$\overline{\partial}$-equation on a lunar domain with mixed boundary conditions

Xiaojun Huang* Xiaoshan Li†

Abstract

In this paper, making use of the method developed by Catlin, we study the L^2-estimate for the $\bar{\partial}$-equation on a lunar manifold with the mixed boundary conditions.

Keywords and Phrases: $\overline{\partial}$-operator, L^2-estimate, $\overline{\partial}$-Dirichlet boundary condition, $\overline{\partial}$-Neumann boundary condition.

Contents

*Supported in part by NSF-1101481

†Supported by the China Scholarship Council and the Fundamental Research Fund for the Central Universities

1 Introduction

In this paper, we study the $\overline{\partial}$-equation for $(0,q)$-forms on a special type of non-smooth domain $S_\varphi^{\epsilon_0}$, called a lunar domain, in a complex manifold with mixed boundary conditions. The domain we are considering here has two pieces of the boundaries M_0 and M_1 intersecting highly tangentially along a smooth real-submanifold E. We assume that M_0 has at least $(q+1)$-positive Levi eigenvalues or $(n-q+1)$-negative Levi eigenvalues. Assume M_1 has the opposite property for the Levi eigenvalues as that for M_0.

We impose the $\overline{\partial}$-Dirichlet boundary condition on M_0 and the $\overline{\partial}$-Neumann boundary condition on M_1. We introduce a Hermitian metric over $S_\varphi^{\epsilon_0}$ such that E can be treated as the infinity of $S_\varphi^{\epsilon_0}$. We will establish an L^2-estimate and derive a Hodge-type decomposition theorem in this setting.

$\overline{\partial}$-equations over such a special type of non-smooth domains, with mixed boundary conditions, are of fundamental importance in understanding many geometric problems. In the deep papers of Catlin [Cat], Cho [Cho] and Catlin-Cho [CC], such equations played a crucial role for studying various extension problems for CR structures, which are directly linked to the local embedding problem of abstract CR manifolds with certain signature conditions. In a paper of Huang-Luk-Yau [HLY], solving such a $\overline{\partial}$-equation for $(0,2)$-forms also played an important role for the study of various deformation problems for compact strongly pseudoconvex CR manifolds of at least five dimension. In the work of Catlin [Cat], Catlin-Cho [CC] and Cho [Cho], the domain encountered is only assumed to be sitting in an almost complex manifold. However, the domain is uniformly scaled such that it is sufficiently close to M_0. In this specific setting, Catlin proved that there is no cohomology obstruction for solving the $\overline{\partial}$-equations. Other related studies for the $\bar{\partial}$-Dirichlet problem can be found in the work of Chakrabarti-Shaw [CSh].

In this paper, we will study the above mentioned $\overline{\partial}$-equation, with the mixed boundary conditions, without any scaling of the domain. Then one does not expect the $\overline{\partial}$-equation is always solvable. However, we will show that the obstruction is of finite dimension. Though we basically follow the approach of Catlin [Cat], one key point in our paper is that we use the property close to the non-smooth corner near E for a different weighted metric to avoid the difficulty which was circumvented in [Cat], [Cho] only by uniformly shrinking the lunar domain $S_{\varphi}^{\epsilon_0}$ toward M_0.

$\overline{\partial}$-equations with various boundary conditions are the basic tools to work on many geometric or analytic problems in Several Complex Variables and Complex Geometry. There is a vast amount of work done in the literature. Here, we only refer the reader to the books by Folland-Kohn [FK], Hörmander [Ho2], Demailly [DE] and Chen-Shaw [CS], as well as, many references therein.

2 Basic set-up and statement of the main theorem

Let M be a smooth hypersurface of real dimension $2n-1 (n \geq 3)$ in a complex manifold X of real dimension $2n$. Let $\varphi \in C^{\infty}(M)$ be a function such that $d\varphi(x) \neq 0$ when $\varphi(x) = 0$. Write $K = \overline{\{x \in M : \varphi(x) > 0\}}$. Assume $K \subset\subset M$ is bounded domain in M with smooth boundary $E = \{x \in M : \varphi(x) = 0\}$.

For a sufficiently small $\epsilon_0 > 0$, define $M^{\epsilon_0} = \{x \in M : |\varphi(x)| < \epsilon_0\}$. Suppose that there exists a tubular neighborhood $\mathcal{N}_{\epsilon_0}$ of M^{ϵ_0} in X and a C^{∞} map Φ such that $\Phi : \mathcal{N}_{\epsilon_0} \to M^{\epsilon_0} \times (-2, 2)$ is a diffeomorphism. Write $\Omega_{\epsilon_0} = M^{\epsilon_0} \times (-2, 2)$, $\mathcal{L} = \Phi_*(T^{1,0}\mathcal{N}_{\epsilon_0})$, where $T^{1,0}\mathcal{N}_{\epsilon_0}$ is the holomorphic tangent bundle of $\mathcal{N}_{\epsilon_0}$. Then $(\Omega_{\epsilon_0}, \mathcal{L})$ is a complex manifold biholomorphic to $(\mathcal{N}_{\epsilon_0}, T^{1,0}\mathcal{N}_{\epsilon_0})$. Also $(M^{\epsilon_0} \times \{0\}, \mathcal{L}|_{(M^{\epsilon_0}} \times \{0\}) \cap \mathbb{C}T(M_0 \times \{0\})$ is a CR hypersurface in $(\Omega^{\epsilon_0}, \mathcal{L})$. Identify $M^{\epsilon_0} \times \{0\}$ with M^{ϵ_0}. Write $S = \mathcal{L}|_{M^{\epsilon_0}} \cap \mathbb{C}TM^{\epsilon_0}$ which is the CR bundle of M^{ϵ_0}. In what follows, when there is no risk of causing confusion, we identify $\mathcal{N}_{\epsilon_0}$ with Ω_{ϵ_0} and objects defined over $\mathcal{N}_{\epsilon_0}$ with those corresponding ones over Ω_{ϵ_0}.

Define $r(x,t) = t\varphi^{-4}(x)$. Assume $S_{\varphi}^{\epsilon_0}$ is a bounded domain in X with two pieces of connected boundaries $M_0 := M \cap \{\varphi > 0\}$ and M_1, whose closures intersect M tangentially along E. Moreover, $S_{\varphi}^{\epsilon_0} \cap \mathcal{N}_{\epsilon_0} := \{(x,t) \in \Omega_{\epsilon_0} | \varphi(x) > 0, -1 < r(x,t) < 0\}$ and $M_1 \cap \mathcal{N}_{\epsilon_0} = \{(x,t) | \varphi(x) > 0, r(x,t) = -1\}$.

Equip X with a Hermitian metric. For any $x_0 \in M_0$ or $x_0 \in M_1$, let $\{L_j\}_{j=1}^{n}$ be a smooth orthornormal basis of the cross sections of $T^{(1,0)}(W(x_0))$, where $W(x_0)$ is sufficiently small neighborhood of x_0 in the ambient space. Let $\{\omega_j\}_{j=1}^n$ be its dual frame. Assume that L_j are tangent to M or M_1, when restricted to M_0 or M_1, for $j \neq n$, respectively. For a $(0,q)$-form with $0 < q \leq n$

$$U = \sum_{j_1 < j_2 < \cdots < j_q} U^{j_1 j_2 \cdots j_q} \overline{\omega_{j_1}} \wedge \cdots \wedge \overline{\omega_{j_q}}$$

defined in the side of $W(x_0) \cap M_0$ or of $W(x_0) \cap M_1$, which is inside $S_\varphi^{\epsilon_0}$, that is smooth up to M_0 or M_1. We say U satisfies the $\overline{\partial}$-Dirichlet condition along M_0 if $U_J|_{M_0} \equiv 0$ whenever $J = (j_1, \cdots, j_q)$ with $j_q \neq n$. We say U satisfies the $\overline{\partial}$-Neumann condition along M_1 if $U_J|_{M_1} \equiv 0$ when $j_q = n$. Apparently, this definition is independent of the choice of the Hermitian metric over X. Indeed, one only needs a smooth Hermitian metric over $\overline{S_\varphi^{\epsilon_0}} \setminus E$ to define the Dirichlet or Neumann boundary conditions along M_0 and M_1.

Following Catlin in [Cat], we write $\mathcal{E}_c^{(0,q)}$ for the collection of smooth $(0,q)$-forms with compact support in $\overline{S_\varphi^{\epsilon_0}} \setminus E$. Write $\mathcal{B}_+^q(S_\varphi^{\epsilon_0})$ for the subset of $\mathcal{E}_c^{(0,q)}$, whose elements satisfy $\overline{\partial}$-Dirichlet boundary condition along M_0. Write $\mathcal{B}_-^q(S_\varphi^{\epsilon_0})$ for the subset of $\mathcal{E}^{(0,q)}$ whose elements satisfy the $\overline{\partial}$-Neumann boundary condition along M_1. Now, we will use the specific Hermitian metric over $S_\varphi^{\epsilon_0}$ to be defined in (3.4) of Section 3, which is smooth up to the boundary $M_0 \cup M_1 \setminus E$ and blows up at a suitable rate when approaching their intersection E. We define $L^2_{(0,q)}(S_\varphi^{\epsilon_0})$ to be the space of $(0,q)$-forms with coefficients being L^2-integrable with respect to this metric. We extend the $\overline{\partial}$-operator to the L^2-space in the following way:

We say that $U \in L^2_{(0,q)}(S_\varphi^{\epsilon_0})$ is in the domain of the operator T with $TU = F$ if for any $V \in \mathcal{B}_-^{q+1}(S_\varphi^{\epsilon_0})$, we have $(U, \overline{\partial}' V) = (F, V)$, where $\overline{\partial}'$ is the standard formal adjoint operator of $\overline{\partial}$ with respect to this specific Hermitian metric. Similarly, we define $S:\ L^2_{(0,q-1)}(S_\varphi^{\epsilon_0}) \to L^2_{(0,q)}(S_\varphi^{\epsilon_0})$ and let T^* and S^* be their Hilbert adjoints. We define $Q(U,U) = \|TU\|^2 + \|S^*U\|^2$ to be the Q-norm associated with the operators T and S^*.

In [Cat] and [Cho], to study the extension of CR structure of M, the authors obtained a standard L^2-estimate with respect to the $\overline{\partial}$-operator with mixed boundary condition when the thickness of $S_\varphi^{\epsilon_0}$ is sufficiently small. (See [Corollary 7.10, Cat]).

In this paper, We consider the L^2-estimate with respect to a $\overline{\partial}$-operator with mixed boundary conditions. However, the thickness of $S_\varphi^{\epsilon_0}$ can be arbitrary. Define $N^+(K)$ (respectively, $N^-(K)$) to be the largest $m \geq 0$ such that the Levi form has at least m positive (respectively, negative) eigenvalues at each $x \in K$ with respect to the domain $S_\varphi^{\epsilon_0}$. Define $N^+(M_1)$ (respectively, $N^-(M_1)$) has the same meaning as for $N^+(K)$ (respectively, for $N^-(K)$) with respect to $S_\varphi^{\epsilon_0}$, too. Then our main theorem is the following:

Theorem 2.1. *Assume the above notations and definitions. If either we have $N^+(K) \geq q+1$ and $N^-(M_1) \geq q+1$ or we have $N^-(K) \geq n-q+1$ and $N^+(M_1) \geq n-q+1$. Then there exists a neighborhood $V_{c,0,1}$ of the boundary of $S_\varphi^{\epsilon_0}$ in $\overline{S_\varphi^{\epsilon_0}}$ and a constant $C > 0$ such that for any $U \in L^2_{(0,q)}(S_\varphi^{\epsilon_0})$ with $U \in Dom(T) \cap Dom(S^*)$, it holds that*

$$\int_{V_{c,0,1}} |U|^2 dV \leq C \left(Q(U,U) + \int_F |U|^2 dV \right), \tag{2.1}$$

where F is a certain compact subset of $S_\varphi^{\epsilon_0}$ independent of U.

