phi_0^{(I_1+2I_n)}\Phi_0^{(2,0)}(z) + 2i\frac{\xi_2}{\mu_2}\Phi_0^{(I_2+2I_n)}\Phi_0^{(2,0)}(z)\Big)\\
&= (|\xi_1|^2 + |\xi_2|^2) + 4\Big|\frac{\xi_1}{\mu_1}e_1^* + \frac{\xi_2}{\mu_2}e_2^*\Big|^2.\\
III &= 2\mathrm{Re}\Big(2i\frac{\xi_1}{\mu_1}e_1^*\phi^{(2,1)}(z) + 2i\frac{\xi_2}{\mu_2}e_2^*\phi^{(2,1)}(z)\Big).
\end{aligned}
\tag{4.62}
$$

The equality for I follows from (3.5) and II follows from (4.26). By (4.50), we get

$$
|z|^2 A(z,z) + 2\mathrm{Re}\big(\Phi_0^{(2,0)}(z)\Phi_0^{(2,2)}(z)\big) + \big(\big|f^{(2,1)}(z)\big|^2 + \big|\phi^{(2,1)}(z)\big|^2\big) = 0. \tag{4.63}
$$

Substituting (3.5), (4.59), (4.61) and (4.62) into (4.63), we get

$$\begin{aligned}&|z|^2A(z,z)-2(|\xi_1|^2+|\xi_2|^2)+(|\xi_1|^2+|\xi_2|^2)+4\Big|\frac{\xi_1}{\mu_1}e_1^*+\frac{\xi_2}{\mu_2}e_2^*\Big|^2\\&+2\mathrm{Re}\Big(2i\frac{\xi_1}{\mu_1}e_1^*\phi^{(2,1)}(z)+2i\frac{\xi_2}{\mu_2}e_2^*\phi^{(2,1)}(z)\Big)\\&+(|\xi_1|^2+|\xi_2|^2)+\big|\phi^{(2,1)}(z)\big|^2=0.\end{aligned}\tag{4.64}$$

Hence we get

$$|z|^2A(z,z)+\left|\phi^{(2,1)}(z)-2i\left(\frac{\xi_1}{\mu_1}e_1^*+\frac{\xi_2}{\mu_2}e_2^*\right)\right|^2=0.\tag{4.65}$$

Substituting (4.23) into (4.65), we get

$$\begin{aligned}&|z|^2A(z,z)+\left|\widetilde{\Phi}_1^{(2,1)}(z)\right|^2-\frac{4}{\mu_1+\mu_2}\Big|\sqrt{\frac{\mu_1}{\mu_2}}z_1f_2^{(1,2)}(z)\\&-\sqrt{\frac{\mu_2}{\mu_1}}z_2f_1^{(1,2)}(z)\Big|^2=0.\end{aligned}\tag{4.66}$$

Making use of (4.46) and Lemma 2.4, we get

$$\begin{aligned}&\widetilde{\phi}_{33}^{(2,1)}(z)=\frac{-2}{\sqrt{\mu_1+\mu_2}}\Big(\sqrt{\frac{\mu_1}{\mu_2}}z_1f_2^{(1,2)}(z)-\sqrt{\frac{\mu_2}{\mu_1}}z_2f_1^{(1,2)}(z)\Big),\\&\widetilde{\phi}_{3j}^{(2,1)}(z)=0\quad\text{for } j>3.\end{aligned}\tag{4.67}$$

By (4.20), the proof of Theorem 4.1 (2) is also complete.

5 Proof of Theorem 1.1

Step (I): An application of a normal form in [22] *for maps with geometric rank 1*: We first consider $F\in Rat(\mathbb{B}^n,\mathbb{B}^N)$ with geometric rank 1. Then by Theorem 1.2 of [22], F is equivalent to a map of the form $\Phi=(z_1,\ldots,z_{n-1},z_nH(z)):=(\phi_1,\ldots,\phi_N)$ with $H\in Rat(\mathbb{B}^n,\mathbb{B}^{N-n+1})$ also of geometric rank one. We first have the following:

Lemma 5.1 *If $H(\mathbb{B}^n)$ is contained in an affine subspace of dimension m in $\mathbb{C}^{N-n+1}$, then $F(\mathbb{B}^n)$ is contained in an affine subspace of dimension $m+n$ in $\mathbb{C}^N$.*

