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Non-embeddable real algebraic hypersurfaces

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Abstract We study various classes of real hypersurfaces that are not embeddable into more special hypersurfaces in higher dimension, such as spheres, real algebraic compact strongly pseudoconvex hypersurfaces or compact pseudoconvex hypersurfaces of finite type. We conclude by stating some open problems.

1 Introduction

This paper is motivated by the following general problem:

Given a real hypersurface M in a complex manifold X, when can it be (holomorphically) embedded into a more special real hypersurface M' in a complex manifold X' of possibly larger dimension? More specifically, which strongly pseudoconvex hypersurfaces can be embedded into a sphere?

By a holomorphic map (resp. embedding) of M into M', we mean a holomorphic map (resp. embedding) of an open neighborhood of M in X into X', sending M into M'. In particular, it follows that a hypersurface holomorphically embeddable into a sphere $\mathbb{S}^{2N-1} := \{\sum_j |z_j|^2 = 1\} \subset \mathbb{C}^N$ is necessarily strongly pseudoconvex and real-analytic. However, not every strongly pseudoconvex real-analytic hypersurface can be even locally embedded into a sphere, as was independently shown by Faran [9] and Forstneric [10]. These results, showing that such hypersurfaces in general position are not embeddable into spheres, were more

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recently further extended and strengthened by Forstneric [11] showing that they also do not admit transversal holomorphic embeddings into a hyperquadric

$$\mathbb{H}_\ell^{2N-1} := \left\{ -\sum_{j\le\ell} |z_j|^2 + \sum_{j>\ell} |z_j|^2 = 1 \right\} \subset \mathbb{C}^N$$

of any signature ℓ. (By a transversal embedding F we mean one not sending the tangent space T_pX into $T_{F(p)}\mathbb{H}_\ell^{2N-1}$ for $p \in M$.)

Explicit examples of non-embeddable strongly pseudoconvex real-analytic hypersurfaces were given by the second author [25] along with explicit invariants serving as obstructions to embeddability. In Theorem 2.1 below we give an example of a *compact* strongly pseudoconvex real-analytic hypersurface in $\mathbb{C}^2$ that does not admit any holomorphic embedding into a sphere (and more generally any transversal holomorphic embedding into a hyperquadric).

The existence of non-embeddable real-analytic hypersurfaces suggests to consider the embeddability problem for the more restricted class of *real-algebraic* hypersurfaces, i.e. ones locally given by real polynomial equations. In this line, Webster [23] showed in 1978 that any Levi-nondegenerate real-algebraic hypersurface does in fact admit transversal holomorphic embeddings into hyperquadrics of suitable dimension and signature. As a consequence of the study of the Chern–Moser–Weyl tensor, Huang and Zhang [16] obtained concrete algebraic Levi non-degenerate hypersurfaces with positive signature which cannot be holomorphic embedded into a hyperquadric (with the same signature) of any dimension.

During the Conference on Several Complex Variables and PDEs in Serra Negra, Brazil, in August 2011, the authors observed that the strongly pseudoconvex (near 0) real-algebraic hypersurface defined by

$$M := \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : \mathsf{Im}\, z_n = \sum_{j=1}^{n-1} |z_j|^2 - |z_1|^4 \right\}, \quad n \ge 3,$$

is not locally (holomorphically) embeddable into any sphere of any dimension nor into any closed strongly pseudoconvex real-algebraic hypersurface $M' \subset \mathbb{C}^N$ for any N. In fact, any such embedding would be algebraic by a result of the first author [12] and hence would extend (as holomorphic embedding into M') to points of M of mixed Levi signature, which is impossible. In Theorem 2.2 below, we state a generalization of this phenomenon leading to many simple examples of strongly pseudoconvex real-algebraic hypersurfaces that are not holomorphically embeddable even into any closed pseudoconvex hypersurface $M' \subset \mathbb{C}^N$ of finite D'Angelo type.

One can similarly construct the following locally non-embeddable example in $\mathbb{C}^2$:

$$M := \{ z \in \mathbb{C}^2 : \mathsf{Im}\, z_2 = |z|^2 - |z|^4 \},$$

where the proof is based on the observation that any potential embedding would be extendable to "large" Levi-degenerate sets, which is impossible (see Theorem 2.4 below). Along the same lines, we further study the property of a class of real algebraic pseudoconvex hypersurfaces discovered by Kohn and Nirenberg [17], not to admit holomorphic supporting functions near certain weakly pseudo-convex points (and hence not locally holomorphically convexifiable near these points). We will prove a general non-embeddability result in Theorem 3.6 below, which is, in addition to the Kohn–Nirenberg property, based on a property stated in Proposition 3.10, which roughly says that in certain situations a holomorphic extension of a local embedding from M into M' even along paths outside M still sends M into M'.

Proposition 3.10 is a generalization of what is called the invariant property for holomorphic correspondences in the literature (see [14]). However, our proof here is more geometric and also simpler even in the case considered in [14]. Our general non-embeddability theorem immediately leads to many examples of compact pseudoconvex real-algebraic hypersurfaces, strongly pseudoconvex away from a single point, that are not locally holomorphically embeddable into any compact strongly pseudoconvex real-algebraic hypersurface of any dimension. We also mention recent related preprint by Ebenfelt and Son [8].

We next address the related problem for hypersurfaces of positive (mixed) Levi signature. That is, whether there exists a *compact Levi nondegenerate real-algebraic hypersurface of signature* $\ell > 0$ that is not transversally embeddable into a hyperquadric $\mathbb{H}_\ell^N$ of higher dimension but the same signature ℓ. Note that Webster's result [23] shows that without the signature restriction, such an embedding is always possible. However, based on a monotonicity property of the *Chern–Moser–Weyl tensor* [16] and algebraicity results in [13] (see also [24] and [5]), we give in Sect. 4 below examples of compact real-algebraic Levi-nondegenerate hypersurfaces of positive Levi signature in the projective space that are not transversally locally holomorphically embeddable into any hyperquadric of any dimension but the same signature. (Note that there is no compact hypersurface in $\mathbb{C}^n$ with positive signature, since any such hypersurface must have a strongly pseudoconvex point.)

Finally we mention some open problems in the last section.

2 Hypersurfaces not embeddable into certain real-algebraic hypersurfaces

We first recall that a smooth real hypersurface in an open subset U of $\mathbb{C}^n$ is called real algebraic, if it has a real-valued polynomial defining function. A real algebraic hypersurface in U has an extension to a real analytic variety in $\mathbb{C}^n$, which may possess singularities. Of course, all real algebraic hypersurfaces in $\mathbb{C}^n$ are automatically smooth and closed.

2.1 A compact strongly pseudoconvex real-analytic hypersurface not embeddable into any strongly pseudoconvex real-algebraic hypersurface

In [24, Corollary 1.2], the second author gave an explicit example of a germ of real-analytic strongly pseudoconvex hypersurface in $\mathbb{C}^2$ that is not transversally holomorphically embeddable into any Levi-nondegenerate real-algebraic hypersurface. By following verbatim the proof of [25, Corollary 1.2] one has:

Theorem 2.1 *For* $0 < \varepsilon \ll 1$,

$$M = \left\{ (z, w) \in \mathbb{C}^2 : |z| < 1,\ |w|^2 + |z|^2 + \varepsilon \mathrm{Re} \sum_{k \geq 2} z^k \bar{z}^{(k+2)!} = 1/2 \right\}$$

is a compact strongly pseudoconvex hypersurface that does not admit a nontrivial holomorphic embedding into any Levi-nondegenerate real-algebraic hypersurface in $\mathbb{C}^N$. *In particular,* M *is not holomorphically embeddable into any strongly pseudoconvex real-algebraic hypersurface in* $\mathbb{C}^N$.

