Flattening of CR singular points and analyticity of the local hull of holomorphy

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Contents

1 Introduction

A primary goal in this paper is to study the question that asks when a real analytic submanifold M of codimension two in $\mathbf{C}^{n+1}$ bounds a real analytic (up to M) Levi-flat hypersurface $\widehat{M}$ near $p \in M$ such that $\widehat{M}$ is foliated by a family of complex hypersurfaces moving along the normal direction of M at p, and gives the invariant local hull of holomorphy of M near p. This question is equivalent to the holomorphic flattening problem for M near p.

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To be more precise, we first discuss some basic holomorphic property for a real submanifold in a complex space. For a point q in a real submanifold $M \subset \mathbb{C}^{n+1}$, there is an immediate holomorphic invariant, namely, the complex dimension $CR_M(q)$ of the tangent space of type $(1,0)$ at q. $CR_M(q)$ is an upper semi-continuous function over M. q is called a CR point of M if $CR_M(q') \equiv CR_M(q)$ for all $q'(\approx q) \in M$. Otherwise, q is called a CR singular point of M. When M near p bounds a Levi-flat hypersurface foliated by a family of complex hypersurfaces moving along the normal direction of M at p, then the tangent space of M at p is a complex hyperplane. In this case p must be a CR singular point unless we are in the trivial and uninteresting situation that M is a complex hypersurface itself.

Investigations for CR manifolds and CR singular manifolds have very different nature. There is a vast amount of work related to the study of various problems for CR manifolds, which goes back to the work of Poincaré [Po], Cartan [Cat] and Chern-Moser [CM]. The study of submanifolds with CR singular points at least dates back to the fundamental paper of Bishop [Bis] in 1965. Since then, many efforts have been paid to understand both the geometric and analytic structures of such manifolds. Here, we mention the papers by Kenig-Webster [KW1-2], Moser-Webster [MW], Bedford-Gaveau [BG], Huang-Krantz [HK], Huang [Hu1], Gong [Gon1-3], Huang-Yin [HY1-2], Stolovitch [Sto], Dobeault-Tomassini-Zaitsev [DTZ1-2], Ahern-Gong [AG], Coffman [Cof1-2], Lebl [Le1-2], Burcea [Va1], etc, and many references therein.

Let $M \subset \mathbb{C}^{n+1}$ be a codimension two real submanifold with *CR singular points*. Then a simple linear algebra computation shows that $CR_M(q) = n-1$ when q is a CR point, and $CR_M(q) = n$ when q is a CR singular point. The general holomorphic (or, formal) flattening problem is then to ask when M can be transformed, by a biholomorphic (formal equivalence, respectively) mapping, to an open piece of the standard Levi-flat hyperplane $(\mathbb{C}^n \times \mathbb{R}^1) \times \{0\} \subset \mathbb{C}^{n+1}$. A good understanding to this problem is crucial for understanding many geometric, analytic and dynamic properties of the manifolds. For instance, by a classical theorem of Cartan, solving the problem when M bounds a real analytic (up to M) Levi-flat hypersurface is equivalent to solving the holomorphic flattening problem of the manifold. Here, we refer the reader to the papers by Kenig-Webster [KW1], Moser-Webster [MW], Huang-Krantz [HK], Gong [Gon1-3], Stolovitch [Sto], Huang-Yin [HY1], Dobeault-Tomassini-Zaitsev[DTZ1], and many references therein, for investigations along these lines.

The major difficulty for getting the flattening property for M lies in the complicated nature of the CR singular points. And, in general, only non-degenerate CR singular points with a rich geometric structure could be flattened. To be more precise, we use $(z_1, \cdots, z_n, w)$ for the complex coordinates of $\mathbb{C}^{n+1}$. We first make the following definition. For related concepts and many intrinsic discussions on this matter, see the work in Stolovitch [Sto], Dobeault-Tomassini-Zaitsev [DTZ1], and Huang-Yin [HY2]:

Definition 1.1. Let M be a codimmension two real submanifold in $\mathbb{C}^{n+1}$. We say $q \in M$ is a non-degenerate CR singular point, or a non-degenerate complex tangent point, if there is a biholomorphic change of coordinates which maps p to 0 and in the new coordinates (z, w), M

is defined near 0 by an equation of the following form:

$$w = \sum_{j=1}^{n} \left(|z_j|^2 + \lambda_j (z_j^2 + \overline{z_j^2}) \right) + o(|z|^2) \tag{1.1}$$

Here, $0 \leq \lambda_1, \cdots, \lambda_n < \infty$. $\{\lambda_1, \cdots, \lambda_n\}$ (counting multiplicity) are called the Bishop invariants of M at 0. We call λ_j an elliptic, parabolic or hyperbolic Bishop invariant of M at 0 in terms of $\lambda_j < 1/2$, $\lambda_j = 1/2$, or $\lambda_j > 1/2$, respectively.

Notice that the set of Bishop invariants at a non-degenerate CR singular point $p \in M$ consists of the only second order biholomorphic invariants of M at $p \in M$. By the results in Moser-Webster [MW] and Huang-Krantz [HK], in the case of complex dimension two ($n+1 = 2$), any real analytic surface near an elliptic CR singular point can be flattened. On the other hand, a generic real analytic surface near a parabolic or hyperbolic CR singular point can not be flattened, though it can be formally flattened whenever the Bishop invariant is not exceptional. See the work of Moser-Webster [MW], Gong [Gon 1-3] and a very recent paper by Ahern-Gong [AG] on many discussions on this matter. Here, we recall that a Bishop invariant λ is called non-exceptional if the following quadratic equation in ν has no roots of unity:

$$\lambda\nu^2 - \nu + \lambda = 0. \tag{1.2}$$

However, the situation for $n > 1$ is very different. Consider the following codimension two real analytic submanifold in $\mathbb{C}^3$:

Example 1.1.

$$M := \{ \ w = \sum_{j=1}^{2} |z_j|^2 + 2\Re \left(\sum_{j_1+j_2 \geq 3} a_{j_1 j_2} z_1^{j_1} z_2^{j_2} \right) + \sqrt{-1} \sum_{j_1 \geq 2, j_2 \geq 2} b_{j_1 \overline{j_2}} z_1^{j_1} \overline{z_2}^{j_2}, \quad b_{j\bar{l}} = \overline{b_{l\bar{j}}}. \} \tag{1.3}$$

M has a non-degenerate CR singular point at 0 and all Bishop invariants of M at 0 are 0 and thus all elliptic. It was shown in Huang-Yin [HY2] ([Remark 2.7, HY2]) that $(M, 0)$ can not even be flattened to the order m if $b_{j_1 \overline{j_2}} \neq 0$ for some $j_1 + j_2 \leq m$. Namely, if $b_{j_1 \overline{j_2}} \neq 0$ for some $j_1 + j_2 \leq m$, then there is no holomorphic change of variables (preserving the origin) such that in the new coordinates, M is defined near 0 by an equation of the form $w = \rho$ with the property that $\Im(\rho)$ vanishes at the origin to the order at least m.