Proof Indeed, we first notice that linear fractional transformations map affine linear subspaces to affine linear subspaces. Also, $F(\mathbb{B}^n)$ is contained in an affine subspace of dimension m, if and only if F is equivalent to a map of the form $(G,0)$ with G having m-components. Now suppose the image of $H=(h_1,\ldots,h_{N-n+1})$ is contained in an affine subspace of dimension $m\le N-n$, then there are $(N-n-m+1)$ linearly independent vectors $\mu_j=(a_{j1},\ldots,a_{jk})$ with $k=N-n+1$ such that

$\sum_{l=1}^k a_{jl}h_l(z) \equiv c_j$ for certain $c_j \in \mathbb{C}$. If $c_j = 0$ for all j, then $\sum_{l=1}^k a_{jl}\phi_{n-1+l}(z) \equiv 0$. Hence $\Phi(\mathbb{B}^n)$ is contained in an affine linear subspace of dimension $m+n-1$. Otherwise, assume without loss of generality that $c_1 = 1$. Then we have $\sum_{l=1}^k (a_{jl} - c_j a_{1l})\phi_{n-1+l}(z) \equiv 0$. Notice that $\{\mu_2 - c_2\mu_1, \ldots, \mu_{N-n-m+1} - c_{N-n-m+1}\mu_1\}$ is also linearly independent, we see that $\Phi(\mathbb{B}^n)$ is contained in an affine subspace of dimension $m+n$. This proves Lemma 5.1. □

Applying the gap rigidity in [22] to H and Lemma 5.1, we see that when $3n+1 \le N \le 4n-5$ and $n \ge 6$, $F(\mathbb{B}^n)$ is contained in an affine linear subspace of dimension $3n$. This proves Theorem 1.1 in case the map F has geometric rank one.

Indeed, by an induction argument, we see that when $N < (k+1)n - \frac{k(k+1)}{2}$, $F(\mathbb{B}^n)$ is contained in a linear affine subspace of dimension kn, if $(k+1)n - \frac{k(k+1)}{2} > 0$.

Step (II): Completion of the Proofs of Theorems 1.1: By Lemma 3.2 in [19], when $N \le 4n-7$, any $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ can only have geometric rank $\kappa_0 = 0, 1$, or 2. When $\kappa_0 = 0$, F is linear and thus Theorem 1.1 follows trivially. When $\kappa_0 = 1$, the proof of Theorem 1.1 is already done in Step (I). The case of Theorem 1.1 for maps with geometric rank two is obviously a special case of the following Theorem 5.2.

Theorem 5.2 *Let F be a proper rational map from $\mathbb{H}_n$ into $\mathbb{H}_N$ with geometric rank $\kappa_0 = 2$. Assume that $n \ge 7$ and $3n \le N \le 4n-6$. Then F is equivalent to a map of the form $(G, 0')$ where G is a proper rational map from $\mathbb{H}_n$ into $\mathbb{H}_{N'}$ with $N' = 3n$.*

Since F is rational, by a result of Cima–Suffridge, the above F extends holomorphically across $\partial\mathbb{H}_n$.

Let N be such that $N \le 4n-6$. Let F be a proper rational holomorphic map from $\mathbb{H}_n$ into $\mathbb{H}_N$ with geometric rank $\kappa_0 = 2$ and $F(0) = 0$. As mentioned in §2, we can assume, without loss of generality, that F satisfies the normalization in Theorem 2.3.

Write $\mathcal{L}_j = \frac{\partial}{\partial z_j} - 2i\overline{z_j}\frac{\partial}{\partial w}$ for $j = 1, \ldots, n-1$, which form a basis of tangent vector fields of type $(1,0)$ along $\partial\mathbb{H}_n$. Let $\mathcal{L}^\alpha$ be defined in the standard way. Notice that for any smooth function h near 0, $\mathcal{L}^\alpha h|_0 = \frac{\partial^{|\alpha|}}{\partial z^\alpha}h|_0 := D_z^\alpha h|_0$.