2.2 Strongly pseudoconvex real-algebraic hypersurfaces not embeddable into finite type real-algebraic hypersurfaces

Recall that a real-analytic hypersurface is of *finite D'Angelo type* if and only if it does not contain any complex curve. By a point of *mixed Levi signature* we mean a point of a real hypersurface, where the Levi form (for a choice of conormal) has both positive and negative eigenvalues. We next state the following:

Theorem 2.2 *Let $M \subset \mathbb{C}^{n+1}$ be a connected real-algebraic hypersurface with a point of mixed Levi signature. Then any holomorphic map sending an open subset of M into any closed pseudoconvex finite D'Angelo type real-algebraic hypersurface $M' \subset \mathbb{C}^{N+1}$ is constant.*

Proof Obviously, M has nonzero Levi form on a dense subset. By [24], [5] (or by [12] when the target is strongly pseudoconvex), any holomorphic map F sending an open subset of M into M' is complex-algebraic. Since M is connected and since the branching variety of F is of complex codimension one (if F is not a single-valued), we can extend F along a path to a neighborhood of a point $p \in M$ of mixed Levi signature, still sending M into M'. Since M' is pseudoconvex, F must be constant near p and hence it is constant. □

Example 2.3 Consider the hypersurface

$$M := \{z \in \mathbf{C}^n : n \geq 3,\ \operatorname{Im} z_n = |z|^2 - |z_1|^4\}. \tag{2.1}$$

Then no open piece of M can be holomorphically embedded into any closed real-algebraic hypersurface of finite D'Angelo type in $\mathbb{C}^N$ for any N.

2.3 Hypersurfaces with large Levi-degenerate set

Theorem 2.4 *Let $M \subset \mathbb{C}^{n+1}$ be a connected real-algebraic hypersurface, Levi-nondegenerate at some point, whose set of Levi-degenerate points contains a real-analytic submanifold that is generic in $\mathbb{C}^{n+1}$. Then any holomorphic map sending an open subset of M into any strongly pseudoconvex real-algebraic hypersurface $M' \subset \mathbb{C}^{N+1}$ is constant.*

Proof It follows from the assumptions that the set $S \subset M$ of Levi-degenerate points is a generic real-analytic submanifold near some point $p \in S$. As in the Proof of Theorem 2.2 any holomorphic map F sending an open subset of M into M' extends holomorphically and algebraically into an open neighborhood of a point $p \in S$ as above, still sending M into M'. (Note that algebraicity here already follows from [12].) Since M' is strongly pseudoconvex, the extension F must have rank less than $n+1$ for all $q \in S$ near p. Since S is a generic submanifold of $\mathbb{C}^{n+1}$, the rank of F is less than $n+1$ in an open neighborhood of p. On the other hand, F is either constant or of full rank $n+1$ at any Levi-nondegenerate point of M. Hence F must be constant. □

Example 2.5 A simple example of M satisfying the assumptions of Theorem 2.4 is

$$M := \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2 - |z|^4\},$$

which is strongly pseudoconvex near 0. The Levi-degenerate set here is $\{(z, w) \in M : |z| = 1/2\}$ and hence Theorem 2.4 applies.

Example 2.6 An example of a compact pseudoconvex M satisfying the assumptions of Theorem 2.4 is the following boundary of a Reinhardt domain:

$$M := \{(z, w) \in \mathbb{C}^2 : (|z|^2 + |w|^2)^4 + (|z|^2 - |w|^2)^4 = 1\}.$$

Away from the cross $zw = 0$, M is locally biholomorphically equivalent to

$$\widetilde{M} := \{(z,w) \in \mathbb{C}^2 : (|z| + |w|)^4 + (|z| - |w|)^4 = 1\},$$

under the finite holomorphic map $(z,w) \mapsto (z^2, w^2)$. The real part of $\widetilde{M}$ is the rotated convex curve $\{x^4 + y^4 = 1/4\}$ whose real part is convex. Then M is pseudoconvex and is Levi-degenerate along the generic submanifold $\{(z,w) \in M : |z| = |w|\}$.

3 Non-embeddable Kohn–Nirenberg type domains

Example 3.1 Given integers $0 < l < k$, consider the following famous Kohn–Nirenberg domain [17]. (In the paper of Kohn–Nirenberg [17], though only the case with $l = 1, k = 4, c = \frac{15}{7}$ was studied, the result in their paper holds, with the same argument, for the following more general domain which we still call the Kohn–Nirenberg domain):

$$\Omega = \{(z,w) \in \mathbf{C}^2 : \rho = -\mathrm{Im}\, w + z^k \bar{z}^k + c\mathrm{Re}\,(z^l \bar{z}^{2k-l}) < 0\}, \quad 2 < |c| < \frac{k^2}{l(2k-l)}. \tag{3.1}$$

Also notice that the boundary of Ω is of type $2k$ at 0 and of bi-type $(l, 2k-l)$. It is easily seen that Ω is smooth. Since the Levi form of $\partial\Omega$ is positive over $\partial\Omega \setminus L_0$ with $L_0 := \{\mathrm{Im}\, w = 0, z = 0\}$, and is semi-definite along L_0, we see that Ω is strongly pseudoconvex away from L_0 and is weakly pseudoconvex of finite type along L_0. Kohn and Nirenberg [17] proved the following basic feature of the boundary of Ω that we call here *Kohn–Nirenberg property*:

Definition 3.2 A real hypersurface $M \subset \mathbb{C}^n$ is said to satisfy the *Kohn–Nirenberg property* at a point $p \in M$ if for any holomorphic function $h \not\equiv 0$ in any neighborhood U of p in $\mathbb{C}^n$ with $h(p) = 0$, the zero set $\mathcal{Z}$ of h intersects M transversally at some smooth point of $\mathcal{Z}$ near p.

In particular, a hypersurface with the Kohn–Nirenberg property at a point is always minimal at that point. (We mention that when $M \cap \mathcal{Z}$ separates $\mathcal{Z}$, it has Hausdorff codimension one and thus must be generically smooth in $\mathcal{Z}$.) See also Example 3.5 for *compact* hypersurfaces with the Kohn–Nirenberg property. The argument in [17] is very general and can be used to obtain further classes of hypersurfaces satisfying the Kohn–Nirenberg property. We mention the paper by Kolar [18] for a discussion of the similar but different property of local holomorphic convexifiability.

We shall also consider local holomorphic supporting functions:

Definition 3.3 A subset $M' \subset \mathbb{C}^N$ is said to admit *local holomorphic supporting functions* if for each $q \in M'$, there is a neighborhood Ω of q in $\mathbb{C}^N$ and a holomorphic function h in Ω such that $h(q) = 0$ but $\mathrm{Im}\, h(z) < 0$ for $z \in M' \cap \Omega$, $z \neq q$.

Remark 3.4 In particular, when M' is a smooth hypersurface of finite D'Angelo type and locally holomorphically convexifiable, it admits local holomorphic supporting functions. This is a consequence of a result of McNeal on the equivalence of linear type and D'Angelo type for convex domains. (See [7], for instance).