Example 1.1 shows that in higher dimensions, the geometry from the nearby CR points also play a role in the flattening problem, while in the two variables case, the nearby points are totally real and can all be locally holomorphically flattened. Thus the nearby points in the two dimension case has no influence for the holomorphic property at a non-degenerate CR singular point. Indeed, suppose M is already flattened and is defined by an equation of the form $u = q(z, \overline{z}), v = 0$, where $w = u + iv$. Then the complex hypersurface $S_{u_0} =: \{w = u_0 + i0\}$ with $u_0 \in \mathbb{R}$ intersects M along a CR submanifold E of CR dimension $(n-1)$ near p_0 if S_{u_0} intersects

M (CR) transversally at p_0. The points where S_{u_0} is (CR) tangent to M are apparently CR singular points of M. Recall a well-known terminology (see [T] and [Tu]): A point p in a CR submanifold N is called a non-minimal point if N contains a proper CR submanifold S containing p such that $T_p^{(1,0)}S = T_p^{(1,0)}N$. Hence, in such a terminology, we have the following simple fact:

If M can be flattened, then all CR points in M are non-minimal CR points.

We mention that the necessary condition for the non-minimality of CR points already appeared in the earlier work of Dobeault-Tomassini-Zaitsev [DTZ1-2] and Lebl [Le1-2] on the study of the general complex Plateau problem, which looks for the Levi-flat varieties (even maybe in the sense of current) bounded by the given manifolds.

Our main results, which we state below, demonstrate that, with the non-minimality assumption at CR points, the existence of one Bishop invariant not being parabolic, namely, not equal to $\frac{1}{2}$, is good enough for the formal flattening and the existence of just one elliptic Bishop invariant suffices for the holomorphic flattening:

Theorem 1.2. *Let $M \subset \mathbb{C}^{n+1}$ with $n > 1$ be a codimension two smooth submanifold with $p \in M$ a non-degenerate complex tangent point $p \in M$. Suppose that one element λ from the set of Bishop invariants of M at p is not parabolic, namely, not equal to $\frac{1}{2}$. Also assume that all CR points of M near p are non-minimal. Then M can be formally flattened near p. Namely, for any positive integer m, there is a holomorphic change of coordinates which maps p to 0 and maps M to a manifold defined by an equation of the form $w = \rho(z, \overline{z})$ with $\Im\rho$ vanishing at least to the order m at the origin.*

We mention that the result in Theorem 1.2 holds even if M is assumed just to be a formal submanifold with the same type of assumptions, or we need only assume that the set of non-minimal CR points over M forms an open subset O with $p \in \overline{O}$. (See Theorem 3.2 and Corollary 3.3.) However, as demonstrated even in the two dimensional case by Moser-Webster [MW] and Gong [Gon1-3], more geometric structure is needed to get the holomorphic flattening in the above theorem. Indeed, making use of the construction of holomorphic disks in Kenig-Webster [KW1] and Huang-Krantz [HK], we have the following convergence result for Theorem 1.2 under the assumption of at least one ellipticity for the Bishop invariants:

Theorem 1.3. *Let $M \subset \mathbb{C}^{n+1}$ with $n > 1$ be a codimension two real analytic CR manifold with $p \in M$ a non-degenerate complex tangent point (namely, a non-degenerate CR singular point). Suppose one of the Bishop invariants λ of M at p is elliptic. Then M near p can be holomorphically flattened if and only if all CR points of M near p are non-minimal.*

As we mentioned above, by the classical Cartan theorem ([Cat]), Theorem 1.3 is equivalent to the following geometric theorem:

Theorem 1.4. *Let $M \subset \mathbb{C}^{n+1}$ with $n > 1$ be a codimension two real analytic CR manifold with $p \in M$ a non-degenerate complex tangent point. Suppose one of the Bishop invariants λ of M*

at p is elliptic. Also assume that all CR points of M near p are non-minimal. Then the local hull of holomorphy $\widehat{M}$ of M near p is a real analytic Levi-flat hypersurface which has M near p as part of its real analytic boundary. Moreover $\widehat{M}$ is foliated by a family of smooth complex hypersurfaces in $\mathbb{C}^{n+1}$, that moves along the transversal direction of the tangent space of M at p.

Example 1.5. Define $M \subset \mathbb{C}^3 =: \{(z_1, z_2, w)\}$ by the following equation near 0:

$$w = q(z, \overline{z}) + p(z, \overline{z}) + iE(z, \overline{z}).$$

Here $q = |z_1|^2 + \lambda_1(z_1^2 + \overline{z_1^2}) + |z_2|^2 + \lambda_2(z_2^2 + \overline{z_2^2})$ with $0 \leq \lambda_1, \lambda_2 < \infty$, and

$$p(z, \overline{z}) + iE(z, \overline{z}) = \mu_1|z_1|^2(z_1 + \lambda_1\overline{z_1}) + \mu_2|z_2|^2(z_2 + \lambda_2\overline{z_2}) + \mu_1 z_1(|z_2|^2 + \lambda_2\overline{z_2^2}) + \mu_2 z_2(|z_1|^2 + \lambda_1\overline{z_1^2}).$$

Here μ_1, μ_2 are two complex numbers. Then, M is non-minimal at its CR points near its non-degenerate CR singular point 0. (See Example 7.2.) Hence, our result says that when one of the λ_1, λ_2 is not $\frac{1}{2}$, then M can be formally flattened at 0; and when one of the λ_1, λ_2 is less than $\frac{1}{2}$, then M can be holomorphically flattened near 0.