Assume the normalization in Corollary 3.4 for F. Also assume that $\varphi_{33}^{(3,0)}(z) \not\equiv 0$. Then

$$\begin{aligned} \mathop{span}_{|\beta|\le 3}\{\mathcal{L}^\beta F|_0\} = span\{&(0, \ldots, 0, 1^{jth}, 0, \ldots, 0),\\ &1 \le j \le n + \sharp\mathcal{S}_0 = n + (n-1) + (n-2) = 3n-3\}. \end{aligned} \tag{5.1}$$

Applying Theorem 4.1 (1), we see that

$$\mathop{span}_{|\beta|\le 4}\{\mathcal{L}^\beta F|_0\} = span\{(0, \ldots, 0, 1^{jth}, 0, \ldots, 0),\ 1 \le j \le n + \sharp\mathcal{S}_0\}. \tag{5.2}$$

Hence

$$\mathop{span}_{|\beta|\le 4}\{\mathcal{L}^\beta F|_0\} = \mathop{span}_{|\beta|\le 3}\{\mathcal{L}^\beta F|_0\}. \tag{5.3}$$

Now, we proceed in a similar way as in [18], though the situation in [18] is harder for the maps there are only assumed to be twice differentiable. Notice that we have assumed that $\phi_{33}^{(3,0)}(z) \not\equiv 0$.

For any $p \in \mathbb{H}_n(\approx 0)$, there exist $\tau_p \in Aut_0(\mathbb{H}_N)$, $\sigma_p \in Aut_0(\mathbb{H}_n)$ such that $G_p = \tau_p \circ F_p \circ \sigma_p$ satisfies the normalization condition in Theorem 2.3. Moreover, we can also have the $\Phi_1^{(3,0)}(z)$ coming from G_p not identically zero, for we can choose τ_p, σ_p to depend smoothly on p. Hence, after applying U_p, a unitary matrix transformation, to normalize the Φ_1-part, we get the normalization as in Corollary 3.4 for the new map with the corresponding $\phi_{33}^{(3,0)}(z) \not\equiv 0$. Notice that for the new G_p, we have

$$\underset{|\beta|\le 4}{span}\{\mathcal{L}^\beta G_p|_0\} = \underset{|\beta|\le 3}{span}\{\mathcal{L}^\beta G_p|_0\}, \text{ or } \underset{|\beta|\le 4}{span}\{D_z^\beta G_p|_0\} = \underset{|\beta|\le 3}{span}\{D_z^\beta G_p|_0\}. \tag{5.4}$$

Also the dimension of the above space is $n + \sharp\mathcal{S}_0$ for any $p \approx 0$.

Still write τ_p for $U_p \circ \tau_p$. Then $F_p = \tau_p^{-1} \circ G_p \circ \sigma_p^{-1}$. Now, for any $|\alpha| = 4$, we claim that

$$\begin{aligned} &D_z^\alpha(\tau_p^{-1} \circ G_p \circ \sigma_p^{-1})|_0 \in \underset{|\beta|\le 3}{span}\{D_z^\beta(\tau_p^{-1} \circ G_p \circ \sigma_p^{-1})|_0\}, \\ &\text{or} \quad \mathcal{L}^\alpha F_p|_0 \in \underset{|\beta|\le 3}{span}\{\mathcal{L}^\beta F_p|_0\}. \end{aligned} \tag{5.5}$$

Here, as defined before, D_z^α is the regular differentiation, with respect to z, of order $|\alpha|$.

Indeed, write

$$\sigma_p^{-1} = \Big(\mu\frac{z - aw}{q(z,w)}A, \mu^2\frac{w}{q(z,w)}\Big), \ \tau_p^{-1} = \Big(\widetilde{\mu}\frac{\widetilde{z} - \widetilde{a}\widetilde{w}}{\widetilde{q}(\widetilde{z},\widetilde{w})}\widetilde{A}, \widetilde{\mu}^2\frac{\widetilde{w}}{\widetilde{q}(\widetilde{z},\widetilde{w})}\Big)$$

with $\mu, \widetilde{\mu} \neq 0,\ A, \widetilde{A}$ unitary matrices, $q(0), \widetilde{q}(0) = 1$.

Write $G_p = (h(z,w), w)$. Then

$$F_p(z,0) = \left(\frac{\widetilde{\mu}}{q^*(z)}h(\frac{\mu z}{q(z,0)}A, 0)\widetilde{A}, 0\right), \tag{5.6}$$

for a certain holomorphic function $q^*(z)$ with $q^*(0) = 1$. Now to show that for any $|\alpha| = 4,\ D_z^\alpha F_p(z,0)|_0 \in span_{|\beta|\le 3}\{D_z^\beta F_p(z,0)|_0\}$, it suffices to show that

$$D_z^\alpha h\big(\frac{\mu z}{q(z,0)}A, 0\big)\big|_0 \in \underset{|\beta|\le 3}{span}\big\{D_z^\beta h\big(\frac{\mu z}{q(z,0)}A, 0\big)\big|_0\big\}.$$