Corollary 2.2. *Write $H^{(0,q)}(S_\varphi^{\epsilon_0})$ for the quotient space N_T/R_S with $N_T = \{U : U \in L^2_{(0,q)}(S_\varphi^{\epsilon_0}),\ TU = 0\}$ and R_S the image of the operator S. Then $H^{(0,q)}(S_\varphi^{\epsilon_0}) = N_T/R_S$ is of finite dimension.*

3 Existence of the special frames on $S_\varphi^{\epsilon_0}$ near E

For the proof of Theorem 2.1, we follow the approach in Catlin [Cat] and Catlin-Cho [CC]. However, we need to choose a different weight of blowing up for the metric near the singular set E of the boundary to deal with the difficulty caused by not shrinking the thickness of the lunar domain. This also requires the modification for the choice of the special frame to study the L^2-estimates later. For the convenience of the reader, we give a detailed exposition on the choice of the frame in this section.

Let M_{t_0} near E be defined by the defining equation $t = t_0$. Define $\eta = -\frac{1}{2}(i\partial t - i\overline{\partial} t)$ near E. Then η is a real-valued 1-form and is a contact form along each M_t near E. Let X_0 be a real-valued smooth vector field tangent to M near E such that $(\eta, X_0) = 1$ over M near E. Extend X_0 to a neighborhood of E in Ω_{ϵ_0}, independent of t, and scale X_0 if needed. Then we can get a real-valued smooth vector field X_0 in a neighborhood of E in Ω such that near E $(\eta, X_0) = 1$ and $X_0(t) \equiv 0$.

We assume, without loss of generality, that the Levi form of M_0 is defined by $\sqrt{-1}\eta([X_1, \overline{X_2}])$, $X_1, X_2 \in S$. Write $S_{(x,t)}$ for the subspace of $\mathcal{L}_{(x,t)}$ that are tangent to M_t near E. Set

$Y_0 = -J_{\mathcal{L}}(X_0)$, so that $X_0 + \sqrt{-1}Y_0$ is a section of $\mathcal{L}$ that is transversal to the level set t. Let $G : \Omega_{\epsilon_0} \cap O(E) \to \Omega_{\epsilon_0} \cap O(E)$ be a diffeomorphism such that G fixes $M \cap O(E)$ and

$$G_* Y_0|_{(x,0)} = \frac{\partial}{\partial t}\Big|_{(x,0)}, \quad x \in M \cap O(E).$$

Here we write $O(E)$ for a small neighborhood of E in Ω_{ϵ_0}. Since $dt(J_{\mathcal{L}}(X_0))$ always has the same sign (If not, $X_0 + \sqrt{-1}Y_0$ is a section that is tangent to the level set), we may assume that $dt(J_{\mathcal{L}}(X_0)) < 0$, thus $dt(Y_0) > 0$ along M_0. Hence G preserves the sides of M_0. Then $\widetilde{Z} = -\sqrt{-1}G_*(X_0 + \sqrt{-1}Y_0)$ is a global section of $S_\varphi^{\epsilon_0}$ near E such that along M_0

$$\widetilde{Z} = -\sqrt{-1}X_0 + \frac{\partial}{\partial t}. \tag{3.1}$$

We write $\widetilde{Z} = \widetilde{X} + g(x,t)\frac{\partial}{\partial t}$, where $\widetilde{X}t \equiv 0$. Then we set $Z_n = X + \frac{\partial}{\partial t}$ near E with $X = g^{-1}(x,t)\widetilde{X}$.

We define another subbundle of $\mathcal{L}$ on $S_\varphi^{\epsilon_0}$ by setting

$$\mathcal{R}_{(x,t)} = \{L \in \mathcal{L}_{(x,t)} : Lr = 0,\ r = t\varphi^{-4}(x)\}. \tag{3.2}$$

Clearly, the map defined by

$$H(L) = L - L(r)(Z_n r)^{-1} Z_n,\ L \in S_{(x,t)}, \tag{3.3}$$

defines an isomorphism of $\mathcal{S} := \cup_{(x,t)\approx E} S_{(x,t)}$ onto $\mathcal{R} := \cup_{(x,t)\approx E} R_{(x,t)}$, where

$$Z_n(r) = \varphi(x)^{-4}\left(1 + (-4t)\varphi(x)^{-1}X\varphi(x)\right).$$

We fix a smooth Hermitian metric $<,>_0$ on $\overline{S_\varphi^{\epsilon_0}}$ that is induced from the Hermitian metric on X such that on Ω_{ϵ_0} we have $< Z_n, Z_n >_0 = 1$ near E. We define a new Hermitian metric $<,>$ on $\overline{S_\varphi^{\epsilon_0}} \setminus E$ such that near E we have the following relations :

$$\begin{aligned} < H(L_1), H(L_2) > &= \varphi^{-4+\lambda}(x) < L_1, L_2 >_0, L_1, L_2 \in \mathcal{S} \\ < H(L_1), Z_n > &= 0, L_1 \in \mathcal{S} \\ < Z_n, Z_n > &= \varphi^{-8+2\lambda}(x) \end{aligned} \tag{3.4}$$

where λ is a constant with $0 < \lambda < \frac{1}{2}$. We now show that $\overline{S_\varphi^{\epsilon_0}} \setminus E$ near E can be covered by special coordinate systems such that on each chart there is an orthonormal frame of $\mathcal{L}$ that satisfies good estimates. This is fundamentally important for it then helps to reduce the non-compact situation to more or less the compact situation. Comparing with the weight in [Cat], we add $\varphi^\lambda(x)$ to take care of the trouble created from the corner near E.

Proposition 3.1. *For any $x_0 \in M_0$ with $0 < \varphi(x_0) \ll 1$, there exists a neighborhood $W(x_0) \subset \overline{S_\varphi^{\epsilon_0}} \setminus E$ with the following properties:*

(i) On $W(x_0)$, there are smooth coordinates $y_1, \ldots, y_{2n}$ so that

$$W(x_0) = \left\{ y : |y'| < \sigma_0, -\varphi^\lambda(x_0) \le y_{2n} \le 0 \right\}, \tag{3.5}$$

where σ_0 is a constant independent of x_0 to be determined later. Also $y' = (y_1, \ldots, y_{2n-1})$ is independent of t and $y_{2n} = t\varphi^{-4}(x)\varphi^\lambda(x_0)$. $M_0 \cap W(x_0)$ and $M_1 \cap W(x_0)$ correspond to points in $W(x_0)$ with $y_{2n} = 0$ and $y_{2n} = -\varphi^\lambda(x_0)$, respectively. Moreover, the point x_0 corresponds to the origin.

(ii) On $W(x_0)$, there exists a smooth orthonormal frame $L_1, \cdots, L_n$ for $\mathcal{L}$ such that if $\omega^1, \cdots, \omega^n$ are the dual frame, and if L_k and ω^k are written as $\sum_{j=1}^{2n} b_{kj} \frac{\partial}{\partial y_j}$ and $\sum_{j=1}^{2n} d_{kj} dy_j$, respectively, then

$$\sup_{y \in W(x_0)} \left\{ |D_y^\alpha b_{kj}(y)| + |D_y^\alpha d_{kj}(y)| \right\} \le C_{|\alpha|}, \quad b_{k2n} = 0 \text{ for } k \ne n. \tag{3.6}$$

where $C_{|\alpha|}$ is independent of x_0, j, k.

(iii) If $d_1 \le d_2 \le \cdots \le d_{n-1}$ are the eigenvalues of the Levi form $\sqrt{-1}\eta([L_1, \overline{L_2}])$ at x_0, then at every point $x \in W(x_0)$, we have the following estimates, which are uniformly on x_0:

$$\begin{aligned} \omega^n([L_i, \overline{L_j}])(y) &= O(\sigma_0), i \ne j, i, j < n, \\ \omega^n([L_j, \overline{L_j}])(y) &= d_j + O(\sigma_0), j < n. \end{aligned} \tag{3.7}$$

Here we use $O(\sigma_0)$ to denote the terms which are bounded by $C\sigma_0$ where C is a constant independent of x_0 and σ_0.

(iv) For each sufficiently small σ_0, there is a countable family $\{W(x_\alpha)\}$ such that it covers a fixed neighborhood of $K \setminus E$ in K near E and for any point $p \in K \setminus E$ in this neighborhood,

there are at most N_0 elements from this family that contain p. Here N_0 is independent of the choice of p and σ_0. Moreover, for each α, there is a function $\xi_\alpha \in C_0^\infty(W(x_\alpha)\cap M)$ such that $\sum_\alpha \xi_\alpha^2 \equiv 1$ and the differentiation of ξ_α with respect to the y'-coordinates is bounded by C/σ_0 with C a fixed constant independent of x_α and σ_0.

Proof. The proof of this proposition is similar to that in [Cat], though adding a new scale $\varphi(x_0)^\lambda$ requires modifications. For convenience of the reader, we include all the details. First, there exists a finite number of coordinate charts $V'_v, v = 1, \cdots, N$ in M that cover K near E in M such that on each V'_v, there exists coordinates $(x_1, \ldots, x_{2n-1})$ with $\frac{\partial}{\partial x_{2n-1}} = -X_0$ at all points in V'_v. Also, V'_v is defined by $|x'| < \epsilon_1$ for a certain fixed small $\epsilon_1 > 0$. Define $V_v := V'_v \times [0, -1]$ and set on V_v, $x_{2n} = t$, $x_k(x', t) = x_k(x')$, $k < 2n$ for $x' \in V'_v$. We can assume that there exists an orthonormal frame $\{L_i^v\}_{i=1}^{n-1}$ of $\mathcal{S}$ with respect to the former fixed Hermitian metric $<,>_0$ of $\mathcal{L}$ in V_v. Let $L_n^v = Z_n$. For any point $x_0 \in M$ with $0 < \varphi(x_0) << 1$, by the Lesbeque covering lemma, we can assume that $x_0 \in V_v$ for a certain v with $|x'(x_0)| < \epsilon_2$, where $0 < \epsilon_2 < \epsilon_1$ is independent of x_0. We can define an affine transformation $C_{x_0}^v : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ so that if $(x'_0, 0) \in \mathbb{R}^{2n}$ is the coordinates of x_0, then

$$C_{x_0}^v(x', x_{2n}) = (P_{x_0}(x' - x'_0), x_{2n}), \tag{3.8}$$

where P_{x_0} is a $(2n-1)\times(2n-1)$ constant matrix such that in the new coordinates $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_{2n})$, we have

$$\begin{aligned} L_k^v|_{x_0} &= \frac{\partial}{\partial \tilde{x}_{2k-1}}\Big|x_0 - \sqrt{-1}\frac{\partial}{\partial \tilde{x}_{2k}}\Big|_{x_0}, (1 \le k \le n-1), \\ X_0\Big|_{x_0} &= -\frac{\partial}{\partial \tilde{x}_{2n-1}}\Big|_{x_0}. \end{aligned} \tag{3.9}$$

Also the domain where $\widetilde{x'}$ is defined contains a fixed ball centered at the origin for any choice of x_0. Notice that the second equality in (3.9) implies that $X_0|_{(x',0)} = -\frac{\partial}{\partial \tilde{x}_{2n-1}}|_{(x',0)}$ at all points of $M \cap V_v$. Hence, along $M \cap V_v$,

$$L_n^v\ |_{(x',0)} = \sqrt{-1}\frac{\partial}{\partial \tilde{x}_{2n-1}}\Big|_{(x',0)} + \frac{\partial}{\partial \tilde{x}_{2n}}\Big|_{(x',0)}. \tag{3.10}$$

We now define a new coordinates $y = (y_1, \cdots, y_{2n})$ by means of a dilation map $D_{x_0} : \mathbb{R}^{2n} \to$

$\mathbb{R}^{2n}$. Set

$$\begin{aligned} y &= D_{x_0}(\tilde{x}) \\ &= \left(\varphi^{-2+\frac{\lambda}{2}}(x_0)\tilde{x}_1, \cdots, \varphi^{-2+\frac{\lambda}{2}}(x_0)\tilde{x}_{2n-2}, \varphi^{-4+\lambda}(x_0)\tilde{x}_{2n-1}, \varphi^{\lambda}(x_0)\varphi^{-4}(x)\tilde{x}_{2n}\right). \end{aligned} \tag{3.11}$$

In terms of the y-coordinates, we define an open set $W(x_0)$ by

$$W(x_0) = \left\{x \in V_v \cap S_\varphi^{\epsilon_0} : |y_k(x)| < \sigma_0, k = 1, \ldots, 2n-1, -\varphi^\lambda(x_0) \le y_{2n} \le 0\right\}. \tag{3.12}$$

When $0 < \varphi(x_0) << 1$, one can apparently find a fixed small number $\sigma_0' > 0$ such that every $W(x_0)$ is contained in some V_v whenever $\sigma_0 < \sigma_0'$. Notice that in $W(x_0)$, the set where $y_{2n} = 0$ and $y_{2n} = -\varphi^\lambda(x_0)$ coincides with the set where $r(x,t) = 0$ and $r(x,t) = -1$, respectively, which represents the two boundaries of $S_\varphi^{\epsilon_0}$.