In this example, $M \setminus \{0\}$ near 0 is foliated by a family of three dimensional strongly pseudoconvex CR manifolds— the intersections of M with real hypersurfaces $K_c : q(z, \overline{z}) = c$ with $c \in \mathbb{R}$. (When both λ_1, λ_2 are elliptic, $c > 0$). Assume that one of the Bishop invariants $\{\lambda_1, \lambda_2\}$ is not elliptic. Then there is an orbit corresponding to $c = 0$, that extends to the CR singular point with it as its non-smooth point. Also none of the orbits closes up near 0.

We next say a few words about the proof of our main geometric result: To prove Theorem 1.4, we first slice M near the complex tangent point p by a family of two dimensional complex planes along the elliptic direction. We then get a family of elliptic Bishop surfaces. Now each one bounds a three dimensional Levi flat CR manifold and their union forms a codimension one subset $\widetilde{M}$ in $\mathbf{C}^{n+1}$ with M as part of its boundary. An analysis, based on Bishop disks, similar to that in Kenig-Webster [KW1], and in particular, in Huang-Krantz [HK], shows that $\widetilde{M}$ is a real analytic hypersurface with M as part of its real analytic boundary. However, all we know from this construction is that $\widetilde{M}$ has only one Levi-flat direction (along the elliptic direction). And it is not clear at all if $\widetilde{M}$ is flat along the parameter directions. In fact, $\widetilde{M}$ can not be Levi flat without the non-minimality property from the nearby CR points. Now, the crucial issue is that, with the assumption of the non-minimality at the nearby CR points, we can find a formal transformation which makes M formally flattened, while any finite order truncation of this transformation preserves $\widetilde{M}$. The existence of this transformation is the content of Theorem 3.1, which is a more general but also more technical version of Theorem 1.2. (Notice that in the two dimensional setting, the uniqueness of $\widetilde{M}$ is done normally by showing that $\widetilde{M}$ is the local hull of holomorphy of M and thus is invariant under biholomorphic transformation. However, this is more or less equivalent to proving that $\widetilde{M}$ is Levi-flat. Hence it can not be achieved in this way in higher dimensions.) After this is done, we see that the Levi-form of $\widetilde{M}$ vanishes to any high order as we like. Since $\widetilde{M}$ is real analytic up to M, by the unique continuation property for real-analytic functions, we conclude that the Levi-form of $\widetilde{M}$ vanishes everywhere. Thus, $\widetilde{M}$ must be Levi-flat everywhere.

Most part of the paper is devoted to the proof Theorem 3.1 (a more general and more technical version of Theorem 1.2). Here one sees an essential difference from arguments in the two dimensional case. Indeed, the phenomenon is also different in this setting as there is no need to impose the non-exceptional property for even a single Bishop invariant. Our basic idea for the proof of Theorem 3.1 goes as follows: Suppose M in Theorem 3.1 is flattened to order $m-1$. We first normalize the m^{th}-order of the imaginary part of the defining function of M to fix all possible free choices of coordinates. This will be done in Theorem 4.2. Then we show that the non-minimality of the nearby CR points forces the vanishing of such a normal form. In §2 and §3, we will derive three basic equations that must be satisfied for M under the non-minimality assumption. These will be used in §4 and §5 to prove Theorem 3.1.

Theorem 1.3 is equivalent to Theorem 1.4 by a classical result of Cartan which states that a real analytic hypersurface is Levi-flat if and only if it can be transformed locally to an open piece of the standard Levi-flat hyperplane defined by $\Im w = 0$. When all Bishop invariants at p are elliptic, we mention that Theorem 1.4 can also be derived by combining the results obtained in Dobeault-Tomassini-Zaitsev [DTZ1-2] and the work in a very recent preprint by Burcea [Bur2] with a different approach. (The work in Dobeault-Tomassini-Zaitsev [DTZ1-2] contains other very nice global results.) The arguments based on Dobeault-Tomassini-Zaitsev [DTZ1-2] and Burcea [Bur2] depend strongly on all the ellipticity of Bishop invariants and requires that the CR orbits in M near the CR singular point form a family of compact strongly pseudoconvex manifolds shrinking down to the complex tangent such that the Harvey-Lawson theorem applies. This is certainly not the case even when one non-elliptic Bishop invariant at the CR singular point appears.

We also include an appendix to give a detailed proof of Theorem 3.1 in the special case of $n = 2$ and $m = 3$. The reader may like to read the Appendix before reading §4 − §6. By including such an appendix, we hope it will help the reader to see the basic ideas, through a simple case, the complicated argument for the proof of Theorem 3.1 in the general setting in §4 − §6.

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2 An immediate consequence for non-minimality near CR points

Let $(M,0)$ be a smooth submanifold of codimension two in $\mathbb{C}^{n+1}$ with $0 \in M$ as a CR singular point. Assume that the CR singular point at $0 \in M$ is non-degenerate as defined in Definition 1.1 such that after a holomorphic change of coordinates, M near 0 is defined by an equation of the form:

$$w = q(z,\overline{z}) + p(z,\overline{z}) + iE(z,\overline{z}), \tag{2.1}$$

where $q(z,\overline{z}) = \sum_{i=1}^{n}(|z_i|^2 + \lambda_i(z_i^2 + \overline{z}_i^2))$ with $0 \le \lambda_1, \cdots, \lambda_n < \infty$ being the Bishop invariants of M at 0, $\mathrm{Ord}(p), \mathrm{Ord}(E) \ge 3$ and both $p(z,\overline{z})$ and $E(z,\overline{z})$ are real-valued smooth functions. For convenience of notation, we also write

$$F(z,\overline{z}) = p(z,\overline{z}) + iE(z,\overline{z}) \text{ and } G(z,\overline{z}) = q(z,\overline{z}) + p(z,\overline{z}). \tag{2.2}$$

Then we have

$$w = q(z,\overline{z}) + F(z,\overline{z}) = G(z,\overline{z}) + iE(z,\overline{z}).$$