Notice that

$$\underset{|\alpha|\le k}{span}\big\{D_z^\alpha h\big(\frac{\mu z}{q(z,0)}A, 0\big)\big|_0\big\} = \underset{|\alpha|\le k}{span}\big\{D_z^\alpha h(z,0)\big|_0\big\}$$

and notice that [by (5.4)]

$$\underset{|\alpha|\le 4}{span}\{D_z^\alpha h(z,0)|_0\} = \underset{|\alpha|\le 3}{span}\{D_z^\alpha h(z,0)|_0\}.$$

We conclude that

$$\underset{|\alpha|\le 4}{span}\{D_z^\alpha F_p(z,0)|_0\} = \underset{|\alpha|\le 3}{span}\{D_z^\alpha F_p(z,0)|_0\}.$$

We thus arrive at a proof for the claim. Moreover, we also conclude from (5.6) that

$$dim\Big(\underset{|\alpha|\le 3}{span}\{D_z^\alpha F_p(z,0)|_0\}\Big) = dim\Big(\underset{|\alpha|\le 3}{span}\{D_z^\alpha G_p(z,0)|_0\}\Big) = n + \#\mathcal{S}_0.$$

Since $\mathcal{L}^\alpha(F_p)|_0 = \mathcal{L}^\alpha(F)(p)$, we get that for $|\alpha| = 4$, $\mathcal{L}^\alpha F(p) \in span_{|\beta|\le 3}\{\mathcal{L}^\beta F(p)\}$. Since $span_{|\beta|\le 3}\{\mathcal{L}^\beta F(p)\}$ has a fixed dimension $n + \#\mathcal{S}_0$ for $p \approx 0$, we can write, for any α, $\mathcal{L}^\alpha F(p)$ as a smooth linear combination of a fixed (smoothly varied) basis from $span_{|\beta|\le 3}\{\mathcal{L}^\beta F(p)\}$. Successively applying $\overline{\mathcal{L}}_j$, $\mathcal{L}_k$ as in the proof of [18, Lemma 4.3] to the so obtained expressions and using the bracket property for such vector fields, we can obtain as in [18] that $D^\alpha F(0) \in span_{|\beta|\le 3}\{D^\beta F(0)\}$ for any multiple index α. Here D^α is the regular total differentiation (not just along the z-directions) of order $|\alpha|$. Thus $F(z,w) \in span_{|\beta|\le 3}\{D^\beta F(0)\}$ by the Taylor expansion for $(z,w) \approx 0$. Now, write as before, $\phi^{(1,1)}(z)w = (e_1^* z_1 + e_2^* z_2)w$. By Theorem 4.1 (2) (3), we see that $span_{|\beta|\le 3}\{D^\beta F(0)\}$ stays in the span of the following vectors:

$$\Big\{(0,\ldots,0,1^{jth},0,\ldots,0), (0,..,0,1),\ (0,0,\ldots,0,\hat{e}_1,0), \\ (0,\ldots,0,\ldots,0,\hat{e}_2,0)\Big\},$$

where $1 \le j \le \big((n-1)+(n-1)+(n-2)\big)+1$. Hence $F(\mathbb{H}_n)$ is contained in a linear subspace with dimension equal to $3n-3+2+1 = 3n$. Hence, we see the proof of Theorem 1.1 in this setting.

Now, if, for a certain $p_0 \approx 0$, the $\phi_{33}^{(3,0)}(z)$ associated with F_{p_0} is not a zero polynomial, then we can consider F_{p_0} instead of F and apply the above argument to conclude the proof of Theorem 5.2. Finally, if after the normalization of F_p to the form as in Corollary 3.4 for any $p \approx 0$, we have $\Phi_1^{(3,0)}(z) \equiv 0$, then a similar method as above shows that $\mathcal{L}^\alpha F(p) \in span_{|\beta|\le 2}\{\mathcal{L}^\beta F(p)\}$ with $dim\Big(span_{|\beta|\le 2}\{\mathcal{L}^\beta F(p)\}\Big) \equiv n + \#\mathcal{S}_0 - 1$. Hence, $F(\mathbb{B}^n)$ is contained in a complex linear subspace of dimension

$$n + (n-1) + (n-2) + 2 = 3n - 1,$$

spanned by

$$\Big\{(0,\ldots,0,1^{jth},0,\ldots,0),(0,..,0,1),\ (0,0,\ldots,0,\hat{e}_1,0),\\ (0,\ldots,0,\ldots,0,\hat{e}_2,0)\Big\},$$

where $1 \le j \le (n-1)+(n-1)+(n-2)$. The proof of Theorem 1.1 is complete now.