Define a frame $L_1, \cdots, L_n$ on $W(x_0)$ by setting

$$\begin{aligned} L_k &= \varphi^{2-\frac{\lambda}{2}}(x)\left(L_k^v - r_k(x)L_n^v\right) = \varphi^{2-\frac{\lambda}{2}}(x)H(L_k^v), k < n, \\ L_n &= \varphi^{4-\lambda}(x)L_n^v, \end{aligned} \tag{3.13}$$

where

$$r_k = (L_k^v r)(L_n^v r)^{-1}. \tag{3.14}$$

Then $\{L_k\}_{k=1}^n$ forms an orthonormal frame on $W(x_0)$ with respect to the scaled Hermitian metric, and $\{L_k\}_{k=1}^{n-1}$ forms an orthonormal basis for $\mathcal{R}$.

If we write L_k^v in terms of the $\tilde{x}$-coordinates corresponding to x_0 as

$$\begin{aligned} L_k^v &= \sum_{l=1}^{2n-1} e_{kl}(\tilde{x})\frac{\partial}{\partial \tilde{x}_l}, k < n, \\ L_n^v &= \frac{\partial}{\partial \tilde{x}_{2n}} + \sum_{l=1}^{2n-1} e_{nl}(\tilde{x})\frac{\partial}{\partial \tilde{x}_l}, \end{aligned} \tag{3.15}$$

and if we set

$$\begin{aligned} E_{kl} &= e_{kl} \circ D_{x_0}^{-1}(y), R_k = r_k \circ D_{x_0}^{-1}(y), \varphi_l = \frac{\partial \varphi}{\partial \tilde{x}_l} \\ \Phi &= \varphi \circ D_{x_0}^{-1}(y), \Phi_l = \varphi_l \circ D_{x_0}^{-1}(y), \end{aligned} \tag{3.16}$$

then a calculation shows that

$$L_k^v - r_k L_n^v = \sum_{l=1}^{2n-1} \left(e_{kl}(\tilde{x}) - r_k(\tilde{x}) e_{nl}(\tilde{x})\right) \frac{\partial}{\partial \tilde{x}_l} - r_k(\tilde{x}) \frac{\partial}{\partial \tilde{x}_{2n}} \tag{3.17}$$

and that the Jacobian matrix $\mathrm{J}ac(D_{x_0})$ of D_{x_0} is

$$\begin{bmatrix} \varphi^{-2+\frac{\lambda}{2}}(x_0) & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \varphi^{-2+\frac{\lambda}{2}}(x_0) & 0 & 0 \\ 0 & \cdots & 0 & \varphi^{-4+\lambda}(x_0) & 0 \\ -4\varphi_1 \varphi^{-5} \varphi^{\lambda}(x_0) \tilde{x}_{2n} & \cdots & -4\varphi_{2n-2} \varphi^{-5} \varphi^{\lambda}(x_0) \tilde{x}_{2n} & -4\varphi_{2n-1} \varphi^{-5} \varphi^{\lambda}(x_0) \tilde{x}_{2n} & \varphi^{\lambda}(x_0) \varphi^{-4} \end{bmatrix} \tag{3.18}$$

We conclude that in the y-coordinates of $W(x_0)$ when $1 \le k \le n-1$,

$$\begin{aligned} L_k = & \sum_{l=1}^{2n-2} \frac{\Phi^{2-\frac{\lambda}{2}}(y)}{\varphi^{2-\frac{\lambda}{2}}(x_0)} \left(E_{kl}(y) - R_k E_{nl}(y)\right) \frac{\partial}{\partial y_l} \\ & + \frac{\Phi^{2-\frac{\lambda}{2}}(y)}{\varphi^{4-\lambda}(x_0)} (E_{k,2n-1}(y) - R_k E_{n,2n-1}(y)) \frac{\partial}{\partial y_{2n-1}}, \\ L_n = & \sum_{l=1}^{2n-2} \frac{\Phi^{4-\lambda}(y)}{\varphi^{2-\frac{\lambda}{2}}(x_0)} E_{nl}(y) \frac{\partial}{\partial y_l} + \frac{\Phi^{4-\lambda}(y)}{\varphi^{4-\lambda}(x_0)} E_{n,2n-1}(y) \frac{\partial}{\partial y_{2n-1}} \\ & + \left(\frac{\varphi^{\lambda}(x_0)}{\Phi^{\lambda}(y)} - 4 \sum_{l=1}^{2n-1} \Phi^{3-\lambda} \Phi_l y_{2n} E_{nl}(y) \right) \frac{\partial}{\partial y_{2n}}. \end{aligned} \tag{3.19}$$

Here

$$R_k = \frac{\sum_{l=1}^{2n-1} (-4) E_{kl}(y) \frac{\Phi^3}{\varphi^{\lambda}(x_0)} \Phi_l(y) y_{2n}}{1 + \sum_{l=1}^{2n-1} (-4) E_{nl}(y) \frac{\Phi^3}{\varphi^{\lambda}(x_0)} \Phi_l(y) y_{2n}}. \tag{3.20}$$

Observe that the diameter in the $\tilde{x}$-coordinates of $W(x_0)$ is of the quantity: $O(\varphi^{2-\frac{\lambda}{2}}(x_0)) \ll \varphi(x_0)$ when $0 < \varphi(x_0) \ll 1$.

Let $f \in C^\infty(W(x_0))$, we define

$$|f|_{m,W(x_0)} = \sup_{y \in W(x_0)} \{|D_y^\alpha f(y)| : |\alpha| \le m\} \tag{3.21}$$

and we can extend this norm to vector fields and 1-forms by using coefficients of $\frac{\partial}{\partial y_j}$ or dy_j. It can be easily verified that

$$\lim_{x_0 \to E} |E_{kl} - b_{kl}|_{m,W(x_0)} = 0, \tag{3.22}$$

where $(b_{kl})_{n\times 2n}$ is a constant matrix given by

$$\begin{aligned} &b_{k,2k-1} = 1, b_{k,2k} = -\sqrt{-1}, k = 1, \ldots, n-1, \\ &b_{n,2n-1} = \sqrt{-1}, b_{n,2n} = 1, \end{aligned} \tag{3.23}$$

and $b_{kl} = 0$ in all other cases. Since $[\frac{\partial \tilde{x}_k}{\partial y_l}]_{2n\times 2n} = \varphi^{2-\frac{\lambda}{2}}(x_0)[O(1)]$ on $W(x_0)$ when x_0 near E,

$$\begin{aligned} \Phi^{2-\frac{\lambda}{2}}(y) &= \varphi^{2-\frac{\lambda}{2}}(x_0) + (2-\frac{\lambda}{2})\Phi^{1-\frac{\lambda}{2}} \sum_{k,l=1}^{2n} D_{\tilde{x}_k}\varphi(\theta)\frac{\partial \tilde{x}_k}{\partial y_l} y_l \\ &= \varphi^{2-\frac{\lambda}{2}}(x_0) + \varphi^{2-\frac{\lambda}{2}}(x_0)o(1) \\ &= \varphi^{2-\frac{\lambda}{2}}(x_0)(1+o(1)) \\ &= \varphi^{2-\frac{\lambda}{2}}(x_0)O(1). \end{aligned} \tag{3.24}$$

where $\theta \in W(x_0)$ and $\Phi^{2-\frac{\lambda}{2}}(y)$ uniformly approximate the same quantity of $\varphi^{2-\frac{\lambda}{2}}(x_0)$. Moreover,

$$\begin{aligned} D_{y_l}\left(\frac{\Phi^{2-\frac{\lambda}{2}}(y)}{\varphi^{2-\frac{\lambda}{2}}(x_0)}\right) &= (2-\frac{\lambda}{2})\frac{\Phi^{1-\frac{\lambda}{2}}(y)}{\varphi^{2-\frac{\lambda}{2}}(x_0)}\sum_{k=1}^{2n}(D_{\tilde{x}_k}\varphi)\left(\frac{\partial \tilde{x}_k}{\partial y_l}\right) = \varphi^{1-\frac{\lambda}{2}}(x_0)O(1), \\ D_{y_l}\left(\frac{\varphi^{\lambda}(x_0)}{\Phi^{\lambda}(y)}\right) &= (-\lambda)\varphi^{\lambda}(x_0)\Phi^{-\lambda-1}(y)\sum_{k=1}^{2n}(D_{\tilde{x}_k}\varphi)\left(\frac{\partial \tilde{x}_k}{\partial y_l}\right) = \varphi^{1-\frac{\lambda}{2}}(x_0)O(1). \end{aligned} \tag{3.25}$$

Since $e_{k,2n-1}(x_0) = 0$ when $k < n$, we have

$$\frac{E_{k,2n-1}(y)}{\varphi^{2-\frac{\lambda}{2}}(x_0)} = \frac{e_{k,2n-1}(\tilde{x}) - e_{k,2n-1}(0)}{\varphi^{2-\frac{\lambda}{2}}(x_0)} = O(1). \tag{3.26}$$

Thus we can write $e_{k,2n-1}(\tilde{x}) = l'_k(\tilde{x}) + O(|\tilde{x}|^2)$, where l'_k is a linear function of $\tilde{x}$. It follows that

$$\lim_{x_0\to E}\left|\varphi^{-2+\frac{\lambda}{2}}(x_0)E_{k,2n-1} - l'_k(y_1, \ldots, y_{2n-2}, 0, 0)\right|_{m,W(x_0)} = 0. \tag{3.27}$$

Similarly, by a direct calculation,

$$\frac{R_k(y)}{\varphi^{2-\frac{\lambda}{2}}(x_0)} \to 0 \tag{3.28}$$

when $x_0 \to E, k = 1, \ldots, n-1$.