In what follows, as is standard in the literature, we write $\chi_\alpha = \frac{\partial \chi}{\partial z_\alpha}$, $\chi_{\overline{\alpha}} = \frac{\partial \chi}{\partial \overline{z}_\alpha}$ with $1 \le \alpha \le n$ for a smooth function $\chi(z,\overline{z})$ in z. For $1 \le j \le n-1$, we define

$$\begin{aligned} L_j =&(G_n - iE_n)\frac{\partial}{\partial z_j} - (G_j - iE_j)\frac{\partial}{\partial z_n} + 2i(G_nE_j - G_jE_n)\frac{\partial}{\partial w} \\ :=&A\frac{\partial}{\partial z_j} - B_{(j)}\frac{\partial}{\partial z_n} + C_{(j)}\frac{\partial}{\partial w}. \end{aligned} \tag{2.3}$$

Then we have

$$\begin{aligned} L_j(-w + G + iE) &= (G_n - iE_n)(G_j + iE_j) - (G_j - iE_j)(G_n + iE_n) \\ &\quad - 2i(G_nE_j - G_jE_n) = 0, \\ L_j(\overline{-w + G + iE}) &= (G_n - iE_n)(G_j - iE_j) - (G_j - iE_j)(G_n - iE_n) = 0. \end{aligned}$$

Hence $L_1, \cdots, L_{n-1}$ are complex tangent vector fields of type $(1,0)$ along M near 0. Moreover, for $1 \le j,k \le n-1$, a straightforward computation shows that

$$\begin{aligned} [L_j, \overline{L}_k] =& \left[A\frac{\partial}{\partial z_j} - B_{(j)}\frac{\partial}{\partial z_n} + C_{(j)}\frac{\partial}{\partial w}, \overline{A}\frac{\partial}{\partial \overline{z}_k} - \overline{B_{(k)}}\frac{\partial}{\partial \overline{z}_n} + \overline{C_{(k)}}\frac{\partial}{\partial \overline{w}}\right] \\ =&\lambda_{(1jk)}\frac{\partial}{\partial \overline{z}_k} + \lambda_{(2jk)}\frac{\partial}{\partial \overline{z}_n} + \lambda_{(3jk)}\frac{\partial}{\partial \overline{w}} + \lambda_{(4jk)}\frac{\partial}{\partial z_j} + \lambda_{(5jk)}\frac{\partial}{\partial z_n} + \lambda_{(6jk)}\frac{\partial}{\partial w}, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \lambda_{(1jk)} = A \cdot (\overline{A})_j - B_{(j)}(\overline{A})_n,\ \lambda_{(4jk)} = -\overline{A} \cdot A_{\overline{k}} + \overline{B_{(k)}}A_{\overline{n}}, \\ \lambda_{(2jk)} = -A \cdot (\overline{B_{(k)}})_j + B_{(j)}(\overline{B_{(k)}})_n,\ \lambda_{(5jk)} = \overline{A} \cdot (B_{(j)})_{\overline{k}} - \overline{B_{(k)}}(B_{(j)})_{\overline{n}}, \\ \lambda_{(3jk)} = A \cdot (\overline{C_{(k)}})_j - B_{(j)}(\overline{C_{(k)}})_n,\ \lambda_{(6jk)} = -\overline{A} \cdot (C_{(j)})_{\overline{k}} + \overline{B_{(k)}}(C_{(j)})_{\overline{n}}. \end{aligned} \tag{2.5}$$

Notice that

$$\lambda_{(1jk)} = -\overline{\lambda_{(4kj)}},\ \lambda_{(2jk)} = -\overline{\lambda_{(5kj)}},\ \lambda_{(3jk)} = -\overline{\lambda_{(6kj)}}. \tag{2.6}$$

In what follows, write $w_j = z_j + 2\lambda_j \overline{z}_j$ for $1 \le j \le n$. Suppose that $E \not\equiv 0$. We write in what follows that

$$\mathrm{Ord}(E) = m \text{ and } H(z,\overline{z}) := E^{(m)}(z,\overline{z}). \tag{2.7}$$

From (2.3), we get the following approximation properties:

$$A = \overline{w}_n + O(2),\ B_{(j)} = \overline{w}_j + O(2),\ C_{(j)} = 2i\overline{\Phi_{(j)}} + O(m+1). \tag{2.8}$$

Here (and also in what follows), we have

$$\Phi_{(j)} = w_n H_{\overline{j}} - w_j H_{\overline{n}},\ \text{and we write } \Phi = \Phi_{(1)}. \tag{2.9}$$

For future applications, we also write

$$\Psi_{(jk)} = w_n \overline{w_n} (\Phi_{(j)})_k - w_n \overline{w_k} (\Phi_{(j)})_n + \overline{w_k} \cdot \Phi_{(j)},\ \Psi = \Psi_{(11)}. \tag{2.10}$$

Substituting these approximation properties to (2.5), we obtain

$$\begin{aligned}
\lambda_{(1jk)} =& \big(\overline{w}_n + O(2)\big) \cdot \big((w_n)_j + O(1)\big) - \big(\overline{w}_j + O(2)\big) \cdot \big((w_n)_n + O(1)\big) = -\overline{w}_j + O(2),\\
\lambda_{(2jk)} =& - \big(\overline{w}_n + O(2)\big) \cdot \big((w_k)_j + O(1)\big) + \big(\overline{w}_j + O(2)\big) \cdot \big((w_k)_n + O(1)\big) = -\delta_{jk}\overline{w}_n + O(2),\\
\lambda_{(3jk)} =& \big(\overline{w}_n + O(2)\big) \cdot \big(- 2i\Phi_{(k)} + O(m+1)\big)_j - \big(\overline{w}_j + O(2)\big) \cdot \big(- 2i\Phi_{(k)} + O(m+1)\big)_n\\
=& - 2i\overline{w}_n (\Phi_{(k)})_j + 2i\overline{w}_j (\Phi_{(k)})_n + O(m+1).
\end{aligned}$$