Theorem 3.6 below implies that no open piece of the boundary of the classical Kohn–Nirenberg domain can be mapped by a non-constant holomorphic map into any connected compact smooth algebraic hypersurface in $\mathbb{C}^n$, that is locally holomorphically convexfiable.

Example 3.5 Consider the following compactified Kohn–Nirenberg type domain:

$$\Omega=\{(z,w)\in\mathbb{C}^2:\rho=\epsilon(|z(w-1)|^2+|z|^{2k}+c|z|^{2l}\operatorname{Re} z^{2k-2l})+|w|^2+|z|^{2k+2}-1<0\}. \tag{3.2}$$

where $0<\epsilon\ll 1$ and l,k,c as in (3.1). Then Ω is a smoothly bounded real-algebraic domain, which is pseudoconvex and strongly pseudoconvex away from $p_0:=(0,1)$. Since the principal terms in ρ at p_0 are the same as those in the classical Kohn–Nirenberg domain case, one still has the Kohn–Nirenberg property at p_0 by the same argument. Again, by Theorem 3.6 below, no open piece of $\partial\Omega$ can be mapped by a non-constant holomorphic map into a smooth compact algebraic hypersurface M', that admits local holomorphic supporting functions.

To get a similar higher dimensional example with the Kohn–Nirenberg property, we need only to find one which includes the boundary of the domain in (3.2) as its CR submanifold. For instance, the boundary of the following domain serves this purpose:

$$\{(z,w)=(z_1,z',w)\in\mathbb{C}\times\mathbb{C}^{n-2}\times\mathbb{C}:\\ \rho=\epsilon(|z_1(w-1)|^2+|z_1|^{2k}+c|z_1|^{2l}\operatorname{Re} z_1^{2k-2l})+|w|^2+|z'|^2+|z_1|^{2k+2}<1\}. \tag{3.3}$$

Theorem 3.6 *Let $M\subset\mathbb{C}^n$ $(n>1)$ be a connected minimal real-algebraic hypersurface, which has the Kohn–Nirenberg property at some point. Let $M'\subset\mathbb{C}^N$ be a compact real-algebraic subset admitting local holomorphic supporting functions at each point. Then any holomorphic map sending an open piece of M into M' is constant.*

The proof is broken up in a sequence of lemmas.

Lemma 3.7 *Let $U\subset\mathbb{C}^n$ be a simply connected open set and $\mathcal{S}\subset U$ a closed complex analytic subset of codimension one. Then for $p\in U\setminus\mathcal{S}$, the fundamental group $\pi_1(U\setminus\mathcal{S},p)$ is generated by loops obtained by concatenating paths $\gamma_1,\gamma_2,\gamma_3$, where γ_1 connects p with a point arbitrarily close to a smooth point $q_0\in\mathcal{S}$, γ_2 is a loop around $\mathcal{S}$ near q_0 and γ_3 is γ_1 reversed.*

Here, by saying that γ_2 goes around q_0, we mean there is a closed embedded real 2-disk $\overline{D}$ in U such that γ_2 is the boundary of $\overline{D}$ and D intersects $\mathcal{S}$ only and transversally at p_0.

Proof Replacing U by $U\setminus Sing(\mathcal{S})$ if needed, we can assume that S is smooth. Here $Sing(\mathcal{S})$ is the singular set of $\mathcal{S}$, which has codimension at least two, hence $U\setminus Sing(\mathcal{S})$ is still simply-connected. Take any loop $\gamma\in\pi_1(U\setminus\mathcal{S},p)$. Since U is simply connected, γ can be contracted to p inside U, i.e. γ viewed as a map from $S^1:=\{|z|=1\}\subset\mathbb{C}$ into U can be continuously extended to the disk $\overline{\Delta}:=\{|z|\le 1\}$. Using Thom's transversality, the disk extension can be approximated by a smooth immersion $\Gamma\colon\overline{\Delta}\to U$ such that $\Gamma|_{S^1}$ is a smooth Jordan loop defining the same class in $\pi_1(U\setminus\mathcal{S},p)$ as γ, and such that $\Gamma(\Delta)$ intersects $\mathcal{S}$ transversally at finitely many smooth points. Since the fundamental group of the disk Δ minus finitely many points is generated by loops going around single points, it is easy to see that $\Gamma|_{S^1}$ and hence γ is generated by loops inside $\Gamma(\overline{\Delta})$ as described in the lemma. □

Let $M\subset U(\subset\mathbb{C}^n)$ be a closed real-analytic subset defined by a family of real-valued real analytic functions $\{\rho_\alpha(z,\overline{z})\}$. Assume that the complexification $\rho_\alpha(z,\xi)$ of $\rho_\alpha(z,\overline{z})$ is holomorphic over $U\times\operatorname{conj}(U)$ with

$$\operatorname{conj}(U):=\{z:\overline{z}\in U\}$$

for each α. Then the complexification $\mathcal{M}$ of M is the complex-analytic subset in $U \times \mathrm{conj}(U)$ defined by $\rho_\alpha(z, \xi) = 0$ for all α. Then for $w \in \mathbb{C}^n$, the Segre variety of M associated with the point w is defined by $Q_w := \{z : (z, \bar{w}) \in \mathcal{M}\}$. Recall the basic properties of the Segre varieties: $z \in Q_w \iff w \in Q_z$ and $z \in Q_z \iff z \in M$. (See [14] for more related notations and definitions.)

Lemma 3.8 *Let $M \subset \mathbb{C}^n$ be a minimal real-analytic hypersurface at a point $p_0 \in M$. Then there exist small open neighborhoods U, $\widetilde{U}$ of p_0 in $\mathbb{C}^n$ with $U \subset\subset \widetilde{U}$ such that the following holds:*

1. *For every $z \in U$, the Segre variety Q_z is a nonempty closed connected smooth hypersurface in $\widetilde{U}$.*
2. *There is no complex hypersurface $H \subset U$ such that $Q_z \equiv Q_w$, when restricted to $\widetilde{U}$, for all $z, w \in H$.*

Proof Let M be a real analytic hypersurface near p_0 as in the lemma with a real analytic defining function ρ near p_0. (1) is a direct consequence of the implicit function theorem and is standard in the literature.

We prove (2) by contradiction assuming there exists H as in the lemma. Suppose $p_0 \in H$. Since $p_0 \in Q_{p_0}$, and for any $w \in H$, we must have $Q_w \equiv Q_{p_0}$. Hence $p_0 \in Q_w$ and therefore $w \in Q_{p_0} \equiv Q_w$. Hence $w \in Q_w$ and thus $H \subset M$, contradicting nonminimality of M.

For H general, and for $z \in H$ $q \in Q_z$, we have $q \in Q_w \equiv Q_z$ for any $w \in H$. Therefore $w \in Q_q$, and thus $H \subset Q_q$, which gives that $H = Q_q$. Hence, by the property of H, we see that $E_q := \cup_{z \in Q_q} Q_z = Q_{z_0}$ for any $z_0 \in H$ and thus is a complex hypersurface.