Combining all the facts above, we conclude that if $k < n$

$$\lim_{x_0 \to E} \left| L_k - \left(\frac{\partial}{\partial y_{2k-1}} - \sqrt{-1} \frac{\partial}{\partial y_{2k}} + l_k \frac{\partial}{\partial y_{2n-1}} \right) \right|_{m, W(x_0)} = 0, \tag{3.29}$$

where $l_k = l'_k(y_1, \ldots, y_{2n-2}, 0, 0)$, and that

$$\lim_{x_0 \to E} \left| L_n - \left(\sqrt{-1} \frac{\partial}{\partial y_{2n-1}} + \frac{\partial}{\partial y_{2n}} \right) \right|_{m, W(x_0)} = 0. \tag{3.30}$$

Write $D = [d_{ij}]_{n \times 2n}$, $B = [b_{ij}]_{n \times 2n}$. Here d_{ij} and b_{ij} are as defined in (ii) of the proposition. Define $\tilde{D} = \begin{bmatrix} D \\ \overline{D} \end{bmatrix}_{2n \times 2n}$, $\tilde{B} = [\ B^t \quad \overline{B}^t\ ]$. Then $\tilde{D} \cdot \tilde{B} = I_{2n \times 2n}$. In order to prove $\{d_{ij}\}_{1 \le i \le n, 1 \le j \le 2n}$ and the derivative of $\{d_{ij}\}_{1 \le i \le n, 1 \le j \le 2n}$ are uniformly bounded, we only need to prove that the absolute value $|det\widetilde{B}|$ of determinant of matrix $\tilde{B}$ has a uniform lower bound. Let $A = [a_{ij}]_{2n \times 2n}$. Here, $a_{2i-1,i} = 1, a_{2i,i} = -\sqrt{-1}, 1 \le i \le n-1, a_{2n-1,n} = \sqrt{-1}, a_{2n,n} = 1, a_{k,n+l} = \overline{a_{k,l}}$ and in other cases $a_{ij} = 0$. Let $C = [1]_{2n \times 2n}$, $E = [e_{kl}]_{2n \times 2n}$, $e_{2n-1,k} = 1, e_{2n-1,k+n} = 1$ and $e_{kl} = 0$ in other cases. Then from (3.29) and (3.30), $\tilde{B} = A + O(\varphi^{2-\frac{\lambda}{2}}(x_0))C + O(\sigma_0)E$. Thus $det(\tilde{B}) = detA \cdot det(I + O(\varphi^{2-\frac{\lambda}{2}}(x_0))A^{-1}C + O(\sigma_0)A^{-1}E)$. Since $|detA| = 2^n \neq 0$, $|det\tilde{B}|$ has a uniform lower bound with respect to $y \in W(x_0)$, when $\varphi(x_0)$ and σ_0 are sufficiently small. This proves (ii).

Since $\{L_i\}_{i=1}^n$ is an orthornormal basis with respect to the scaled Hermitian metric we have defined on $W(x_0)$ near E, and from (3.29), (3.30) we see that the metric tensor and any order of its covariant differentiation on $W(x_0) \cap M$ induced from the Hermitian metric on $\overline{S_\varphi^{\epsilon_0}} \setminus E$ near E must have uniform bounds. It is easy to see that there is a constant $0 < k_0 << 1$ such that any ball in the induced metric over $M_0 \setminus E$ of radius $k_0 \sigma'_0$ is contained in some $W(x_0) \cap M$. From (3.29), (3.30) or by what we argued above, the volume form in $W(x_0) \cap M$ in terms of the y'-coordinates is uniformly bounded from above and from below by a positive constant independent of x_0. We thus see that the volume of the ball $B_{\mu\sigma'_0}$ in M with radius $\mu\sigma'_0$ and $\mu \le k_0$ has the estimate: $C_1(\mu\sigma'_0)^{2n-1} \le \mathrm{Vol}(B_{\mu\sigma'_0}) \le C_2(\mu\sigma'_0)^{2n-1}$ with C_1 and C_2 two positive constants.

Write $W'(x_0) := \{y' : (y', 0) \in W(x_0), |y'| < \sigma_0/2\}$. When σ_0 is smaller than a certain fixed number, we can assume that $W(x_0) \cap M$ is contained in $B_{\mu'\sigma_0'}$ and $W'(x_0)$ contains $B_{\mu''\sigma_0'}$, where $0 < \mu'' < \mu' < k_0$ are constants depending only on σ_0, but not x_0. We can choose a family of $\{W(x_\alpha)\}$ near E such that (1). $\{W'(x_\alpha)\}$ covers $M_0 \setminus E$ near E, (2). The distance between any two centers in the induced metric over $M_0 \setminus E$ is at least $\mu''\sigma_0'$. (The existence of such a family follows from a simple construction based on the Zorn lemma). By the just mentioned volume estimates, one conclude that such a cover is a Besicovitch covering. Namely, there is a constant N_0, independent of σ_0, such that any point is contained in at most N_0-charts. Let $\widetilde{\xi_\alpha} \in C_0^\infty(W(x_\alpha) \cap M)$ be such that $\widetilde{\xi_a} \equiv 1$ over $W'(x_\alpha)$ and the differentiation in the y'-coordinates is bounded by $4/\sigma_0$. Define $\xi_\alpha = \frac{\widetilde{\xi_a}}{\sqrt{\sum_\alpha \widetilde{\xi_a^2}}}$. Then $\xi_\alpha \in C_0^\infty(W(x_\alpha) \cap M)$, $\sum_\alpha \xi_a^2 \equiv 1$ and $|D_{y'}\xi_a| = O(\sigma_0^{-1})$. This proves (iv).

Finally, we note that if L_j^v, $j = 1, \ldots, n-1$, are replaced by

$$X_j^v = \sum_{k=1}^{n-1} U_{jk} L_k^v,$$

where $[U_{jk}]$ is a suitably chosen unitary matrix such that

$$\partial\overline{\partial}t(X_i^v, \overline{X_j^v})|_{x_0} = d_{ij}, 1 \le i, j \le n-1, \tag{3.31}$$

where $d_{ij} = 0, i \ne j; d_{jj} = d_j, j < n$.

Since $r_k = \frac{X_k^v(r)}{X_n(r)} = t\frac{X_k^v \varphi^{-4}(x)}{X_n^v(r)}$, we have $r_k = 0$ along $M \cap W(x_0)$. This shows that along $M \cap W(x_0)$,

$$L_j = \varphi^{2-\frac{\lambda}{2}} X_j^v, j < n;\ \omega^n = \frac{1}{2}\varphi^{-4+\lambda}(x)(dt + \sqrt{-1}\eta) \tag{3.32}$$

Then

$$\begin{aligned}
\omega^n([L_k, \overline{L_n}])(x_0) &= -d\omega^n(L_k, \overline{L_n})(x_0) \\
&= \varphi^{6-\frac{3}{2}\lambda}(x_0)(-d\omega^n)(X_i^v, \overline{Z_n})(x_0) \\
&= -\varphi^{6-\frac{3}{2}\lambda}(x_0)\left\{X_i^v(\omega^n(\overline{Z_n})) - \overline{Z_n}\omega^n(X_i^v) - \omega^n([X_i^v, \overline{Z_n}])\right\}(x_0) \\
&= O(\varphi^{1-\frac{\lambda}{2}}(x_0))
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\omega^n([L_i,\overline{L}_j])(x_0) &= -d\omega^n(L_i,\overline{L}_j)(x_0)\\ &= -d\omega^n\left(\varphi^{2-\frac{\lambda}{2}}(x_0)X_i^v(x_0),\varphi^{2-\frac{\lambda}{2}}(x_0)\overline{X_j^v}(x_0)\right)\\ &= \varphi^{4-\lambda}(x_0)\omega^n([X_i^v,\overline{X_j^v}])(x_0)\\ &= \frac{1}{2}(dt+\sqrt{-1}\eta)([X_i^v,\overline{X_j^v}])(x_0)\\ &= \partial t([X_i^v,\overline{X_j^v}])(x_0)\\ &= \partial\overline{\partial} t(X_i^v,\overline{X_j^v})(x_0).\end{aligned} \tag{3.34}$$

It follows that

$$\omega^n([L_i,\overline{L_j}])(x_0) = d_{ij}, 1\le i,j\le n-1, \tag{3.35}$$

and thus

$$\begin{aligned}&\omega^n([L_i,\overline{L_j}])(y) = O(\sigma_0),\ i\ne j, 1\le i,j\le n-1;\\ &\omega^n([L_j,\overline{L_j}])(y) = d_j + O(\sigma_0),\ j<n\end{aligned} \tag{3.36}$$

in $W(x_0)$. Thus we obtain (iii). The proof of the Proposition 3.1 is complete. □

Let dV denote the volume form associated with the Hermitian metric defined before. In the coordinates $(y_1,\ldots,y_{2n})$ over $W(x_0)$, write $dV = V(y)dy$, where $dy = dy_1\cdots dy_{2n}$. Then we have, as mentioned before, that $V(y)$ satisfies the following:

$$|V(y)|_{1,W(x_0)} \le a_1,\ \inf_{y\in W(x_0)} V(y) > a_2 > 0, \tag{3.37}$$

where a_1, a_2 are constants independent of x_0.

We will define inner product for two functions $g,h\in C_c^\infty(\overline{S_\varphi^{\epsilon_0}}\setminus E)$ by

$$(g,h) = \int g\overline{h}dV.$$

Let N be a submanifold of dimension $2n-1$ in $W(x_0)$ and let ds be the volume form of N that comes from Euclidean metric in $(y_1,\ldots,y_{2n})$-variables. The following is the divergence theorem:

Theorem 3.2. (*Divergence theorem*) *Let $D \subset\subset \mathbb{R}^N$ be a smoothly bounded domain, $g, h, v \in C^\infty(\overline{D})$, $V \neq 0$ on $\overline{D}, dV = V dx, dS = V ds$. If $L = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$ is a smooth vector field,then*

$$\int_D (Lg)\overline{h} dV = -\int_D g\overline{\overline{L}h} dV - \int_D eg\overline{h} dV + \int_{\partial D} g\overline{h} < L, \mathbf{n} > dS, \tag{3.38}$$

where $e = \frac{LV}{V} + \sum_{j=1}^{N} \frac{\partial b_j}{\partial x_j}$, $\mathbf{n}$ is the unit outward normal vector at the boundary points, and $<,>$ is the Euclidean inner product in $\mathbb{R}^N$.

Applying the above divergence theorem to the our situation as in [Cat], one obtains the following: (See [Lemma 5.7, Cat])

Lemma 3.3. *Let $L_1, \dots, L_n$ be the frame constructed in $W(x_0)$, then there exists functions $e_j \in C^\infty(W(x_0))$ and a function $P \in C^\infty(W(x_0))$ such that for all $g, h \in C_c^\infty(W(x_0))$*

$$\begin{aligned} &(L_j g, h) = -(g, \overline{L_j} h) - (e_j g, h), j = 1, \dots, n-1. \\ &(L_n g, h) = -(g, \overline{L_n} h) - (e_n g, h) + \int_{M_0} P g\overline{h} dS - \int_{M_1} P g\overline{h} dS, \end{aligned} \tag{3.39}$$

where $dS = V ds, M_0 = \{z : r(z) = 0\}, M_1 = \{z : r(z) = -1\}$. The real part and imaginary part of the function P satisfy: $0 < c < Re(P(y)) < C$, $|Im(P(y))| \ll 1$ for $y \in W(x_0)$ with $0 < \varphi(x_0) \ll 1$. Here c and C are constants independent of x_0. Moreover, $|e_j|_{m,W(x_0)} \leq C_m$ for $0 < \varphi(x_0) \ll 1$. Here C_m is a constant independent of x_0.

Proof. Applying the divergence theorem above, one can see the above mentioned expressions hold with

$$e_j = \sum_{k=1}^{2n} \frac{\partial b_{jk}}{\partial y_k} + \sum_{k=1}^{2n} b_{jk} \frac{\partial V(y)}{\partial y_k} \frac{1}{V(y)}, \tag{3.40}$$

and

$$P = L_n(y_{2n}) = \frac{\varphi^\lambda(x_0)}{\Phi^\lambda(y)} - 4 \sum_{l=1}^{2n-1} y_{2n} \Phi^{3-\lambda} \Phi_l E_{nl}.$$

Since $\frac{\varphi^\lambda(x_0)}{\Phi^\lambda(y)}$ approaches uniformly to 1 and $\sum_{l=1}^{2n-1} y_{2n} \Phi^{3-\lambda} \Phi_l E_{nl}$ approaches uniformly to zero as x_0 approaches to E, we conclude the proof of the lemma. □

Now, suppose that when $0 < \varphi(x_0) < \varepsilon_1 << 1$, we have constructed the special coordinates and special frame on $W(x_0)$ as in Proposition 3.1. Notice that the subset

$$\mathcal{K}_{\varepsilon_1} := \overline{S_\varphi^{\epsilon_0}} \setminus \left(\left\{x|\varphi(x) < \frac{\varepsilon_1}{2}\right\} \times [-1,0]\right)$$

is compact in $S_\varphi^{\epsilon_0}$. Here, as we mentioned in §2, we identify $\mathcal{N}_{\epsilon_0}$ with Ω_{ϵ_0}. We can then cover $\mathcal{K}_{\varepsilon_1}$ with finitely many coordinate charts.