Combining these relations with (2.6), we get

$$\begin{aligned}
\lambda_{(1jk)} =& - \overline{w}_j + O(2),\ \lambda_{(2jk)} = -\delta_{jk}\overline{w}_n + O(2),\\
\lambda_{(4jk)} =& w_k + O(2), \qquad \lambda_{(5jk)} = \delta_{jk} w_n + O(2),\\
\lambda_{(3jk)} =& - 2i\overline{w}_n (\Phi_{(k)})_j + 2i\overline{w}_j (\Phi_{(k)})_n + O(m+1),\\
\lambda_{(6jk)} =& - 2i w_n (\overline{\Phi_{(j)}})_{\overline{k}} + 2i w_k (\overline{\Phi_{(j)}})_{\overline{n}} + O(m+1).
\end{aligned} \tag{2.11}$$

In what follows, we further assume that M is non-minimal at its CR points. Write $\mathcal{S}$ for the set of CR singular points of M near 0. Suppose that 0 is not an isolated point in $\mathcal{S}$. Notice that $T_0^{(1,0)}M = \mathrm{span}\{\frac{\partial}{\partial z_1}|_0, \cdots, \frac{\partial}{\partial z_n}|_0\}$. For $p_0 \in \mathcal{S}$ close to 0, we easily see that $T_{p_0}^{(1,0)}M = \mathrm{span}\{X_1, \cdots, X_n\}$ for certain tangent vectors of type $(1,0)$ of the form: $X_j = \frac{\partial}{\partial z_j}|_{p_0} + b_j \frac{\partial}{\partial w}|_{p_0}$, $j = 1, \cdots, n$. Since $X_j(\overline{-w+q+F}) = 0$, we get $\overline{z_j} + 2\lambda_j z_j = O(|z|^2)$ when $q_0 \approx 0$. Write $z_j = x_j + \sqrt{-1}y_j$. Then $(1+2\lambda_j)x_j = O(|z|^2), (1-2\lambda_j)y_j = O(|z|^2)$ for $j = 1, \cdots, n$. By the implicit function theorem, we conclude that $\mathcal{S}$ is contained in a submanifold of M near 0 which has at most real dimension n; and when none of the Bishop invariants of M at 0 is parabolic, the only CR singular point of M near 0 is 0 itself.

We next claim that there is an open dense subset $\mathcal{O}_i$ of M near 0 such that for any $q_0 \in \mathcal{O}_i$, at q_0, it holds that

$$[L_i, \overline{L_i}] \notin \mathrm{Span}\{L_j, \overline{L_j}\}_{1 \le j \le n-1} \text{ for } 1 \le i \le n-1. \tag{2.12}$$

We prove the claim by contradiction. Suppose that we have at $q_0 \in \mathcal{O}_i$ the following:

$$\begin{aligned}[L_i, \overline{L}_i] &= \sum_{l=1}^{n-1} \big(\hat{\alpha}_l L_l + \hat{\beta}_l \overline{L}_l\big) \\ &= \sum_{l=1}^{n-1} \hat{\alpha}_l \big(A\frac{\partial}{\partial z_l} - B_{(l)}\frac{\partial}{\partial z_n} + C_{(l)}\frac{\partial}{\partial w}\big) + \sum_{l=1}^{n-1} \hat{\beta}_l \big(\overline{A}\frac{\partial}{\partial \overline{z}_l} - \overline{B_{(l)}}\frac{\partial}{\partial \overline{z}_n} + \overline{C_{(l)}}\frac{\partial}{\partial \overline{w}}\big).\end{aligned} \tag{2.13}$$

Then by considering the coefficients of $\frac{\partial}{\partial z_l}$ and $\frac{\partial}{\partial \overline{z}_l}$ for $1 \le l \le n-1$, we obtain

$$\hat{\alpha}_l = \hat{\beta}_l = 0 \text{ for } l \ne i, \lambda_{(4ii)} = A\hat{\alpha}_i, \ \lambda_{(5ii)} = -B_{(i)}\hat{\alpha}_i.$$

Eliminating $\hat{\alpha}_i$ from the above, we get $A\lambda_{(5ii)} + B_{(i)}\lambda_{(4ii)} = 0$. Combining this with the approximation properties (2.8) and (2.11), we get $|w_i|^2 + |w_n|^2 + O(3) = 0$ at $q_0 \in \mathcal{O}_i$. Now, write this equation as

$$|w_i|^2 + (1+2\lambda_n)^2 x_n^2 + (1-2\lambda_n)^2 y_n^2 + h_0 + h_1 x_n + h_2 x_n^2 + O(x_n^3) = 0. \tag{2.14}$$

Notice that when M is real analytic, it defines a closed proper analytic variety over M and $\mathcal{O}_i$ can be simply defined as its compliment. In general, suppose it defines a subset which contains an open subset V_i with $0 \in \overline{V_i}$. Differentiating (2.14) with respect to x_n, we get the following over V_i:

$$\big(2(1+2\lambda_n)^2 + 2h_2\big)\, x_n = -h_1 + O(x_n^2).$$

Since $h_2 = o(1)$, by the implicit function theorem, the above defines a proper submanifold in M. This is contradiction.

In the following, we write $\mathcal{O} = \cap_{i=1}^{n-1}\mathcal{O}_i \setminus \mathcal{S}$, which $\mathcal{O}$ is an open dense subset of M near 0. In particular, at $q_0(\approx 0) \in \mathcal{O}$, we have

$$T := [L_1, \overline{L}_1] \notin \mathrm{Span}\{L_j, \overline{L_j}\}_{1 \le j \le n-1}.$$

Hence, by the Frobenius theorem, the non-minimality at the subset $\mathcal{O}$ of the CR points (sufficiently close to 0) is equivalent to the following property when restricted to the subset $\mathcal{O}$:

$$[L_i, \overline{L_j}], [[L_i, \overline{L_j}], L_k] \in \mathrm{Span}\big\{\{L_h, \overline{L_h}\}_{1 \le h \le n-1}, T\big\} \text{ for } 1 \le i, j, k \le n-1. \tag{2.15}$$

Recall the following notation we set up before:

$$T = \lambda_{(111)}\frac{\partial}{\partial \overline{z}_1} + \lambda_{(211)}\frac{\partial}{\partial \overline{z}_n} + \lambda_{(311)}\frac{\partial}{\partial \overline{w}} + \lambda_{(411)}\frac{\partial}{\partial z_1} + \lambda_{(511)}\frac{\partial}{\partial z_n} + \lambda_{(611)}\frac{\partial}{\partial w}.$$

Next we give equivalent conditions for (2.15), which are much easier to apply.