On the other hand, assume without loss of generality that $p_0 = 0$ and $\frac{\partial \rho}{\partial z_n}(0) \neq 0$ for a real-analytic defining function of M. Then there is a holomorphic function Ψ in its variables such that E_q is defined, in the (ξ', ξ_n)-coordinates, near 0 by $\xi_n = \Psi(\overline{z'}, q, \xi')$ with parameter $z' \approx 0$. The latter notation here means that z' is sufficiently close to 0 and we shall use it in the sequel. Now, suppose that the statement in (2) fails no matter how we shrink U. Then we have a sequence $q \to 0$ such that E_q is simply defined by $\xi_n = \Psi(0, q, \xi')$. Passing to the limit, we get E_0 is defined by $\xi_n = \Psi(0, 0, \xi')$. This contradicts the minimality as argued above. □

Lemma 3.9 *Let $M \subset \mathbb{C}^n$ be a minimal real-analytic hypersurface at a point $p_0 \in M$ and $\mathcal{S}$ a closed proper complex analytic subset in a neighborhood of p_0 with $p_0 \in \mathcal{S}$. Then there exists a small (simply-connected) open neighborhood U of p_0 in $\mathbb{C}^n$, such that the following holds.*

Take $p \in (M \cap U) \setminus \mathcal{S}$ and let $\gamma \in \pi_1(U \setminus \mathcal{S}, p)$ be obtained by concatenation of $\gamma_1, \gamma_2, \gamma_3$ as in Lemma 3.7, where γ_2 is a small loop around $\mathcal{S}$ near a smooth point $q_0 \in \mathcal{S} \cap U$. Then γ can be slightly perturbed to a homotopic loop $\widetilde{\gamma}(t)$ in $\pi_1(U \setminus \mathcal{S}, p)$ such that there exists a null-homotopic loop $\lambda(t)$ in $\pi_1(U \setminus \mathcal{S}, p)$ with $(\lambda(t), \overline{\widetilde{\gamma}(t)})$ contained in the complexification $\mathcal{M}$ of M for all t. Also, for any element $\hat{\gamma} \in \pi_1(U \setminus \mathcal{S}, p)$, after a small perturbation of $\hat{\gamma}$ if needed, we can find a null-homotopic loop $\hat{\lambda} \in \pi_1(U \setminus \mathcal{S}, p)$ such that $(\hat{\gamma}, \overline{\hat{\lambda}}) \subset \mathcal{M}$.

Proof Let $p_0 \in U \subset \widetilde{U}$ be satisfying the conclusion of Lemma 3.8. Shrinking U if necessary, we may assume that there exists a real analytic (reflection) map $\mathcal{R}\colon U \to U$ with $\mathcal{R}^2 = \mathrm{id}$, $\mathcal{R}|_M = \mathrm{id}$ and $(\mathcal{R}(z), \overline{z}) \in \mathcal{M}$ for all $z \in U$, namely, $\mathcal{R}(z) \in Q_z$. In fact, the map $\mathcal{R}$ can be obtained by slicing M transversally by a family of parallel complex lines $\{L\}$ near p_0 and then taking the Schwarz reflection about $M \cap L$ inside each L of the family. More precisely, let L_0 be a complex line through p_0 intersecting M transversally at p_0. Then sufficiently small neighborhood U of p_0 is foliated by lines L parallel to L_0, which still intersect M transversally. Shrinking U suitably, we may assume that the Schwarz reflection about $M \cap L$ is defined in $U \cap L$ and leaves the latter invariant. Then define $\mathcal{R}$ to be the Schwarz reflection along each line L. (We can of course arrange U such that for any line L, the pair of the reflecting points with respect to $U \cap L$ stays inside U and thus $\mathcal{R}(U) = U$.)

We now claim that we can slightly perturb $q_0 \in \mathcal{S}$ and the direction of the parallel lines (and hence $\mathcal{R}$) such that $\mathcal{R}(q_0) \notin \mathcal{S}$. Indeed, by (2) of Lemma 3.8 applied to $H = \mathcal{S}$, we conclude that $Q_q \not\equiv Q_{q'}$ for two generic $q, q' \in \mathcal{S}$ arbitrarily close to q_0. Then either Q_{q_0} contains points away from $\mathcal{S}$ arbitrarily close to $q_0' := \mathcal{R}(q_0)$ or an open piece of Q_{q_0} is contained in $\mathcal{S}$. But the latter case together with $Q_q \not\equiv Q_{q'}$ with $q, q'(\in \mathcal{S}) \approx q_0$ implies that Q_q cannot contain an open piece of $\mathcal{S}$ for a generic $q \approx q_0$. Then we can choose such q and $q' \in Q_q \setminus \mathcal{S}$ arbitrarily close to q_0 and $q_0' = \mathcal{R}(q_0)$, respectively. Considering the line through q and q' and using the lines parallel to this one to redefine $\mathcal{R}$, this proves the claim.

After slightly perturbing q_0 and $\mathcal{R}$ as in the above, it follows that there exists a sufficiently small open ball Ω containing q_0 such that $\mathcal{R}(\Omega) \cap \mathcal{S} = \emptyset$. Then the paths $\gamma_1, \gamma_2, \gamma_3$ can be perturbed homotopically into $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$ respectively, where $\widetilde{\gamma}_1$ connects p with a point in Ω, $\widetilde{\gamma}_2$ is a loop around $\mathcal{S}$ inside Ω and also sufficiently close to q_0, and $\widetilde{\gamma}_3$ is $\widetilde{\gamma}_1$ reversed such that the loop $\widetilde{\gamma}$ obtained by concatenation of $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$ is homotopic to γ in $\pi_1(U \setminus \mathcal{S}, p)$ and we can take $\lambda(t) := \mathcal{R}(\widetilde{\gamma}(t))$. (Of course, we may need to slightly perturb $\widetilde{\gamma}_1$ to make sure that λ avoids $\mathcal{S}$.) Then λ is null-homotopic in $\pi_1(U \setminus \mathcal{S}, p)$ since $\mathcal{R}(\Omega)$ does not intersect $\mathcal{S}$.

The last statement in the lemma follows from the symmetric property of the reflection map (Segre varieties) and what we just proved. The proof is complete. □

We now choose $\mathcal{R}$ as in the above proof above, defined in a neighborhood of a point $p_0 \in M$.

Proposition 3.10 *Let $\Omega \subset \mathbb{C}^n$, $V \subset \mathbb{C}^N$ be connected open sets, $M \subset \Omega$ a real analytic hypersurface, $M' \subset V$ a real-analytic subset defined by a set of real valued real analytic functions $\{\rho_\alpha\}$ over V, $\mathcal{S} \subset \Omega$ a proper closed complex analytic subset and $\mathcal{F} \subset (\Omega \setminus \mathcal{S}) \times V$ a complex submanifold whose projection to $\Omega \setminus \mathcal{S}$ is a finite sheeted covering. Suppose that:*

1. *M is minimal at $p_0 \in M$.*
2. *The complexification $\rho_\alpha(z, \xi)$ for each α is holomorphic over $V \times$* conj(V).