Write $\mathcal{E}^{0,q}(S_\varphi^{\epsilon_0})$ for the set of smooth $(0,q)$-forms over $S_\varphi^{\epsilon_0}$. And write $\mathcal{E}_c^{0,q}(S_\varphi^{\epsilon_0})$ for the set of smooth $(0,q)$-forms U over $S_\varphi^{\epsilon_0}$ with compact support in $\overline{S_\varphi^{\epsilon_0}} \setminus E$. Let $\mathcal{E}_0^{0,q}(S_\varphi^{\epsilon_0})$ denote the set of smooth sections of $\Gamma^{(0,q)}(S_\varphi^{\epsilon_0})$ with compact support in the interior of $S_\varphi^{\epsilon_0}$. We define the space $L^2_{(0,q)}(S_\varphi^{\epsilon_0})$ of L^2-integral $(0,q)$-forms by using the scaled Hermitian metric.

Suppose $U \in \mathcal{E}_c^{(0,q)}(S_\varphi^{\epsilon_0}), U = \sum\limits_{|J|=q} U_J\overline{\omega^J}$. Then

$$\begin{aligned}\overline{\partial}U &= \sum_{j=1}^{n}\sum_{|J|=q}(\overline{L_j}U^J)\overline{\omega}^j \wedge \overline{\omega}^J + \cdots\,,\\ \overline{\partial}'U &= -\sum_{j=1}^{n}\sum_{|K|=q-1} L_jU^{jK}\overline{\omega^K} + \cdots\,,\end{aligned} \tag{3.41}$$

where $\overline{\partial}'$ is the formal adjoint operator of $\overline{\partial}$ and dots indicate terms where no derivatives of U^J occur. We extend $\overline{\partial}$ and $\overline{\partial}'$ to the L^2-space as in the introduction. Then as in [Cat], define $\mathcal{B}^q$ for a subspace of $\mathcal{E}_c^{(0,q)}(S_\varphi^{\epsilon_0})$, whose elements satisfy the $\overline{\partial}$-Dirichlet condition defined in §2 along M_0 and the $\overline{\partial}$-Neumann condition along M_1. Then $\mathrm{Dom}(T)\cap\mathrm{Dom}(S^*)\cap\mathcal{E}_c^{(0,q)}(S_\varphi^{\epsilon_0}) = \mathcal{B}^q$. Moreover, as in the [Cat] (Lemma 6.4 of [Cat]), the Hörmander-Friderichs smooth lemma also holds in this setting:

Let $U \in \mathrm{Dom}(S^*) \cap \mathrm{Dom}(T)$. Then there exists $U_\mu \in \mathcal{B}^q$ such that

$$\lim_{\mu\to\infty} \|U_\mu - U\| + \|S^*U_\mu - S^*U\| + \|TU_\mu - TU\| = 0. \tag{3.42}$$

Hence, in what follows, we need only to prove the estimate in our main theorem for $U \in \mathcal{B}^q$.

4 The L^2-estimate for the operator T near the corner E

In this section, we establish the estimate near E for forms in $\mathcal{B}^q$. We follow the known procedure to compute the the Q norms as in [Ho1], [FK] and [Cat]. In particular, we follow the computation in [Cat] and make the needed modification to fit our situation here.

We first suppose $U \in \mathcal{B}^q$ with supp U a compact subset of $W(x_0)$ for some $x_0 \in M_0$ with $0 < \varphi(x_0) \ll 1$. Then

$$\begin{aligned} TU = \overline{\partial} U &= \sum_{|J|=q} \sum_{j=1}^{n} (\overline{L_j} U^J) \overline{\omega}^{jJ} + \cdots , \\ S^* U &= - \sum_{|K|=q-1} \sum_{j=1}^{n} L_j U^{jK} \overline{\omega}^K + \cdots . \end{aligned} \tag{4.1}$$

Define

$$Q(U,U) = \|TU\|^2 + \|S^*U\|^2$$

and let

$$\begin{aligned} AU &= \sum_{|J|=q} \sum_{j=1}^{n} (\overline{L_j} U^J) \overline{\omega}^{jJ}, \\ BU &= - \sum_{|K|=q-1} \sum_{j=1}^{n} L_j U^{jK} \overline{\omega}^K . \end{aligned} \tag{4.2}$$

Immediately, we have

$$2\|S^*U\|^2 + 2\|TU\|^2 + C_0\|U\|^2 \geq \|AU\|^2 + \|BU\|^2, \tag{4.3}$$

where C_0 is a constant only depend on the coefficients of L_j and ω_j and independent of x_0 and U.

Notice that

$$\|AU\|^2 = \sum_{(j,J)\neq(n,Kn)}^{n} \|\overline{L_j} U^J\|^2 - \sum_{|K|=q-1} \sum_{(j,k)\neq(n,n)} (\overline{L_j} U^{kK}, \overline{L_k} U^{jK})$$

where the property that $(j, J) \neq (n, Kn)$ means that we exclude those terms where $j = n$ or $n \in J$.

We also notice that

$$\|BU\|^2 = \sum_{j,k=1}^{n} (L_j U^{jK}, L_k U^{kK}). \tag{4.4}$$

To compute $\|AU\|^2+\|BU\|^2$, we follow the computation of Catlin in [pp504-506, Cat] as follows: First we calculate $\|BU\|^2$ by several steps. Notice that if $(j,k)\neq(n,n)$, $|K|=q-1$, then

$$
\begin{aligned}
(L_kU^{kK},L_jU^{jK}) &= (\overline{L_j}U^{kK},\overline{L_k}U^{jK})+(\overline{e_j}L_kU^{kK},U^{jK})\\
&\quad -(e_k\overline{L_j}U^{kK},U^{jK})+([L_k,\overline{L_j}]U^{kK},U^{jK}).
\end{aligned}
\tag{4.5}
$$

Define

$$
L(U)=\sum_{|K|=q-1}\|L_nU^{nK}\|^2+\sum_{|J|=q,n\notin J}\|\overline{L_n}U^J\|^2+\sum_{j=1}^{n-1}\sum_{|J|=q}(\|L_jU^J\|^2+\|\overline{L_j}U^J\|^2). \tag{4.6}
$$

From (3.6) and by the standard big-small constant argument, it follows that

$$
\left|(\overline{e_j}L_kU^{kK},U^{jK})\right|\leq\frac{C_1}{K_0}L(U)+C_1K_0\|U\|^2. \tag{4.7}
$$

Since

$$
\left|(e_k\overline{L_j}U^{kK},U^{jK})\right|=|(\overline{e_j}e_kU^{kK},U^{jK})-(e_kU^{kK},L_jU^{jK})-(\overline{L_j}(e_k)U^{kK},U^{jK})|,
$$

we have

$$
\left|(e_k\overline{L_j}U^{kK},U^{jK})\right|\leq\frac{C_1}{K_0}L(U)+C_1K_0\|U\|^2. \tag{4.8}
$$

Here K_0, C_js are constant independent of the choices of x_0 and U, which may be different in different contexts. K_0 is supposed to be sufficiently large. Notice that

$$
([L_k,\overline{L_j}]U^{kK},U^{jK})=\sum_{i=1}^{n}(C^i_{kj}L_iU^{kK},U^{jK})+\sum_{i=1}^{n}(d^i_{kj}\overline{L_i}U^{kK},U^{jK}), \tag{4.9}
$$

where

$$
C^i_{kj}=\omega^i([L_k,\overline{L_j}]),\ d^i_{kj}=\overline{\omega^i}([L_k,\overline{L_j}]). \tag{4.10}
$$

(a) If $i<n$, then

$$
|(C^i_{kj}L_iU^{kK},U^{jK})|+|(d^i_{kj}\overline{L_i}U^{kK},U^{jK})|\leq\frac{C_1}{K_0}L(U)+C_1K_0\|U\|^2. \tag{4.11}
$$

(b) If $i=n,k=n$, and $j\neq n$

$$
|(C^n_{nj}L_nU^{nK},U^{jK})|\leq\frac{C_1}{K_0}L(U)+C_1K_0\|U\|^2, \tag{4.12}
$$

Hence, the remaining terms to be estimated include the following

$$|(d^n_{nj}\overline{L_n}U^{nK}, U^{jK})|. \tag{4.13}$$

(c) If $i = n, j = n$, and $k \neq n$, we have

$$|(d^n_{kn}\overline{L_n}U^{kK}, U^{nK})| \leq \frac{C_1}{K_0}L(U) + C_1K_0\|U\|^2. \tag{4.14}$$

Thus it suffices to estimate:

$$|(C^n_{kn}L_nU^{kK}, U^{nK})|. \tag{4.15}$$

As in [Cat], we use the following result on the standard uniform sub-elliptic estimate to handle it: (See [CS], for instance.)

Lemma 4.1. *Let Ω be a bounded open neighborhood of the origin in $\mathbb{R}^n$. Let $X_i, 1 \leq i \leq k, k \leq n$ be vector fields with smooth real coefficients up to $\overline{\Omega}$. Denote $\mathcal{L}_1$ to be collection of the $X_i^s, 1 \leq i \leq k$ and $\mathcal{L}_2$ to be collection of $\mathcal{L}_1$ and the vectors of the form $[X, Y]$ with $X, Y \in \mathcal{L}_1$. If $\mathcal{L}_2$ span the tangent space of Ω, then there exists $C > 0$ such that*

$$\|u\|^2_{\frac{1}{2}} \leq C\left(\sum_{i=1}^{k}\|X_iu\|^2 + \|u\|^2\right), u \in C^\infty_0(\Omega). \tag{4.16}$$

Here C only depends on the coefficients of the vector fields.

For any $f \in C^\infty_c(W(x_0))$, we define the tangential Fourier transform for f in $W(x_0)$ by

$$\hat{f}(\xi, y_{2n}) = \int_{\mathbb{R}^{2n-1}} e^{-i<y',\xi>}f(y', y_{2n})dy_1\cdots dy_{2n-1}, \tag{4.17}$$

where $\xi = (\xi_1, \cdots, \xi_{2n-1})$ and $< y', \xi >= y_1\xi_1 + \cdots + y_{2n-1}\xi_{2n-1}$. We define the tangential Sobolev norm $|||f|||_s$ by

$$|||f|||^2_s = \int_{-\varphi^\lambda(x_0)}^{0}\int_{\mathbb{R}^{2n-1}} |\hat{f}(\xi, y_{2n})|^2(1 + |\xi|^2)^s d\xi_1\cdots d\xi_{2n-1}dy_{2n}, s \in \mathbb{R}. \tag{4.18}$$

For more discussions of the tangential Fourier transform and tangential Sobolev norms, we refer the reader to [FK] and [CS] . From Lemma 4.1 and the uniform estimate of the coefficients of the vector fields $\{L_i\}_{i=1}^n$, there exists a constant C_2, which does not depend on x_0 such that for all $f \in C_c^\infty(W(x_0))$,

$$|||f|||_{\frac{1}{2}}^2 \leq C_2 \sum_{k=1}^{n-1} \left(\|L_k f\|^2 + \|\overline{L_k} f\|^2\right) + C_2' \|f\|^2. \tag{4.19}$$

Lemma 4.2. *Suppose $U \in \mathcal{B}^q(S_\varphi^{\epsilon_0})$ and supp $U \subset\subset W(x_0)$ with $0 < \varphi(x_0) \ll 1$. Then*

$$|(C_{kn}^n L_n U^{kK}, U^{nK})| \leq \frac{C_3}{K_0} L(U) + C_3 K_0 \|U\|^2, \ k \neq n, \tag{4.20}$$

$$|(d_{nj}^n \overline{L_n} U^{nK}, U^{jK})| \leq \frac{C_3}{K_0} L(U) + C_3 K_0 \|U\|^2, \ j \neq n. \tag{4.21}$$