First, since $[L_j, \overline{L}_k] \in \mathrm{Span}\{\{L_h, \overline{L_h}\}_{1\leq h\leq n-1}, T\}$ with $1 \leq j, k \leq n-1$ over $M \setminus \mathcal{S}$, we have, over $M \setminus \mathcal{S}$, the following

$$[L_j, \overline{L}_k] = \sum_{l=1}^{n-1}(\alpha_l L_l + \beta_l \overline{L}_l) + \gamma T \text{ for some coefficients } \alpha_l, \beta_l, \gamma.$$

Namely, we have

$$\begin{aligned}
&\lambda_{(1jk)}\frac{\partial}{\partial \overline{z}_k} + \lambda_{(2jk)}\frac{\partial}{\partial \overline{z}_n} + \lambda_{(3jk)}\frac{\partial}{\partial \overline{w}} + \lambda_{(4jk)}\frac{\partial}{\partial z_j} + \lambda_{(5jk)}\frac{\partial}{\partial z_n} + \lambda_{(6jk)}\frac{\partial}{\partial w}\\
=&\sum_{l=1}^{n-1}\alpha_l\left(A\frac{\partial}{\partial z_l} - B_{(l)}\frac{\partial}{\partial z_n} + C_{(l)}\frac{\partial}{\partial w}\right) + \sum_{l=1}^{n-1}\beta_l\left(\overline{A}\frac{\partial}{\partial \overline{z}_l} - \overline{B_{(l)}}\frac{\partial}{\partial \overline{z}_n} + \overline{C_{(l)}}\frac{\partial}{\partial \overline{w}}\right)\\
&+\gamma\left(\lambda_{(111)}\frac{\partial}{\partial \overline{z}_1} + \lambda_{(211)}\frac{\partial}{\partial \overline{z}_n} + \lambda_{(311)}\frac{\partial}{\partial \overline{w}} + \lambda_{(411)}\frac{\partial}{\partial z_1} + \lambda_{(511)}\frac{\partial}{\partial z_n} + \lambda_{(611)}\frac{\partial}{\partial w}\right).
\end{aligned}$$

Comparing the coefficients of $\{\frac{\partial}{\partial z_h}, \frac{\partial}{\partial \overline{z}_h}\}_{1\leq h\leq n-1}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}$, respectively, we get, over $\mathcal{O}$, the following:

(I) If $j \neq 1$ and $k \neq 1$, then we have $\alpha_l, \beta_l = 0$ for $l \neq 1, l \neq j, l \neq k$. Moreover, we have

$$\begin{aligned}
\overline{A}\cdot\beta_1 + \gamma\cdot\lambda_{(111)} = 0,\ A\cdot\alpha_1 + \gamma\cdot\lambda_{(411)} = 0,\ \lambda_{(1jk)} = \overline{A}\cdot\beta_k,\ \lambda_{(4jk)} = A\cdot\alpha_j,\\
\lambda_{(2jk)} = -\beta_1\cdot\overline{B_{(1)}} - \beta_k\cdot\overline{B_{(k)}} + \gamma\cdot\lambda_{(211)},\ \lambda_{(3jk)} = \beta_1\cdot\overline{C_{(1)}} + \beta_k\cdot\overline{C_{(k)}} + \gamma\cdot\lambda_{(311)},\\
\lambda_{(5jk)} = -\alpha_1\cdot B_{(1)} - \alpha_j\cdot B_{(j)} + \gamma\cdot\lambda_{(511)},\ \lambda_{(6jk)} = \alpha_1\cdot C_{(1)} + \alpha_j\cdot C_{(j)} + \gamma\cdot\lambda_{(611)}.
\end{aligned} \tag{2.16}$$

(II) If $j = 1$ but $k \neq 1$, then we get (i) $\alpha_l = 0$ for $l \not= 1$ and (ii) $\beta_l = 0$ for $l \neq 1, l \neq k$. Moreover, we have

$$\begin{aligned}
&\overline{A}\cdot\beta_1 + \gamma\cdot\lambda_{(111)} = 0,\ \lambda_{(41k)} = A\cdot\alpha_1 + \gamma\cdot\lambda_{(411)},\ \lambda_{(11k)} = \overline{A}\cdot\beta_k,\\
&\lambda_{(21k)} = -\beta_1\cdot\overline{B_{(1)}} - \beta_k\cdot\overline{B_{(k)}} + \gamma\cdot\lambda_{(211)},\ \lambda_{(31k)} = \beta_1\cdot\overline{C_{(1)}} + \beta_k\cdot\overline{C_{(k)}} + \gamma\cdot\lambda_{(311)},\\
&\lambda_{(51k)} = -\alpha_1\cdot B_{(1)} + \gamma\cdot\lambda_{(511)},\ \lambda_{(61k)} = \alpha_1\cdot C_{(1)} + \gamma\cdot\lambda_{(611)}.
\end{aligned} \tag{2.17}$$

Back to (2.16), we get from its first line that

$$\overline{A}\beta_1 = -\gamma\cdot\lambda_{(111)},\ A\alpha_1 = -\gamma\cdot\lambda_{(411)},\ \overline{A}\beta_k = \lambda_{(1jk)},\ A\alpha_j = \lambda_{(4jk)}.$$

Multiplying $\overline{A}$ and A to the second and the third lines in (2.16), respectively, and making use of the just obtained relations, we obtain over $\mathcal{O}$ the following:

$$\begin{aligned}
\overline{A}\lambda_{(2jk)} &= \gamma\cdot\lambda_{(111)}\cdot\overline{B_{(1)}} - \lambda_{(1jk)}\cdot\overline{B_{(k)}} + \overline{A}\gamma\cdot\lambda_{(211)},\\
\overline{A}\lambda_{(3jk)} &= -\gamma\cdot\lambda_{(111)}\cdot\overline{C_{(1)}} + \lambda_{(1jk)}\cdot\overline{C_{(k)}} + \overline{A}\gamma\cdot\lambda_{(311)},\\
A\lambda_{(5jk)} &= \gamma\cdot\lambda_{(411)}\cdot B_{(1)} - \lambda_{(4jk)}\cdot B_{(j)} + A\gamma\cdot\lambda_{(511)},\\
A\lambda_{(6jk)} &= -\gamma\cdot\lambda_{(411)}\cdot C_{(1)} + \lambda_{(4jk)}\cdot C_{(j)} + A\gamma\cdot\lambda_{(611)}.
\end{aligned}$$