Remark A Consider the map defined in (1.1):

$F = \big(z_1,\ldots,z_{n-2},\lambda z_{n-1},z_n,\sqrt{1-\lambda^2}z_{n-1}(z_1,\ldots,z_{n-1},\mu z_n,\sqrt{1-\mu^2}z_n z)\big)$, $\lambda,\mu \in (0,1)$.

The map is apparently a proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^{3n}$. Write $F = (f_1,\ldots,f_{3n})$. We claim that F is not equivalent to a map of the form $(G,0)$. Otherwise, there are complex numbers $\{a_j\}_{j=1}^{3n}$, not all zeros, such that $\sum_{j=1}^{3n} a_j f_j \equiv 0$, which is obviously impossible just by comparing the coefficients of degree 3, 2, 1, respectively.

The map F is of degree three. It has geometric rank two just by observing that the largest dimension of the affine subspaces where F is linear is of codimension two. (By a result in [19], this codimension is the same as the geometric rank of the map.)

As we discussed above, the span of the first and the second jets has dimension $3n-1$. That means we have one more independent element from the third jet.

Hence $\phi_{33}^{(3,0)} \not\equiv 0$ for such a map (after transforming to the Heisenberg hypersurface and after the normalization) in a generic position.

Remark B We mention that even for $N \ge 3n-2$, there are many rational proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ that are not equivalent to any polynomial maps as shown in a paper by Faran–Huang–Ji–Zhang [13]. The following is one of the examples provided in [13]:

Let $F(z',z_n) = \left(z', z_n z', z_n^2(\frac{\sqrt{1-|a|^2}z'}{1-\bar{a}z_n}, \frac{z_n-a}{1-\bar{a}z_n})\right)$ with $|a|<1$, which is a proper rational holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{3n-2}$. Then F has geometric rank 1 and is linear along each hyperplane defined by $z_n = constant$. F is equivalent to a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^{3n-2}$ if and only if $a=0$.

This example gives an indication that it is unpractical to achieve a precise classification for proper rational proper maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \in \mathcal{I}_3$ to get the gap rigidity.

Remark C This paper is a simplified version of the authors' early preprint. Theorem 1.1 was first announced in [24] (Theorem 2.9 in [24]).

Acknowledgements The major part of the paper was completed when the first two authors were visiting the School of Mathematics and Statistics, Wuhan University, in the summer of 2009. These two authors would like to express their gratitude to this institute for the hospitality during this visit.