Proof. We follow the proof of [Lemma 7.8, Cat]. In [Cat], a useful fact is that the domain can be uniformly shrunk toward M, that helps to get such types of estimates. In our situation, the domain is fixed. However we go close and close to E such that the quantity

$$|C_{kn}^n|_{W(x_0)} = O(\varphi^{1-\frac{\lambda}{2}}(x_0) + \sigma_0) \tag{4.22}$$

is sufficiently small when $0 < \varphi(x_0) \ll 1, \sigma_0 \ll 1$ to get the desired estimates. Write $L_n = (L_n y_{2n})(\overline{L_n} y_{2n}) \overline{L_n} + \tilde{L}_n$, then $\tilde{L}_n(y_{2n}) = 0$ and

$$\tilde{L}_n = \sum_{j=1}^{2n-1} \left\{ L_n(y_j) - \left(\frac{L_n y_{2n}}{\overline{L_n} y_{2n}}\right) \overline{L_n}(y_j) \right\} \frac{\partial}{\partial y_j}, \tag{4.23}$$

$$\frac{L_n y_{2n}}{\overline{L_n} y_{2n}} = \frac{1 - 4 \sum_{l=1}^{2n-1} \frac{\Phi^3}{\varphi^\lambda(x_0)} y_{2n} E_{nl}(y)}{1 - 4 \sum_{l=1}^{2n-1} \frac{\Phi^3}{\varphi^\lambda(x_0)} y_{2n} \overline{E_{nl}}(y)}. \tag{4.24}$$

Let $\tilde{L}_n = \sum_{j=1}^{2n-1} a_j(y) \frac{\partial}{\partial y_j}$. Then (3.19), (3.33), (4.23), (4.24) give that

$$\begin{aligned} &|a_j(y)|_{n+1, W(x_0)} = O(\varphi^{2-\frac{\lambda}{2}}(x_0)), j < 2n-1 \\ &|a_{2n-1}(y)|_{n+1, W(x_0)} = O(1), |C_{kn}^n|_{W(x_0)} = O(\varphi^{1-\frac{\lambda}{2}}(x_0) + \sigma_0). \end{aligned} \tag{4.25}$$

Then

$$(C_{kn}^n L_n U^{kK}, U^{nK}) = \left(C_{kn}^n \frac{L_n(y_{2n})}{\overline{L_n}(y_{2n})} \overline{L_n} U^{kK}, U^{nK}\right) + (C_{kn}^n \tilde{L}_n U^{kK}, U^{nK}). \tag{4.26}$$

Now

$$
\begin{aligned}
(C^n_{kn}\tilde{L}_n U^{kK}, U^{nK}) &\le |C^n_{kn}|_{W(x_0)} \cdot ||| \sum_{j=1}^{2n-1} a_j(y)\frac{\partial U^{kK}}{\partial y_j} |||_{-\frac{1}{2}} \cdot |||U|||_{\frac{1}{2}} \\
&\le C_3 |C^n_{kn}|_{W(x_0)} \cdot |||U|||^2_{\frac{1}{2}},
\end{aligned}
\tag{4.27}
$$

On the other hand, since $\|\overline{L_n}U^{kK}\| \le L(U)$ then

$$
\left(C^n_{kn}\frac{L_n(y_{2n})}{\overline{L_n}(y_{2n})}\overline{L_n}U^{kK}, U^{nK} \right) \le \frac{C_3}{K_0}L(U) + C_3K_0\|U\|^2, k < n.
$$

Notice that we can make $|C^n_{kn}|_{W(x_0)}$ sufficiently small by letting x_0 close to E. Combining (4.19), (4.27) and the just obtained estimate, we conclude the estimate in (4.20). The proof for (4.21) is similar. □

(d) If $i = n$, and $1 \le j, k \le n-1$, we need to control the following terms:

$$
(C^n_{kj}L_nU^{kK}, U^{jK}), (d^n_{kj}\overline{L_n}U^{kK}, U^{jK}).
$$

For $n \in K$, it holds that

$$
|C^n_{kj}L_nU^{kK}, U^{jK})| \le \frac{C_2}{K_0}L(U) + C_2K_0\|U\|^2. \tag{4.28}
$$

When $n \notin K$, we have

$$
|(d^n_{kj}\overline{L_n}U^{kK}, U^{jK})| \le \frac{C_2}{K_0}L(U) + C_2K_0\|U\|^2. \tag{4.29}
$$

The only remaining two cases are (i): For $n \notin K$, we need to control

$$
(C^n_{kj}L_nU^{kK}, U^{jK})
$$

and (ii): For $n \in K$, we need to control

$$
(d^n_{kj}\overline{L_n}U^{kK}, U^{jK}).
$$

Since $C^n_{kj}(y) = \omega^n([L_k, \overline{L_j}])(y) = d_{kj}(x_0) + O(\sigma_0), d^n_{kj}(y) = -d_{kj}(x_0) + O(\sigma_0)$, thus

$$
(C^n_{kj}L_nU^{kK}, U^{jK}) = d_{kj}(x_0)(L_nU^{kK}, U^{jK}) + (O(\sigma_0)L_nU^{kK}, U^{jK}). \tag{4.30}
$$

Define

$$E(x_0,U,K_0)=\frac{1}{K_0}L(U)+K_0\|U\|^2+\sigma_0\int_{M_0}|U|^2dS+\sigma_0\int_{M_1}|U|^2dS.$$

When $n\notin K$, integrating by part, we have from (4.30)

$$\begin{aligned}&(C^n_{jj}L_nU^{jK},U^{jK})=-d_j(x_0)\int_{M_1}P|U^{jK}|^2dS+O(E(x_0,U,K_0)),\\&(C^n_{kj}L_nU^{kK},U^{jK})=O(E(x_0,U,K_0)).\end{aligned}\tag{4.31}$$

When $n\in K$, integrating by part, we have from (4.30)

$$\begin{aligned}&(d^n_{jj}\overline{L_n}U^{jK},U^{jK})=-d_j(x_0)\int_{M_0}P|U^{jK}|^2dS+O(E(x_0,U,K_0)),\\&(d^n_{kj}L_nU^{kK},U^{jK})=O(E(x_0,U,K_0)).\end{aligned}\tag{4.32}$$

Then

$$\begin{aligned}\|AU\|^2+\|BU\|^2=&\sum_{|J|=q,n\notin J}\|\overline{L_n}U^J\|^2+\sum_{|K|=q-1}\|L_nU^{nK}\|^2+\sum_{j=1}^{n-1}\sum_{|J|=q}\|\overline{L_j}U^J\|^2\\&-\sum_{|J|=q}\sum_{j\in J}\left(d_j(x_0)\int_{M_0}P|U^J|^2dS+d_j(x_0)\int_{M_1}P|U^J|^2dS\right)\\&+O(E(x_0,U,K_0))\\=&\sum_{|J|=q,n\notin J}\|\overline{L_n}U^J\|^2+\sum_{|K|=q-1}\|L_nU^{nK}\|^2+\sum_{j=1}^{n-1}\sum_{|J|=q}\|\overline{L_j}U^J\|^2\\&-\sum_{|J|=q}\sum_{j\in J}\left(d_j(x_0)\int_{M_0}Re(P)|U^J|^2dS+d_j(x_0)\int_{M_1}Re(P)|U^J|^2dS\right)\\&+O(E(x_0,U,K_0)).\end{aligned}\tag{4.33}$$

Making use of the sign condition on the Levi forms and by a standard argument (see [pp 62, FK] and [CS], for instance), we obtain from the above

$$\|AU\|^2+\|BU\|^2\geq c\left(L(U)+\int_{M_0}|U|^2dS+\int_{M_1}|U|^2dS\right)+O(K_0\|U\|^2).\tag{4.34}$$

Hence, there exist positive constants c and C both indpendent of the choices of x_0 and U such that

$$\|AU\|^2+\|BU\|^2\geq cL(U)-C\|U\|^2.\tag{4.35}$$

The following Lemma from [Cat] is a fundamental fact, by which the mixed boundary conditions enters the estimate. It is Lemma 7.7 of [Cat] with σ^3 being replaced by $\varphi^\lambda(x_0)$.

Lemma 4.3. *Suppose $f \in C_c^\infty(W(x_0))$ with $\varphi(x_0) \ll 1$, and f vanishes either on M_0 or M_1. Then there exists a constant $\hat{C}_1$ independent of x_0 and U, so that*

$$\begin{aligned} \varphi^{-\lambda}(x_0)\|f\|^2 &\le \hat{C}_1 \left(\|L_n f\|^2 + \sum_{k=1}^{n-1} \left(\|L_k f\|^2 + \|\overline{L_k} f\|^2 \right) \right) \\ \varphi^{-\lambda}(x_0)\|f\|^2 &\le \hat{C}_1 \left(\|\overline{L_n} f\|^2 + \sum_{k=1}^{n-1} \left(\|L_k f\|^2 + \|\overline{L_k} f\|^2 \right) \right) \end{aligned} \tag{4.36}$$

Combining this Lemma 4.3 with (4.35), we proved Part (1) of the following theorem:

Theorem 4.4. *(1). There exists a constant $0 < \varepsilon_2 \ll 1$ independent of x_0 and a constant $\tilde{C}$ independent of x_0 and ε_2 such that if $0 < \varphi^\lambda(x_0) \le \varepsilon_2$ and $U \in \mathcal{B}^q$ with supp $U \subset\subset W(x_0)$, then*

$$\varphi^{-\lambda}(x_0)\|U\|^2 \le \tilde{C}\left(\|TU\|^2 + \|S^*U\|^2\right). \tag{4.37}$$

(2). There exists a small neighborhood V_c of E in $\overline{S_\varphi^{\epsilon_0}}$ such that for any $U \in \mathcal{B}^q$ it holds that

$$\int_{V_c} |U|^2 dV \le \|TU\|^2 + \|S^*U\|^2 + \int_{S_\varphi^{\epsilon_0}\setminus V_c} |U|^2 dV. \tag{4.38}$$

Proof. Let $V_c = \{(x,t) \in \mathcal{N}_{\epsilon_0} \cap S_\varphi^{\epsilon_0} : 0 < \varphi^\lambda(x) \le \varepsilon_3\}$, $\varepsilon_3 < \varepsilon_2$. Let $\{W(x_\alpha)\}$, $\{\xi_\alpha\}$ be as in Proposition 3.1 (iv), where $\{W(x_\alpha)\}$ is a Besicotvich cover of V_c for ε_3 sufficiently small.

$$\begin{aligned} \int_{V_c} |U|^2 dV &= \sum_\alpha \int_{V_c} |\xi_\alpha U|^2 dV \le \sum_\alpha \int_{W(x_\alpha)} |\xi_\alpha U|^2 \le \tilde{C} \sum_\alpha \varphi^\lambda(x_\alpha)(\|T\xi_\alpha U\|^2 + \|S^*\xi_\alpha U\|^2) \\ &\le \tilde{C} \sum_\alpha \varphi^\lambda(x_\alpha)(\|\xi_\alpha TU\|^2 + \|\xi_\alpha S^*U\|^2) + \tilde{C}\frac{c^2}{\sigma_0^2} \sum_\alpha \varphi^\lambda(x_\alpha) \int_{W(x_\alpha)} |U|^2 dV \\ &\le \tilde{C}\varepsilon_3\left(\|TU\|^2 + \|S^*U\|^2\right) + \tilde{C}\varepsilon_3 \frac{c^2}{\sigma_0^2} N_0 \int_{S_\varphi^{\epsilon_0}} |U|^2 dV. \end{aligned} \tag{4.39}$$

Here the N_0 is as in Proposition 3.1 (iv). When ε_3 is sufficiently small such that

$$\max\left\{\tilde{C}\varepsilon_3, \tilde{C}\varepsilon_3 \frac{c^2}{\sigma_0^2} N_0\right\} \le \frac{1}{2},$$

then we get

$$\int_{V_c} |U|^2 dV \le \|TU\|^2 + \|S^*U\|^2 + \int_{S_\varphi^{\epsilon_0}\setminus V_c} |U|^2 dV. \tag{4.40}$$

□

5 Proof of Theorem 2.1

We now give a proof of Theorem 2.1. We use the notations set up above. By the Hörmander-Friderichs approximation theorem mentioned in the end of §3, we need only work on forms in $\mathcal{B}^q$.