After rewriting, we get

$$\begin{aligned}
\big(\overline{A}\cdot\lambda_{(211)}+\overline{B_{(1)}}\cdot\lambda_{(111)}\big)\cdot\gamma =&\overline{A}\lambda_{(2jk)}+\lambda_{(1jk)}\cdot\overline{B_{(k)}},\\
\big(\overline{A}\cdot\lambda_{(311)}-\overline{C_{(1)}}\cdot\lambda_{(111)}\big)\cdot\gamma =&\overline{A}\lambda_{(3jk)}-\lambda_{(1jk)}\cdot\overline{C_{(k)}},\\
\big(A\cdot\lambda_{(511)}+B_{(1)}\cdot\lambda_{(411)}\big)\cdot\gamma =&A\lambda_{(5jk)}+\lambda_{(4jk)}\cdot B_{(j)},\\
\big(A\cdot\lambda_{(611)}-C_{(1)}\cdot\lambda_{(411)}\big)\cdot\gamma =&A\lambda_{(6jk)}-\lambda_{(4jk)}\cdot C_{(j)}.
\end{aligned}\tag{2.18}$$

Thus we get from the first and the last equations in (2.18) the following relation over $\mathcal{O}$:

$$\begin{aligned}
&(\overline{A}\cdot\lambda_{(2jk)}+\overline{B_{(k)}}\cdot\lambda_{(1jk)})\cdot(A\cdot\lambda_{(611)}-C_{(1)}\cdot\lambda_{(411)})\\
&=(\overline{A}\cdot\lambda_{(211)}+\overline{B_{(1)}}\cdot\lambda_{(111)})\cdot(A\cdot\lambda_{(6jk)}-C_{(j)}\cdot\lambda_{(4jk)})
\end{aligned}\tag{2.19}$$

After taking a limit, we see the equation in (2.19) holds over M near 0.

Next, we solve (2.17) by the same argument as that used to solve (2.16). In fact, we get from the first line of (2.17) that

$$\overline{A}\beta_1=-\gamma\cdot\lambda_{(111)},\ A\alpha_1=\lambda_{(41k)}-\gamma\cdot\lambda_{(411)},\ \overline{A}\beta_k=\lambda_{(11k)}.$$

Multiplying $\overline{A}$ and A to the second and the third lines of the equations in (2.17), respectively, and using the just obtained relations, we have over $M\setminus\mathcal{S}$

$$\begin{aligned}
\overline{A}\lambda_{(21k)}&=\gamma\cdot\lambda_{(111)}\cdot\overline{B_{(1)}}-\lambda_{(11k)}\cdot\overline{B_{(k)}}+\overline{A}\gamma\cdot\lambda_{(211)},\\
\overline{A}\lambda_{(31k)}&=-\gamma\cdot\lambda_{(111)}\cdot\overline{C_{(1)}}+\lambda_{(11k)}\cdot\overline{C_{(k)}}+\overline{A}\gamma\cdot\lambda_{(311)},\\
A\lambda_{(51k)}&=-(\lambda_{(41k)}-\gamma\cdot\lambda_{(411)})\cdot B_{(1)}+A\gamma\cdot\lambda_{(511)},\\
A\lambda_{(61k)}&=(\lambda_{(41k)}-\gamma\cdot\lambda_{(411)})\cdot C_{(1)}+A\gamma\cdot\lambda_{(611)}.
\end{aligned}$$

Rearranging the terms and replacing γ by $\hat{\gamma}$, we get over $M\setminus\mathcal{S}$

$$\begin{aligned}
\big(\overline{A}\lambda_{(211)}+\lambda_{(111)}\cdot\overline{B_{(1)}}\big)\cdot\hat{\gamma} =&\overline{A}\lambda_{(21k)}+\lambda_{(11k)}\cdot\overline{B_{(k)}},\\
\big(\overline{A}\lambda_{(311)}-\lambda_{(111)}\cdot\overline{C_{(1)}}\big)\cdot\hat{\gamma} =&\overline{A}\lambda_{(31k)}-\lambda_{(11k)}\cdot\overline{C_{(k)}},\\
\big(A\lambda_{(511)}+\lambda_{(411)}\cdot B_{(1)}\big)\cdot\hat{\gamma} =&A\lambda_{(51k)}+\lambda_{(41k)}\cdot B_{(1)},\\
\big(A\lambda_{(611)}-\lambda_{(411)}\cdot C_{(1)}\big)\cdot\hat{\gamma} =&A\lambda_{(61k)}-\lambda_{(41k)}\cdot C_{(1)}.
\end{aligned}\tag{2.20}$$

From the first and the last equations in (2.20), we see that, after taking a limit, the following equation holds near 0:

$$\begin{aligned}
&(\overline{A}\cdot\lambda_{(21k)}+\overline{B_{(k)}}\cdot\lambda_{(11k)})\cdot(A\cdot\lambda_{(611)}-C_{(1)}\cdot\lambda_{(411)})\\
&=(\overline{A}\cdot\lambda_{(211)}+\overline{B_{(1)}}\cdot\lambda_{(111)})\cdot(A\cdot\lambda_{(61k)}-C_{(1)}\cdot\lambda_{(41k)}).
\end{aligned}\tag{2.21}$$