Then there exists a neighborhood U of p_0 in Ω, depending only on M and p_0, such that if a certain local branch F of $\mathcal{F}$, defined over a subdomain $D \subset U \setminus \mathcal{S}$ with $D \cap M \neq \emptyset$ sends a $D \cap M$ into M', then any other branch of $\mathcal{F}$ obtained by continuing F along paths in $U \setminus \mathcal{S}$ also sends M into M'. Equivalently, if $\mathcal{F}'$ is the connected component of $\mathcal{F} \cap (U \setminus \mathcal{S}) \times V$ containing the graph of F, then for any $(z, w) \in \mathcal{F}'$ with $z \in M$, we have $w \in M'$. More generally, slightly perturbing $\mathcal{R}$ if needed, we have $F_1(Q_z \cap O(\mathcal{R}(z))) \in Q'_{F_2(z)}$ for any $(z, F_1(z)), (z, F_2(z)) \in \mathcal{F}'$ with $z, \mathcal{R}(z) \in U \setminus \mathcal{S}$. Here for any $w \in V$, $Q'_w := \{z \in V : \rho_\alpha(z, \overline{w}) = 0 \ \forall \alpha\}$, and $O(a)$ denotes a small neighborhood of a in $\mathbb{C}^n$.

Proof Let U be a simply connected neighborhood of p_0 in Ω. We need only consider the case when $\mathcal{S}$ is of codimension one in Ω, for, otherwise, $U \setminus \mathcal{S}$ is also simply connected. Hence, the continuation of F along curves in $U \setminus \mathcal{S}$ defines a holomorphic map over $U \setminus \mathcal{S}$ and all the properties stated in the Proposition follows easily. We also choose $U \subset \Omega$ such that the conclusion of Lemma 3.9 is satisfied. In addition, we can choose U such that $M \cap U$ is connected and minimal.

Denote by $F : D \subset U \setminus \mathcal{S} \to V$ a local branch of $\mathcal{F}$, where D is a domain with $D \cap M \neq \emptyset$, sending $D \cap M$ into M'. Let $p \in D \cap M$. Then $(F, \overline{F})\colon D \times \mathrm{conj}(D) \to V \times \mathrm{conj}(V)$ sends an open neighborhood of $(p, \bar{p})$ in the complexification $\mathcal{M}$ of M into $\mathcal{M}'$.

Let $F_1 : D_1 \subset U \setminus \mathcal{S} \to V$ be a branch of $\mathcal{F}$ with some point $p_1 \in M \cap D_1$, obtained by continuing F along a path in $U \setminus \mathcal{S}$, connecting p_1 with p. Since $M \cap U$ is connected and minimal, $(M \cap U) \setminus \mathcal{S}$ is also connected. Hence there exists a path γ in $(M \cap U) \setminus \mathcal{S}$ connecting p with p_1. Then by the analyticity of M', the branch F_2 of $\mathcal{F}$ obtained by continuing F along γ is sending a neighborhood of p in M into M'.

Hence $(F_2, \overline{F_2}) := (F_2(\cdot), \overline{F_2(\cdot)})$ sends a neighborhood of $(p_1, \overline{p}_1)$ in $\mathcal{M}$ into $\mathcal{M}'$. Now the branch F_1 is obtained from F_2 by continuation along a certain loop γ in $\pi_1(U \setminus \mathcal{S}, p_1)$. Notice that $\mathcal{R}^2 = \mathsf{id}$. By Lemmas 3.7 and 3.9, slightly perturbing γ and $\mathcal{R}$ if needed, we can assume that $\lambda(t) = \mathcal{R}(\gamma(t))$ is a null homotopic loop in $U \setminus \mathcal{S}$. Notice that $\gamma = \mathcal{R}(\lambda)$. Applying the holomorphic continuation along the loop $(\gamma, \overline{\lambda})$ in $\mathcal{M}$ for $\rho_\alpha(F_1, \overline{F}_1)$ for each α, one concludes by the uniqueness of analytic functions that $(F_1, \overline{F}_2)$ also sends a neighborhood of $(p_1, \overline{p}_1)$ in $\mathcal{M}$ into $\mathcal{M}'$. Namely, for any z near p_1, we have $F_1(Q_z \cap O(\mathcal{R}(z))) \subset Q'_{F_2(z)}$, where $O(a)$ as before denotes a small neighborhood of a. Now, applying the holomorphic continuation along the loop $(\lambda, \overline{\gamma})$ in $\mathcal{M}$ for $\rho_\alpha(F_1, \overline{F}_2)$ for each α, one concludes by the uniqueness of analytic functions that $(F_1, \overline{F}_1)$ sends a neighborhood of $(p_1, \overline{p}_1)$ in $\mathcal{M}$ into $\mathcal{M}'$. In particular, F_1 maps a neighborhood of p_1 in M into M'. (Cf. Lemma 2.1 of [15]). The last assertion in the proposition can be proved with a similar argument based on the holomorphic continuation and the uniqueness property for analytic functions. □

Lemma 3.11 *For an open set $U \subset \mathbb{C}^n$, consider the complex analytic subset*

$$\mathcal{F} := \left\{ (z, w) \in U \times \mathbb{C} : \sum_{l=0}^{m} a_l(z) w^l = 0 \right\}, \tag{3.4}$$

where $a_0(z), \ldots, a_m(z)$ are holomorphic functions in U that do not simultaneously vanish on a (possibly singular) complex hypersurface. Suppose that $M \subset U$ is a real-analytic hypersurface and $C > 0$ is a constant such that $|F(z)| \leq C$ for any branch $F : \Omega \to \mathbb{C}$ of $\mathcal{F}$ and any $z \in M \cap \Omega$. Write $\mathcal{S}' := \{z \in U : a_m(z) = 0\}$. Then $M \cap \mathcal{S}'$ is contained in a complex analytic subset of $\mathcal{S}'$ of positive codimension.

Proof Since $a_0(z), \ldots, a_m(z)$ do not simultaneously vanish on a (possibly singular) complex hypersurface, for each non-empty irreducible component $\mathcal{C}$ of $\mathcal{S}'$, there exists $j < m$ such that $a_j(z)$, does not vanish identically on $\mathcal{C}$. Hence $\{a_j = 0\}$ defines a complex analytic subset of $\mathcal{S}'$ of positive codimension.

We claim that $M \cap \mathcal{C} \subset \{a_j = 0\}$. Indeed, suppose on the contrary, there exists $z_0 \in M \cap \mathcal{C}$ with $a_j(z_0) \neq 0$. Since M is a real hypersurface, there exists a sequence $z_k \in M \setminus \mathcal{S}'$ converging to z_0 as $k \to \infty$ such that $\mathcal{F}$ has m branches (counted with multiplicity) around each z_k. Since all branches of $\mathcal{F}$ are uniformly bounded on M by our assumption, the same is true for their symmetric functions. In particular, $\frac{a_j(z)}{a_m(z)}$ is also bounded along z_k. On the other hand $a_m(z_0) = 0,\ a_j(z_0) \neq 0$ imply $\frac{a_j(z_k)}{a_m(z_k)} \to \infty$ as $k \to \infty$, which is a contradiction.