References

1. Alexander, H.: Proper holomorphic mappings in $\mathbf{C}^n$. Indiana Univ. Math. J. **26**, 137–146 (1977)
2. Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: Real submanifolds in complex space and their mappings. In: Princeton Mathematical Series, vol. 47. Princeton University Press, Princeton (1999)
3. Baouendi, M.S., Huang, X., Rothschild, L.: Regularity of CR mappings between algebraic hypersurfaces. Invent. Math. **125**, 13–36 (1996)
4. Cao, H., Mok, N.: Holomorphic immersions between compact hyperbolic space forms. Invent. Math. **100**, 49–61 (1990)
5. Catlin, D., D'Angelo, J.: A stabilization theorem for Hermitian forms and applications to holomorphic mappings. Math Res. Lett. **3**, 149–166 (1996)
6. Cima, J., Suffridge, T.J.: A reflection principle with applications to proper holomorphic mappings. Math Ann. **265**, 489–500 (1983)
7. Cima, J., Suffridge, T.J.: Boundary behavior of rational proper maps. Duke Math. J. **60**, 135–138 (1990)
8. D'Angelo, J.P.: Several Complex Variables and the Geometry of Real Hypersurfaces. CRC Press, Boca Raton (1993)
9. D'Angelo, J.P., Lebl, J.: Complexity results for CR mappings between spheres. Int. J. Math. **20**(2), 149–166 (2009)
10. D'Angelo, J.P., Kos, S., Riehl, E.: A sharp bound for the degree of proper monomial mappings between balls. J. Geom. Anal. **13**(4), 581–593 (2003)
11. Faran, J.: Maps from the two ball to the three ball. Invent. Math. **68**, 441–475 (1982)
12. Faran, J.: On the linearity of proper holomorphic maps between balls in the low codimensional case. J. Diff. Geom. **24**, 15–17 (1986)
13. Faran, J., Huang, X., Ji, S., Zhang, Y.: Polynomial and rational maps between balls. Pure Appl. Math. Quart **6**(3), 829–842 (2010)
14. Forstneric, F.: Extending proper holomorphic mappings of positive codimension. Invent. Math. **95**(1), 31–62 (1989)
15. Forstneric, F.: A survey on proper holomorphic mappings. In: Proceeding of Year in SCVs at Mittag-Leffler Institute, Math. Notes 38. Princeton University Press, Princeton (1992)
16. Hakim, M., Sibony, N.: Fonctions holomorphes bornées sur la boule unité de $\mathbf{C}^n$. Invent. Math. **67**(2), 213–222 (1982)
17. Hamada, H.: Rational proper holomorphic maps from B^n into B^{2n}'. Math. Ann. **331**(no.3), 693–711 (2005)
18. Huang, X.: On a linearity problem for proper holomorphic maps between balls in complex spaces of different dimensions. J. Differ. Geom. **51**, 13–33 (1999)
19. Huang, X.: On a semi-rigidity property for holomorphic maps. Asian J. Math. **7**(4): 463–492 (2003) (A special issue in honor of Professor Y-T Siu's 60th birthday)
20. Huang, X.: On some problems in several complex variables and Cauchy-Riemann Geometry. In: Yau, S. T., Yang L. (eds.) Proceedings of the first ICCM, December, 1998. AMS/IP Studies in Advanced Mathematics, vol. 20, pp. 383–396 (2001) (http://www.math.rutgers.edu/~huangx/iccm1998.pdf)
21. Huang, X., Ji, S.: Mapping B^n into $\mathbb{B}^{2n-1}$. Invent. Math. **145**(no. 2), 219–250 (2001)
22. Huang, X., Ji, S., Xu, D.: A new gap phenomenon for proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^N$. Math Res. Lett. **3**(4), 515–529 (2006)
23. Huang, X., Ji, S., Xu, D.: Several results for holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$. Geometric analysis of PDE and several complex variables, 267292, Contemp. Math., 368. Am. Math. Soc., Providence (2005)
24. Huang,X., Ji, S., Yin, W.: Recent progress on two problems in several complex variables. In: Proceedings of International Congress of Chinese Mathematicians 2007, vol. I, pp. 563–575. International Press (2009) (http://www.math.rutgers.edu/~huangx/iccm2007.pdf)
25. Lamel, B., Mir, N.: Parametrization of local CR automorphisms by finite jets and applications. J. Am. Math. Soc. **20**(2), 519–572 (2007)
26. Lebl, J., Peters, H.: Polynomials constant on a hyperplane and CR maps of spheres. Ill. J. Math. (2013, to appear)
27. Low, E.: Embeddings and proper holomorphic maps of strictly pseudoconvex domains into polydiscs and balls. Math Z. **190**(3), 401–410 (1985)
28. Meylan, F., Mir, N., Zaitsev, D.: Holomorphic extension of smooth CR-mappings between real-analytic and real-algebraic CR-manifolds. Asian J. Math. **7**(4), 493–509 (2003)

29. Mir, N.: Analytic regularity of CR maps into spheres. Math. Res. Lett. **10**(4), 447–457 (2003)
30. Mok, N.: Metric rigidity theorems on Hermitian locally symmetric manifolds. In: Series in Pure Mathematics, vol. 6. World Scientific Publishing Co., Inc., Teaneck (1989)
31. Poincaré, H.: Les fonctions analytiques de deux variables et la représentation conforme. Ren. Cire. Mat. Palermo, II. Ser. **2**3, 185–220 (1907)
32. Putinar, M.: Sums of Hermitian squares: old and new. In: Bleckherman, G., Parrilo, P., Thomas, R. (eds.) Semidefinite optimization and convex algebraic geometry. SIAM, Philadelphia (2013, to appear)
33. Quillen, D.G.: On the representation of hermitian forms as sums of squares. Invent. Math. **5**, 237–242 (1968)
34. Stensones, B.: Proper maps which are Lipschitz α up to the boundary. J. Geom. Anal. **6**(2), 317–390 (1996)
35. Webster, S.: On mapping an n-ball into an (n+1)-ball in the complex space. Pac. J. Math. **81**, 267–272 (1979)