Let $V_{c_1} = \{(x,t) \in \mathcal{N}_{\varepsilon_0} \cap S_\varphi^{\epsilon_0} : \varphi^\lambda(x) < \varepsilon_4\}$ with $\varepsilon_4 < \varepsilon_3$ be a smaller neighborhood of the corner E in $\overline{S_\varphi^{\epsilon_0}}$ with $V_{c_1} \subset\subset V_c$. Let

$$M_{c,0} = M_0 \setminus V_c, M_{c_1,0} = M_0 \setminus V_{c_1}.$$

Suppose that there exists a tubular neighborhood $\mathcal{N}_{c_1}$ of $M_{c_1,0}$ in X and a C^∞ map Φ_{c_1} such that $\Phi_{c_1} : \mathcal{N}_{c_1} \to M_{c_1,0} \times (-2,2)$ is a diffemorphism. Let $\mathcal{L}_{c_1} = \Phi_{c_1 *}(T^{1,0}\mathcal{N}_{c_1})$, where $T^{1,0}\mathcal{N}_{c_1}$ is the holomorphic tangent bundle of $\mathcal{N}_{c_1}$. Write $\Omega_{c_1} = M_{c_1} \times (-2,2)$, then $(\Omega_{c_1}, \mathcal{L}_{c_1})$ is a complex manifold biholomorphic to $(\mathcal{N}_{c_1}, T^{1,0}\mathcal{N}_{c_1})$. Also $M_{c_1,0}$ is a hypersurface in $(\Omega_{c_1}, \mathcal{L}_{c_1})$. In what follows, as before, when there is no risk of causing confusion, we identify $\mathcal{N}_{c_1}$ with Ω_{c_1} and objects defined over $\mathcal{N}_{c_1}$ with those corresponding to Ω_{c_1}. We define two subdomains of $\mathcal{N}_{c_1}$ as follows:

$$\begin{aligned} &\tilde{S}_{c,\varepsilon} = M_{c,0} \times [-\varepsilon^3, 0], \tilde{S}_{c_1,\varepsilon} = M_{c_1,0} \times [-\varepsilon^3, 0], \\ &M_{c,\varepsilon} := M_{c,0} \times \{-\varepsilon^3\}, M_{c_1,\varepsilon} := M_{c_1,0} \times \{-\varepsilon^3\}. \end{aligned} \tag{5.1}$$

We can assume that $\tilde{S}_{c_1,\varepsilon}$ is contained in $\mathcal{N}_{c_1} \cap S_\varphi^{\epsilon_0}$ by making ε sufficiently small. First, we choose a finite cover $\{V_v\}_{v=1}^m$ of $M_{c_1,0}$. With the same argument as in Proposition 3.1 (*iv*) we can assume that for any $x_0 \in M_{c,0}$, there is a coordinate neighborhood

$$V(x_0) = \{(x', x_{2n}) \in M_{c_1,0} \times [-\varepsilon^3, 0] : |x_1| < \varepsilon, \cdots, |x_{2n-1}| < \varepsilon, -\varepsilon^3 \le x_{2n} \le 0\} \tag{5.2}$$

with $V(x_0)$ contained in a certain V_v when ε is sufficiently small, where x_0 corresponds to the origin and $(x_1, \cdots, x_{2n-1})$ is the coordinates of $V(x_0) \cap M_{c_1,0}$. There exist a special orthnormal frame $\{L_i\}_{i=1}^n$ and its dual frame $\{\omega_i\}_{i=1}^n$ satisfying the type of estimates as in Proposition 3.1 (iii).

Let $U \in \mathcal{E}_c^{(0,q)}(S_\varphi^{\epsilon_0})$ with compact support in one of the above constructed coordinate neighborhoods $V(x_0)$ such that it satisfies the $\overline{\partial}$-Dirichlet condition on $M_{c_1,0}$ and vanishes near $M_{c_1,\varepsilon}$. Then we have the following estimate

Lemma 5.1. *There exists a constant $\tilde{C}_1$ independent of x_0 and U such that*

$$\varepsilon^{-3}\|U\|^2 \leq \tilde{C}_1 Q(U,U) \tag{5.3}$$

for ε sufficiently small.

In order to prove Lemma 5.1, we first recall the definition of condition $Z(q)$ from [FK] and [CS].

Definition 5.2. *Let D be a relatively compact subset with C^∞ boundary in a complex Hermitian manifold of complex dimension $n \geq 2$. We say the boundary ∂D satisfies condition $Z(q)$ if for $1 \leq q \leq n-1$, the Levi form associated with D has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at every boundary point.*

Proof of Lemma 5.1: We will use the property that U vanishes near $M_{c_1,\varepsilon}$ to obtain the estimate in (5.3). We define the Hodge star operator $\star$ with respect to the scaled metric as follows: For any $U_1, U_2 \in \mathcal{E}_c^{p,q}(S_\varphi^{\epsilon_0})$, we have

$$(U_1, U_2) = \int_{S_\varphi^{\epsilon_0}} U_1 \wedge \star U_2 dV. \tag{5.4}$$

Since $U \in \mathcal{E}_c^{(0,q)}(S_\varphi^{\epsilon_0})$ with supp$U \subset\subset V(x_0)$ and U satisfies the $\overline{\partial}$-Dirichlet condition on $M_{c_1,0}$, vanishes near $M_{c_1,\varepsilon}$, $\star U$ will be a $(n, n-q)$-form which satisfies the $\overline{\partial}$-Neumann condition on $M_{c_1,0}$ and vanishes near $M_{c_1,\varepsilon}$. Let $V = \star U$. We then have $V = \sum_{|J|=n-q} V^J \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^J$. Then

$$\begin{aligned}\|\overline{\partial}V\|^2 + \|\overline{\partial}^* V\|^2 = &\sum_{j,|J|=n-q} \|\overline{L}_j V^J\|^2 + \sum_{j,k,L} \int_{M_0 \cap V(x_0)} C_{jk}^n P V^{jL} \overline{V^{kL}} dS \\ &+ O(l(V) \cdot \|V\|) + O(\|V\|^2)\end{aligned} \tag{5.5}$$

where $l(V) = \sum_{j,J} \|\overline{L}_j V^J\|$ and $C^n_{jk} = w^n([L_j, \overline{L}_k])$. It follows that there exist constant a_1, a_2 which are independent of x_0 and $V(x_0)$ such that

$$\begin{aligned} \|\overline{\partial} V\|^2 + \|\overline{\partial}^* V\|^2 \geq a_1 l(V) - a_2\|V\|^2 + \sum_{|J|=n-q} \sum_{j\in J} \int_{\partial M_{c_1,0}} d_j(x_0) P|V^J|^2 dS \\ + O(\varepsilon) \sum_{|J|=n-q} \sum_{j\in J} \int_{\partial M_{c_1,0}} d_j(x_0) P|V^J|^2 dS, \end{aligned} \tag{5.6}$$

where $\{d_j(x_0)\}_{j=1}^{n-1}$ are the eigenvalues of the Levi-form on M_0. By the assumption of M_0, we know that M_0 satisfies Condition $Z(n-q)$. Proceeding in the standard way as in [FK] and [CS], we see that there exist constant a_3, a_4 such that

$$\|\overline{\partial} V\|^2 + \|\overline{\partial}^* V\|^2 \geq a_3 \left(\sum_{j,J} \|\overline{L}_j V^J\|^2 + \sum_{j,J} \|L_j V^J\|^2 \right) - a_4\|V\|^2. \tag{5.7}$$

Since V vanish near $M_{c_1,\varepsilon}$, from Lemma 4.3 we have

$$\sum_{j,J} \|\overline{L}_j V^J\|^2 + \sum_{j,J} \|L_j V^J\|^2 \geq \varepsilon^{-3}\|V\|^2. \tag{5.8}$$

Combing (5.7) and (5.8) and when ε is sufficiently small, we have

$$\|\overline{\partial} V\|^2 + \|\overline{\partial}^* V\|^2 \geq a_5 \varepsilon^{-3}\|V\|^2 \tag{5.9}$$

when ε is sufficiently small. Substituting $V = \star U$ to (5.9), we have

$$\|\overline{\partial} \star U\|^2 + \|\overline{\partial}^* \star U\|^2 \geq a_5 \varepsilon^{-3}\| \star U\|^2 \tag{5.10}$$

and since the Hodge star operator $\star$ is an isometry operator in L^2-space, we have

$$\| \star \overline{\partial} \star U\|^2 + \| \star \overline{\partial}^* \star U\|^2 \geq a_5 \varepsilon^{-3}\| \star U\|^2 \tag{5.11}$$

Substituting the identity $\overline{\partial}^* = - \star \overline{\partial} \star$ and $\overline{\partial} = \star \overline{\partial}^* \star$ to (5.11), we complete the proof of the Lemma 5.1. (For detailed discussions of the Hodge star operator in the L^2-space, we also refer the reader to [CSh].) ∎

Moreover, we can choose a set I such that $\{V(x_i)\}_{i\in I}$ is a covering of $\tilde{S}_{c,\varepsilon}$. Moreover, $V(x_i) \subset\subset \tilde{S}_{c_1,\varepsilon}$ and there exists an integer $\hat{N}$, independent of ε, such that no point of $\tilde{S}_{c,\varepsilon}$ lies in more than $\hat{N}$ of such $V(x_i)$'s. We choose functions $\rho_i \in C_c^\infty(V(x_i))$ such that $\sum_{i\in I}\rho_i^2 = 1$ in a neighborhood of $\tilde{S}_{c,\varepsilon}$ and $|\rho_i|_{1,V(x_i)} \le \tilde{c}_1\varepsilon^{-1}$, where $\tilde{c}$ is a constant independent of x_i. Then for all $U \in \mathcal{E}_c^{(0,q)}(S_\varphi^{\epsilon_0})$ which satisfies the $\overline{\partial}$-Dirichlet condition on $M_{c_1,0}$ and vanishes near $M_{c_1,\varepsilon}$, we have

$$\begin{aligned} \varepsilon^{-3}\int_{\tilde{S}_{c,\varepsilon}} |U|^2 dV &\le \varepsilon^{-3}\sum_{i\in I}\int_{V(x_i)} |\rho_i U|^2 dV \le \tilde{C}_1 \sum_{i\in I} Q(\rho_i U, \rho_i U) \\ &\le \tilde{C}_1 Q(U,U) + \frac{\tilde{C}_2\tilde{c}_1^2\hat{N}}{\varepsilon^2}\int_{\tilde{S}_{c_1,\varepsilon}} |U|^2 dV \end{aligned} \tag{5.12}$$

Thus

$$\int_{\tilde{S}_{c,\varepsilon}} |U|^2 dV \le \tilde{C}_1\varepsilon^3 Q(U,U) + \tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon\int_{\tilde{S}_{c_1,\varepsilon}} |U|^2 dV \tag{5.13}$$

and

$$(1-\tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon)\int_{\tilde{S}_{c,\varepsilon}} |U|^2 dV \le \tilde{C}_1\varepsilon^3 Q(U,U) + \tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon\int_{\tilde{S}_{c_1,\varepsilon}\setminus\tilde{S}_{c,\varepsilon}} |U|^2 dV \tag{5.14}$$