This proves the claim and therefore, $M \cap \mathcal{C}$ is contained in the set $\{a_j = 0\}$ of positive codimension. Since $\mathcal{C}$ is an arbitrary irreducible component of $\mathcal{S}'$, the proof is complete. □

Corollary 3.12 *In addition to the assumptions of Lemma 3.11, suppose that M satisfies the Kohn–Nirenberg property at $p \in M$ (see Definition 3.2). Then $a_m(p) \neq 0$.*

Proof Indeed, otherwise by the Kohn–Nirenberg property, the zero set $\mathcal{S}'$ of $a_m(z)$ must intersect M transversally at some smooth point. The latter implies that $M \cap \mathcal{S}'$ is a real hypersurface in $\mathcal{S}'$ at such point, which contradicts the conclusion of Lemma 3.11. □

Proof of Theorem 3.6 Assume the hypotheses in Theorem 3.6. Suppose that there is a non-constant holomorphic map F sending an open piece M into M'. By a result of Diederich and Fornaess [6], M' does not contain non-trivial holomorphic curves. Since M is minimal, by [24] and [5], F is complex algebraic. In particular, F extends holomorphically along any path away from a proper complex algebraic subset $\mathcal{S} \subset \mathbb{C}^n$. We need only prove the theorem assuming that $\mathcal{S}$ is a codimension one complex analytic variety near $p_0 \in M$ with the Kohn–Nirenberg property.

Since M is minimal and connected, $M \setminus \mathcal{S}$ is also connected. Then F has holomorphic extensions to points of $M \setminus \mathcal{S}$ arbitrarily close to a point $p_0 \in M$ with the Kohn–Nirenberg property, sending M into M'. Now Proposition 3.10 implies that there exists a neighborhood U of p_0 in $\mathbb{C}^n$ and an extension $\widetilde{F}$ of F to a point in $M \cap U$ such that any extension of $\widetilde{F}$ along a path in $U \setminus \mathcal{S}$ sends M into M'. Consider the (n-dimensional) Zariski closure $\widetilde{\mathcal{F}} \subset \mathbb{C}^n \times \mathbb{C}^N$ of the graph of $\widetilde{F}$ and denote by $\widehat{\mathcal{F}}$ the analytic irreducible component of $\widetilde{\mathcal{F}} \cap (U \times \mathbb{C}^N)$ containing the graph of $\widetilde{F}$. In particular, $\widehat{\mathcal{F}} \setminus (\mathcal{S} \times \mathbb{C}^N)$ is connected and therefore each branch of $\widehat{\mathcal{F}}$ away from $\mathcal{S}$ sends M into M'.

Since M' is compact, it follows that all branches of $\widehat{\mathcal{F}}$ are uniformly bounded on M. Then Corollary 3.12 implies that, after possible shrinking U around p_0, $\widehat{\mathcal{F}}$ becomes bounded. Furthermore, by further shrinking U, we may assume that $\widehat{\mathcal{F}} \cap (\{p_0\} \times \mathbb{C}^N) = \{(p_0, p_0')\}$ for some $p_0' \in M'$.

Since M' has local holomorphic supporting functions, there exists a holomorphic function h in a neighborhood of p_0' in $\mathbb{C}^N$ such that $h(p_0') = 0$ and $\operatorname{Im} h < 0$ on M' away from p_0'. Let $F_1, \ldots, F_m$ be local branches of $\widehat{\mathcal{F}}$ at $z \in U \setminus \mathcal{S}$, counted with multiplicity. Define $h^* := \sum_{j=1}^m h \circ F_j$. Then h^* is well-defined away from $\mathcal{S}$ and extends holomorphically to p_0 with $h^*(p_0) = 0$. Furthermore, since all branches of $\widehat{\mathcal{F}}$ send M into M', we have $\operatorname{Im} h^*(z) < 0$ for generic $z \in M$ unless $\widehat{\mathcal{F}} \cap (\{z\} \times \mathbb{C}^N) = \{(z, p_0')\}$. Since F is assumed to be non-constant, so is $\widetilde{F}$. Hence there exist points $z \in M$ arbitrarily close to p_0 with $\operatorname{Im} h^*(z) < 0$. In particular, $h^* \not\equiv 0$ and hence, by the Kohn–Nirenberg property, the zero set $\mathcal{Z} := \{h^* = 0\}$ intersects M transversally at some smooth points of $\mathcal{Z}$ arbitrarily close to p_0.

Since M is minimal, one-sided neighborhood D of p_0 is filled by small analytic disks in U attached to M by a result of Trépreau [21] (see also [22]). Therefore we have $\operatorname{Im} h^* \le 0$ in D by the maximum principle. Since $\mathcal{Z}$ intersects M transversally at some points close to p_0, it also intersects D. That is, $\operatorname{Im} h^*(z) = 0$ for some $z \in D$. Now it follows from the maximum principle that $h^* \equiv 0$ in D and therefore in M. But then, as mentioned before, we must have $\widehat{\mathcal{F}} \cap (\{z\} \times \mathbb{C}^N) = \{(z, p_0')\}$ for all $z \in M$, implying that $\widetilde{F}$ and hence F are constant. This completes the proof. □

Remark 3.13 (a) Assume that there exists $\varepsilon > 0$, such that for any $p \in M'$ and z in the ball $B_\varepsilon(p)$, it holds that $M' \cap Q_z' \cap B_\varepsilon(p) = \{z\}$, for instance, if M' is a strongly pseudoconvex hypersurface. Then if $F_1(z)$ and $F_2(z)$ in Proposition 3.10, are sufficiently close for some $z \in M \setminus \mathcal{S}$, it follows that $F_1 \equiv F_2$. In particular, F cannot be extended as correspondence with a non-empty (non-blowing-up) branch locus intersecting M.

(b) We also mention a paper by Shafikov in [19] where more detailed studies in the equidimensional case ($N = n$) were addressed.

4 Hypersurfaces of positive Levi signature

Fix two integers n, ℓ with $1 < \ell \le n/2$. For any ϵ, define

$$M_\epsilon := \left\{ [z_0, \ldots, z_{n+1}] \in \mathbf{CP}^{n+1} : |z|^2 \left(-\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 \right) + \epsilon \left(|z_1|^4 - |z_{n+1}|^4 \right) = 0 \right\}.$$

Here $|z|^2 = \sum_{j=0}^{n+1} |z_j|^2$ as usual. For $\epsilon = 0$, M_ϵ reduces to the generalized sphere with signature ℓ, which is the boundary of the generalized ball

$$\mathbf{B}_\ell^{n+1} := \left\{ [z_0, \ldots, z_{n+1}] \in \mathbf{CP}^{n+1} : -\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 < 0 \right\}.$$

The boundary $\partial \mathbf{B}_\ell^{n+1}$ is locally holomorphically equivalent to the hyperquadric $\mathbf{H}_\ell^{n+1} \subset \mathbf{C}^{n+1}$ of signature ℓ defined by $\mathsf{Im}\, z_{n+1} = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2$, where $(z_1, \ldots, z_{n+1})$ is the coordinates of $\mathbf{C}^{n+1}$.

For $0 < \epsilon \ll 1$, M_ϵ is a compact smooth real-algebraic hypersurface with Levi form nondegenerate of the same signature ℓ. We now state our next theorem:

Theorem 4.1 *There is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following holds: M_ϵ is a smooth real-algebraic hypersurface in $\mathbf{CP}^{n+1}$ with nondegenerate Levi form of signature ℓ at every point. Moreover for any point $p \in M_\epsilon$ and a holomorphic map F from a neighborhood U of p in $\mathbf{C}^{n+1}$ into $\mathbf{C}^{N+1}$ sending M_ϵ into higher dimensional hyperquadric $\mathbf{H}_\ell^{N+1}$ of the same signature, it follows that F must be totally degenerate in the sense that $F(U) \subset \mathbf{H}_\ell^{n+1}$. In particular, there does not exist any holomorphic embedding from any open piece of M_ϵ into $\mathbf{H}_\ell^{N+1}$.*

There are two main ingredients in our proof: the Chern–Moser–Weyl tensor and an algebraicity theorem of the first author in [13]. We first recall the related concept for the Chern–Moser–Weyl tensor. For a more detailed account on this matter, the reader is referred to [3] and [16].