First, we choose ε such that $1-\tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon \ge \frac{1}{2}$, then

$$\begin{aligned} \int_{\tilde{S}_{c,\varepsilon}} |U|^2 dV &\le 2\tilde{C}_1\varepsilon^3 Q(U,U) + 2\tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon\int_{\tilde{S}_{c_1,\varepsilon}\setminus\tilde{S}_{c,\varepsilon}} |U|^2 dV \\ &\le 2\tilde{C}_1\varepsilon^3 Q(U,U) + 2\tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon\int_{V_c} |U|^2 dV \end{aligned} \tag{5.15}$$

Next, let

$$M_{c,1} = M_1 \cap (S_\varphi^{\epsilon_0}\setminus V_c), M_{c_1,1} = M_1 \cap (S_\varphi^{\epsilon_0}\setminus V_{c_1}).$$

By the same construction as above we can find a small neighborhood $\mathcal{T}_{c_1,\varepsilon}$ of $M_{c_1,1}$ in $S_\varphi^{\epsilon_0}$ which is biholomorphic to $M_{c_1,1}\times[0,\varepsilon^3]$. We similarly write

$$\begin{aligned} &\mathcal{T}_{c,\varepsilon} = M_{c,1}\times[0,\varepsilon^3], \mathcal{T}_{c_1,\varepsilon} = M_{c_1,1}\times[0,\varepsilon^3] \\ &\widehat{M}_{c,0} = M_{c,1}\times\{0\}, \widehat{M}_{c,\varepsilon} = M_{c,1}\times\{\varepsilon^3\} \\ &\widehat{M}_{c_1,0} = M_{c_1,1}\times\{0\}, \widehat{M}_{c_1,\varepsilon} = M_{c_1,1}\times\{\varepsilon^3\} \end{aligned} \tag{5.16}$$

For all $x_0 \in \widehat{M}_{c,0}$ we choose a coordinate neighborhood $V(x_0) = \{(x', x_{2n}) \in M_{c_1,0}\times[0,\varepsilon^3] : x' = (x_1,\cdots,x_{2n-1}), |x_1| < \varepsilon,\cdots,|x_{2n-1}| < \varepsilon, 0 \le x_{2n} \le \varepsilon^3\}$, where x_0 corresponds to the origin and $(x_1, \cdots, x_{2n-1})$ is the coordinates of $V(x_0) \cap \widehat{M}_{c,0}$. There exist a special frame $\{L_i\}_{i=1}^n$ and their dual frame $\{\omega_i\}_{i=1}^n$, that satisfy good estimates as in Proposition 3.1 with $L_n(x_{2n}) = -1$ on $V(x_0) \cap M_1$. Moreover,

$$\omega^n([L_i, \overline{L_j}])(x) = d_{ij}(x_0) + O(\varepsilon), 1 \leq, i, j \leq n-1, \tag{5.17}$$

where $d_{ij}(x_0) = 0$ when $i \neq j$, $d_{ij}(x_0) = d_j(x_0)$ when $i = j$ and $\{d_j(x_0)\}_{j=1}^{n-1}$ are the Levi eigenvalues of the Levi form on M_1 with respect to the domain. Notice that we now have $P = -L_n(x_{2n}) = 1$ on $V(x_0) \cap M_1$.

By assumption of M_1, we have that M_1 satisfies $Z(q)$ condition. Let $U \in \mathcal{E}_c^{0,q}(S_\varphi^{\epsilon_0})$ satisfy the $\overline{\partial}$-Neumann condition on $\widehat{M}_{c_1,0}$ and vanish near $\widehat{M}_{c_1,1}$. Assume that U has a compact support in some $V(x_0)$. Then by a similar argument as in Lemma 5.1 and when ε is sufficiently small we have the following L^2-estimate

$$\varepsilon^{-3}\|U\|^2 \leq \tilde{C}_3 Q(U, U). \tag{5.18}$$

Here the constant $\tilde{C}_3$ is independent of U and x_0. Then choosing a covering $\{V(x_\lambda)\}_{\lambda \in \Lambda}$ of $\mathcal{T}_{c,\varepsilon}$ with the same property as the covering $\{V(x_i)\}_{i \in I}$ in the proof of Lemma 5.1 and also using a partition of unity with the same property with respect to such covering, we have

$$\int_{\mathcal{T}_{c,\varepsilon}} |U|^2 dV \leq 2\tilde{C}_3\varepsilon^3 Q(U,U) + 2\tilde{C}_4\tilde{c}_2^2\hat{N}\varepsilon \int_{\mathcal{T}_{c_1,\varepsilon}\setminus\mathcal{T}_{c,\varepsilon}} |U|^2 dV \leq 2\tilde{C}_3\varepsilon^3 Q(U,U) + 2\tilde{C}_4\tilde{c}_2^2\hat{N}\varepsilon \int_{V_c} |U|^2 dV \tag{5.19}$$

for some constants $\tilde{C}_3, \tilde{C}_4, \tilde{c}_2$ which do not depend on U.

We choose a cut-off function χ_ε such that $\chi_\varepsilon \equiv 1$ in a small neighborhood O_ε of $M_{c_1,0} \cup M_{c_1,1} \cup V_{c_1}$ and χ_ε equals to zero near $M_{c,\varepsilon}$ and $\widehat{M}_{c,\varepsilon}$. Then there is a neighborhood $V_{c,0,\varepsilon}$ of $M_{c,0}$ in $\tilde{S}_{c,\varepsilon}$ such that when χ_ε restricted to $V_{c,0,\varepsilon}$ equals to 1. Then

$$\begin{aligned} \int_{V_{c,0,\varepsilon}} |U|^2 dV \leq \int_{\tilde{S}_{c,\varepsilon}} |\chi_\varepsilon U|^2 &\leq 2\tilde{C}_1\varepsilon^3 Q(\chi_\varepsilon U, \chi_\varepsilon U) + 2\tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon \int_{V_c} |\chi_\varepsilon U|^2 dV \\ &\leq 2\tilde{C}_1\varepsilon^3 Q(U,U) + C(\varepsilon)\int_{K_\varepsilon} |U|^2 dV + 2\tilde{C}_2\tilde{c}_1^2\hat{N}\varepsilon \int_{V_c} |U|^2 dV, \end{aligned} \tag{5.20}$$

where $C(\varepsilon)$ is a constant depending on ε and K_ε is the compliment of O_ε in the domain, which is a compact subset of $S_\varphi^{\epsilon_0}$.

Similarly, there is a neighborhood $V_{c,1,\varepsilon}$ of $M_{c,1}$ in $\mathcal{T}_{c,\varepsilon}$ such that χ_ε is identically one when restricted to $V_{c,1,\varepsilon}$. Then

$$\begin{aligned}\int_{V_{c,1,\varepsilon}}|U|^2dV &\le \int_{\mathcal{T}_{c,\varepsilon}}|\chi_\varepsilon U|^2 \le 2\tilde{C}_3\varepsilon^3 Q(\chi_\varepsilon U,\chi_\varepsilon U)+2\tilde{C}_4\tilde{c}_2^2\hat{N}\varepsilon\int_{V_c}|\chi_\varepsilon U|^2dV\\ &\le 2\tilde{C}_3\varepsilon^3Q(U,U)+C(\varepsilon)\int_{K_\varepsilon}|U|^2dV+2\tilde{C}_4\tilde{c}_2^2\hat{N}\varepsilon\int_{V_c}|U|^2dV.\end{aligned} \tag{5.21}$$

Thus

$$\begin{aligned}&\int_{V_c}|U|^2dV+\int_{V_{c,0,\varepsilon}}|U|^2dV+\int_{V_{c,1,\varepsilon}}|U|^2dV\\ &\le (1+4\tilde{C}_1\varepsilon^3+4\tilde{C}_3\varepsilon^3)Q(U,U)+4C(\varepsilon)\int_{K_\varepsilon}|U|^2dV\\ &+\int_{K_\varepsilon^*}|U|^2dV+(4\tilde{C}_2\tilde{c}_1^2+4\tilde{C}_4\tilde{c}_2^2)\hat{N}\varepsilon\int_{V_c}|U|^2dV,\end{aligned} \tag{5.22}$$

where $K_\varepsilon^*=S_\varphi^{\epsilon_0}\setminus\{V_c\cup V_{c,0,\varepsilon}\cup V_{c,1,\varepsilon}\}$ which is a compact subset of $S_\varphi^{\epsilon_0}$. We choose ε such that $(4\tilde{C}_2\tilde{c}_1^2+4\tilde{C}_4\tilde{c}_2^2)\hat{N}\varepsilon\le\frac{1}{2}$. Then

$$\begin{aligned}&\int_{V_c}|U|^2dV+\int_{V_{c,0,\varepsilon}}|U|^2dV+\int_{V_{c,1,\varepsilon}}|U|^2dV\\ &\le (2+8\tilde{C}_1\varepsilon^3+8\tilde{C}_3\varepsilon^3)Q(U,U)+4C(\varepsilon)\int_{K_\varepsilon}|U|^2dV+2\int_{K_\varepsilon^*}|U|^2dV.\end{aligned} \tag{5.23}$$

Let $V_{c,0,1}=V_c\cup V_{c,0,\varepsilon}\cup V_{c,1,\varepsilon}$, $F=K_\varepsilon\cup K_\varepsilon^*$. Applying the approximation theorem by the smooth forms as mentioned at the end of §3, we thus complete the proof of Theorem 2.1. The proof of Corollary 2.2 follows from Theorem 2.1 by the standard argument as in Hörmander [Ho1].

Remark 5.3. *From the proof of the main theorem, we actually obtained the following stronger estimate containing the boundary term for* $U\in\mathcal{E}_c^{0,q}\cap Dom(T)\cap DomS^*$*:*

$$\int_{\partial S_\varphi^{\epsilon_0}}|U|^2dV+\int_{V_{c,0,1}}|U^2dV\le C\left(Q(U,U)+\int_F|U|^2dV\right), \tag{5.24}$$

where F *is a certain fixed compact subset of* $S_\varphi^{\epsilon_0}$*. Other related sub-elliptic estimates will be discussed in [Li].*

Acknowledgement: The authors thank the referee for many helpful comments.

References

[Cat] D. Catlin, *Sufficient Conditions for the Extension of CR Structures*, *Jour. Geom. Aanlysis*(4) 4, 467-538(1994).

[Cho] S. Cho, *Extension of CR Structure on Pseudoconvex CR Manifolds with One Degenerate Eigenvalues* , Tohoku Math. J. 55, 321-360(2003).

[CC] D. Catlin and S. Cho, *Extension of CR Structure on three dimensional Compact pseudoconvex CR manifolds*, Math. Ann. 334, 253-280(2006).

[CS] S. C. Chen, M. -C. Shaw, *Partial Differential Equations in Several Complex Variables*, AMS/IP Studies in Advanced Mathematics, 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.

[CSh] D. Chakrabarti and M. -C. Shaw, *L^2 Serre Duality on Domains in Complex Manifolds and Applications*, Trans. Amer. Math. Soc. 364, 3529-3554(2012).

[DE] J. P. Demailly, *Complex Analytic and Differential Geometry*, Monograph Grenoble, 1997.

[FK] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex,* Annals of Math. Studies 75. Princeton Universities Press, Princeton, NJ, 1972.

[HLY] X. Huang, S. Luk and S. S. T. Yau, *On a CR Family of Compact Strongly Pseudoconvex CR Manifolds,* J.Differential Geometry. 72, 353-379(2006).

[Ho1] L. Hörmander, *L^2 Estimates and Existence Theorems for the $\overline{\partial}$ operator*, Acta Math. 113, 89-152 (1965).

[Ho2] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Third Edition, North-Holland, 1990.

[Li] X. Li, Ph. D. Thesis, Wuhan University, 2014. (to appear)

Xiaojun Huang (huangx@math.rutgers.edu), School of Mathematics and Statistics, Wuhan University, Hubei 430072, China; and Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA.

Xiaoshan Li (xiaoshanli@whu.edu.cn), School of Mathematics and Statistics, Wuhan University, Hubei 430072, China.