We use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. We always assume that $n \geq 2$. Let M be a smooth real hypersurface. M is said to be Levi non-degenerate at $p \in M$ with signature $\ell \leq n/2$ if there is a local holomorphic change of coordinates, that maps p to the origin, such that in the new coordinates, M is defined near 0 by an equation of the form:

$$r = v - |z|_\ell^2 + o(|z|^2 + |u|) = 0. \tag{4.1}$$

Here, we write $u = \mathsf{Re}\, w$, $v = \mathsf{Im}\, w$ and

$$\langle a, \bar{b} \rangle_\ell = -\sum_{j=0}^{\ell} a_j \bar{b}_j + \sum_{j=\ell+1}^{n} a_j \bar{b}_j, \quad |z|_\ell^2 = \langle z, \bar{z} \rangle_\ell.$$

When $\ell = 0$, we regard $\sum_{j \leq \ell} a_j = 0$.

Assume that M is Levi non-degenerate with the same signature ℓ at any point with $\ell \leq n/2$. A contact form θ over M is said to be appropriate if the Levi form $L_{\theta|_p}$ associated with θ at any point $p \in M$ has ℓ negative eigenvalues and $n - \ell$ positive eigenvalues. Let θ be an appropriate contact form over M. Then from the Chern–Moser Theory, there is a unique 4th order curvature tensor $\mathcal{S}_\theta$ associated with θ [3], which we call the Chern–Moser–Weyl tensor with respect to the contact form θ along M. $\mathcal{S}_\theta$ can be regarded as a section in

$$T^{*(1,0)}M \otimes T^{*(0,1)}M \otimes T^{*(1,0)}M \otimes T^{*(0,1)}M.$$

We write $\sigma S_{\theta|_p}$ for the restriction of σS_θ at $p \in M$. For another appropriate contact for $\widetilde{\theta}$, we have $\widetilde{\theta} = k\theta$ with $k \neq 0$. Notice that $k > 0$ when $\ell \neq n/2$. Then $\sigma S_{\widetilde{\theta}} = k \sigma S_\theta$, i.e. the Chern–Moser–Weyl tensor at a point $p \in M$ can be invariantly seen as multilinear map

$$S\colon T_p^{(1,0)}M \times T^{(0,1)}M_p \times T^{(1,0)}M_p \times T_p^{(0,1)}M \to \mathbb{C} \otimes T_pM/(T_p^{(1,0)}M + T_p^{(0,1)}M).$$

The Chern–Moser–Weyl tensor has particularly simple expression in the normal coordinates, which we give as follows: By the Chern–Moser normal form theory [3], there is a holomorphic coordinates system in which M is defined near 0 by an equation of the following form (see [3, (6.25), (6.30)]):

$$r = v - |z|_\ell^2 + \frac{1}{4}s(z,\bar z) + o(|z|^4) = v - |z|_\ell^2 + \frac{1}{4}\sum s_{\alpha\bar\beta\gamma\bar\delta} z_\alpha \bar z_\beta z_\gamma \bar z_\delta + o(|z|^4) = 0 \quad (4.2)$$

where s satisfies the trace condition

$$\triangle_\ell s(z,\bar z) \equiv 0,$$

with $\triangle_\ell := -\sum_{j\le\ell} \frac{\partial^2}{\partial z_j \partial \bar z_j} + \sum_{j=\ell+1}^n \frac{\partial^2}{\partial z_j \partial \bar z_j}$. Here $s(z,\bar z) = \sum s_{\alpha\bar\beta\gamma\bar\delta} z_\alpha \bar z_\beta z_\gamma \bar z_\delta$, $\theta = i\partial r$, $s_{\alpha\bar\beta\gamma\bar\delta} = s_{\gamma\bar\beta\alpha\bar\delta} = s_{\gamma\bar\delta\alpha\bar\beta}$, $\overline{s_{\alpha\bar\beta\gamma\bar\delta}} = s_{\beta\bar\alpha\delta\bar\gamma}$ and the trace condition is equivalent to $\sum_{\alpha,\beta=1}^n s_{\alpha\bar\beta\gamma\bar\delta} g^{\bar\beta\alpha} = 0$ where $g^{\bar\beta\alpha} = 0$ for $\beta \ne \alpha$, $g^{\bar\beta\beta} = 1$ for $\beta > \ell$, $g^{\bar\beta\beta} = -1$ for $\beta \le \ell$. Then

$$s_{\alpha\bar\beta\gamma\bar\delta} = \sigma S_{\theta|_0}\left(\left.\frac{\partial}{\partial z_\alpha}\right|_0, \left.\frac{\partial}{\partial \bar z_\beta}\right|_0, \left.\frac{\partial}{\partial z_\gamma}\right|_0, \left.\frac{\partial}{\partial \bar z_\delta}\right|_0 \right).$$

We also write $s_{\theta|_0}(z,\bar z)$ for $s(z,\bar z)$. Consider the Levi null-cone

$$\mathcal{C}_\ell = \{z \in \mathbb{C}^n : |z|_\ell = 0\}.$$

Then $\mathcal{C}_\ell$ is a real-algebraic variety of real codimension 1 in $\mathbb{C}^n$ for $\ell > 0$ with the only singularity at 0. For each $p \in M$, write

$$\mathcal{C}_\ell T_p^{(1,0)} M = \{v_p \in T_p^{(1,0)} M : \langle d\theta_p, v_p \wedge \bar v_p \rangle = 0\}.$$

$\mathcal{C}_l T_p^{(1,0)} M$ is independent of the choice of θ. Let F be a CR diffeomorphism from M to M'. We also have $F_*(\mathcal{C}_\ell T_p^{(1,0)} M) = \mathcal{C}_\ell T_{F(p)}^{(1,0)} M'$. Write $\mathcal{C}_\ell T^{(1,0)} M = \coprod_{p\in M} \mathcal{C}_\ell T_p^{(1,0)} M$ with the natural projection π to M. We say that X is a smooth section of $\sigma C_\ell T^{(1,0)} M$ if X is a smooth vector field of type $(1,0)$ along M such that $X|_p \in \sigma C_\ell T_p^{(1,0)} M$ for each $p \in M$.

We say that the Chern–Moser–Weyl curvature tensor σS_θ is *pseudo positive semi-definite* (resp. *pseudo negative semi-definite*) at $p \in M$ if $\sigma S_{\theta|_p}(X, \overline{X}, X, \overline{X}) \ge 0$ (resp. $\sigma S_{\theta|_p}(X, \overline{X}, X, \overline{X}) \le 0$) for all $X \in \sigma C_\ell T_p^{(1,0)} M$). We say that σS_θ is *pseudo positive definite* (resp. *pseudo negative definite*) at $p \in M$ if $\sigma S_{\theta|_p}(X, \overline{X}, X, \overline{X}) > 0$ (resp. $\sigma S_{\theta|_p}(X, \overline{X}, X, \overline{X}) < 0$) for all $X \in \sigma C_\ell T_p^{(1,0)} M \setminus 0$). We use the terminology *pseudo semi-definite* to mean either pseudo positive semi-definite or pseudo negative semi-definite.

The following will be used later:

Theorem 4.2 ([16], Corollary 3.3) *Let $M \subset \mathbb{C}^{n+1}$ be a Levi non-degenerate real hypersurface of signature ℓ. Suppose that F is a holomorphic mapping defined in a (connected) open neighborhood U of M in $\mathbf{C}^{n+1}$ that sends M into $\mathbf{H}_\ell^{N+1} \subset \mathbb{C}^{N+1}$. Assume that $F(U) \not\subset \mathbf{H}_\ell^{N+1}$. Then when $\ell < \frac{n}{2}$, the Chern–Moser–Weyl curvature tensor with respect to any appropriate contact form θ is pseudo negative semi-definite. When $\ell = \frac{n}{2}$, along any contact form θ, σS_θ is pseudo semi-definite.*

We next state the following algebraicity theorem:

Theorem 4.3 ([13], Corollary in §2.3.5) *Let $M_1 \subset \mathbf{C}^n$ and $M_2 \subset \mathbf{C}^N$ with $N \ge n \ge 2$ be two Levi non-degenerate real-algebraic hypersurfaces. Let $p \in M_1$ and U_p be a small connected open neighborhood of p in $\mathbf{C}^n$ and F be a holomorphic map from U_p into $\mathbf{C}^N$*

such that $F(U_p \cap M_1) \subset M_2$ and $F(U_p) \not\subset M_2$. Suppose that M_1 and M_2 have the same signature ℓ at p and $F(p)$, respectively. Then F is algebraic in the sense that each component of F satisfies a nontrivial holomorphic polynomial equation.

It was first proved in [12] when $\ell = 0$, namely, the strongly pseudo-convex case. The case with $\ell > 0$ was done in the first author's thesis [13, §2.3.5]. It also follows from a more general algebraicity theorem of the second author in [24, Corollary 1.6].

The proof of the above theorem follows from the same proof as in the signature zero case [12], except in the $\ell > 0$ case, we have now the Hopf lemma property as part of the assumption and that the proof of [12, Lemma 2.8] (or [13, Lemma 2.8]) needs to be replaced by the following simple linear algebra lemma:

Lemma 4.4 (Lemma 2.8′, [13]) *Assume that V is a smooth complex-analytic hypersurface in a neighborhood of 0 in $\mathbf{C}^{n+1}$. Assume that M' is a Levi non-degenerate hypersurface of signature $\ell > 0$ at 0 and $T_0^{(1,0)} M' \neq T_0^{(1,0)} V$. Assume that $M' \cap V$ contains a Levi non-degenerate CR submanifold of hypersurface type with signature ℓ through 0. Then $M' \cap V$ is a Levi non-degenerate hypersurface of signature ℓ in V near 0.*

Proof of Theorem 4.1 We first note that for small ϵ, M_ϵ is a small perturbation of the compact quadric M_0 of signature ℓ. Hence there exists a positive $0 < \epsilon_0$ such that whenever $0 < \epsilon < \epsilon_0$, M_ϵ is everywhere Levi non-degenerate with the same signature ℓ.

Now, we compute the Chern–Moser–Weyl tensor of M_ϵ at the point

$$P_0 := [\xi_0^0, \dots, \xi_{n+1}^0], \quad \xi_j^0 = 0 \text{ for } j \neq 0, \ell+1, \ \xi_0^0 = 1, \ \xi_{\ell+1}^0 = 1,$$

and consider the coordinates

$$\xi_0 = 1, \ \xi_j = \frac{\eta_j}{1+\sigma}, \quad j = 1, \dots, \ell, \ \xi_{\ell+1} = \frac{1-\sigma}{1+\sigma}, \ \xi_{j+1} = \frac{\eta_j}{1+\sigma}, \ j = \ell+1, \dots, n.$$

Then in the (η, σ)-coordinates, P_0 becomes the origin and M_ϵ is defined near the origin by an equation in the form:

$$\rho = -4\mathsf{Re}\,\sigma - \sum_{j=1}^{\ell} |\eta_j|^2 + \sum_{j=\ell+1}^{n} |\eta_j|^2 + a(|\eta_1|^4 - |\eta_n|^4) + o(|\eta|^4) = 0, \tag{4.3}$$

for some $a > 0$. Now, let $Q(\eta, \overline{\eta}) = -a(|\eta_1|^4 - |\eta_n|^4)$ and make a standard ℓ-harmonic decomposition [SW]:

$$Q(\eta, \overline{\eta}) = N^{(2,2)}(\eta, \overline{\eta}) + A^{(1,1)}(\eta, \overline{\eta})|\eta|_\ell^2. \tag{4.4}$$

Here $N^{(2,2)}(\eta, \eta)$ is a $(2, 2)$-homogeneous polynomial in $(\eta, \overline{\eta})$ such that $\Delta_\ell N^{(2,2)}(\eta, \overline{\eta}) = 0$ with Δ_ℓ as before. Now $N^{(2,2)}$ is the Chern–Moser–Weyl tensor of M_ϵ at 0 (with respect to an obvious contact form) with $N^{(2,2)}(\eta, \overline{\eta}) = Q(\eta, \overline{\eta})$ for any $\eta \in \mathcal{C}T_0^{(1,0)} M_e$. Now the value of the Chern–Moser–Weyl tension has negative and positive value at $X_1 = \frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_{\ell+1}}|_0$ and $X_2 = \frac{\partial}{\partial \eta_2} + \frac{\partial}{\partial \eta_n}|_0$, respectively. If $\ell > 1$, then both X_1 and X_2 are in $\mathcal{C}T_0^{(1,0)} M_e$. We see that the Chern–Moser–Weyl tensor can not be semi-definite near the origin in such a coordinate system.

Next, suppose an open piece U of M_ϵ can be holomorphically and transversally embedded into the $\mathbf{H}_\ell^{N+1}$ for $N > n$ by F. Then by the algebraicity result in Theorem 4.3, F is algebraic. Since the branching points of F and the points where F is not defined (poles or points of indeterminancy of F) are contained in a complex-algebraic variety of codimension at most one, F extends holomorphically along a smooth curve γ starting from some point in U and ending up at some point $p^*(\approx 0) \in M_\epsilon$ in the (η, σ)-space where the Chern–Moser–Weyl tensor of M_ϵ is not pseudo-semi-definite. By the uniqueness of real-analytic functions, the extension of F must also map an open piece of p^* into $\mathbf{H}_\ell^{N+1}$. The extension is not totally degenerate. By Theorem 4.2, we get a contradiction. □

5 Open problems

We mention here the following questions that still seem to be open.

Question 1 *Is there any example of a compact strongly pseudoconvex real-algebraic hypersurface in* $\mathbb{C}^n$ *that is not holomorphically embeddable into a sphere of any dimension?*

In fact, all known examples of hypersurfaces that are not embeddable into spheres are also not embeddable into strongly pseudoconvex real-algebraic hypersurfaces. It remains unknown whether these two classes are different, more precisely.

Question 2 *Is there any example of a (not necessarily compact) strongly pseudoconvex real-algebraic hypersurface in* $\mathbb{C}^n$ *that is holomorphically embeddable into a compact strongly pseudoconvex real-algebraic hypersurface but is not holomorphically embeddable into a sphere of any dimension?*

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