Next we will examine $[L_1,T]$. A direct computation shows that

$$[L_1,T]=\Gamma_{(1)}\frac{\partial}{\partial\overline{z}_1}+\Gamma_{(2)}\frac{\partial}{\partial\overline{z}_n}+\Gamma_{(3)}\frac{\partial}{\partial\overline{w}}+\Gamma_{(4)}\frac{\partial}{\partial z_1}+\Gamma_{(5)}\frac{\partial}{\partial z_n}+\Gamma_{(6)}\frac{\partial}{\partial w}.\tag{2.22}$$

where

$$\Gamma_{(1)} = A(\lambda_{(111)})_1 - B_{(1)}(\lambda_{(111)})_n,$$
$$\Gamma_{(2)} = A(\lambda_{(211)})_1 - B_{(1)}(\lambda_{(211)})_n,$$
$$\Gamma_{(3)} = A(\lambda_{(311)})_1 - B_{(1)}(\lambda_{(311)})_n,$$
$$\Gamma_{(4)} = A(\lambda_{(411)})_1 - B_{(1)}(\lambda_{(411)})_n - \lambda_{(111)}A_{\overline{1}} - \lambda_{(211)}A_{\overline{n}} - \lambda_{(411)}A_1 - \lambda_{(511)}A_n,$$
$$\Gamma_{(5)} = A(\lambda_{(511)})_1 - B_{(1)}(\lambda_{(511)})_n + \lambda_{(111)}(B_{(1)})_{\overline{1}} + \lambda_{(211)}(B_{(1)})_{\overline{n}} + \lambda_{(411)}(B_{(1)})_1 + \lambda_{(511)}(B_{(1)})_n,$$
$$\Gamma_{(6)} = A(\lambda_{(611)})_1 - B_{(1)}(\lambda_{(611)})_n - \lambda_{(111)}(C_{(1)})_{\overline{1}} - \lambda_{(211)}(C_{(1)})_{\overline{n}} - \lambda_{(411)}(C_{(1)})_1 - \lambda_{(511)}(C_{(1)})_n. \tag{2.23}$$

Suppose that over $\mathcal{O}$ we have

$$\begin{aligned}
[L_1, T] =& \sum_{h=1}^{n-1} \big(\kappa_{(h)} L_h + \sigma_{(h)} \overline{L_h}\big) + \tau T \\
=& \sum_{h=1}^{n-1} \kappa_{(h)} \big(A\frac{\partial}{\partial z_h} - B_{(h)}\frac{\partial}{\partial z_n} + C_{(h)}\frac{\partial}{\partial w}\big) + \sum_{h=1}^{n-1} \sigma_{(h)} \big(\overline{A}\frac{\partial}{\partial \overline{z}_h} - \overline{B_{(h)}}\frac{\partial}{\partial \overline{z}_n} + \overline{C_{(h)}}\frac{\partial}{\partial \overline{w}}\big) \\
&+ \tau\big(\lambda_{(111)}\frac{\partial}{\partial \overline{z}_1} + \lambda_{(211)}\frac{\partial}{\partial \overline{z}_n} + \lambda_{(311)}\frac{\partial}{\partial \overline{w}} + \lambda_{(411)}\frac{\partial}{\partial z_1} + \lambda_{(511)}\frac{\partial}{\partial z_n} + \lambda_{(611)}\frac{\partial}{\partial w}\big).
\end{aligned}$$

Then combining this with (2.22), we get $\kappa_{(h)} = 0, \sigma_{(h)} = 0$ for $h \neq 1$ and

$$\begin{aligned}
&\Gamma_{(1)} = \overline{A} \cdot \sigma_{(1)} + \tau \cdot \lambda_{(111)},\ \Gamma_{(4)} = A \cdot \kappa_{(1)} + \tau \cdot \lambda_{(411)}, \\
&\Gamma_{(2)} = -\overline{B_{(1)}} \cdot \sigma_{(1)} + \tau \cdot \lambda_{(211)},\ \Gamma_{(3)} = \overline{C_{(1)}} \cdot \sigma_{(1)} + \tau \cdot \lambda_{(311)}, \\
&\Gamma_{(5)} = -B_{(1)} \cdot \kappa_{(1)} + \tau \cdot \lambda_{(511)},\ \Gamma_{(6)} = C_{(1)} \cdot \kappa_{(1)} + \tau \cdot \lambda_{(611)}.
\end{aligned} \tag{2.24}$$

We get from the first line in (2.24) that

$$\overline{A}\sigma_{(1)} = \Gamma_{(1)} - \tau \cdot \lambda_{(111)},\ A\kappa_{(1)} = \Gamma_{(4)} - \tau \cdot \lambda_{(411)}.$$

Multiplying $\overline{A}$ and A to the second and the third lines of the equations in (2.24), respectively, and using the just obtained relations, we have over $\mathcal{O}$

$$\begin{aligned}
\overline{A}\Gamma_{(2)} &= -\overline{B_{(1)}} \cdot (\Gamma_{(1)} - \tau \cdot \lambda_{(111)}) + \overline{A}\tau \cdot \lambda_{(211)}, \\
\overline{A}\Gamma_{(3)} &= \overline{C_{(1)}} \cdot (\Gamma_{(1)} - \tau \cdot \lambda_{(111)}) + \overline{A}\tau \cdot \lambda_{(311)}, \\
A\Gamma_{(5)} &= -B_{(1)} \cdot (\Gamma_{(4)} - \tau \cdot \lambda_{(411)}) + A\tau \cdot \lambda_{(511)}, \\
A\Gamma_{(6)} &= C_{(1)} \cdot (\Gamma_{(4)} - \tau \cdot \lambda_{(411)}) + A\tau \cdot \lambda_{(611)}.
\end{aligned}$$

Rearranging the terms, we get over $\mathcal{O}$

$$\begin{aligned}
\big(\overline{A}\lambda_{(211)} + \lambda_{(111)} \cdot \overline{B_{(1)}}\big) \cdot \tau =& \overline{A}\Gamma_{(2)} + \Gamma_{(1)} \cdot \overline{B_{(1)}}, \\
\big(\overline{A}\lambda_{(311)} - \lambda_{(111)} \cdot \overline{C_{(1)}}\big) \cdot \tau =& \overline{A}\Gamma_{(3)} - \Gamma_{(1)} \cdot \overline{C_{(1)}}, \\
\big(A\lambda_{(511)} + \lambda_{(411)} \cdot B_{(1)}\big) \cdot \tau =& A\Gamma_{(5)} + \Gamma_{(4)} \cdot B_{(1)}, \\
\big(A\lambda_{(611)} - \lambda_{(411)} \cdot C_{(1)}\big) \cdot \tau =& A\Gamma_{(6)} - \Gamma_{(4)} \cdot C_{(1)}.
\end{aligned} \tag{2.25}$$

As before, from the first two equations in (2.25), we obtain, near 0, the following:

$$\begin{aligned}&(\overline{A}\cdot\Gamma_{(2)}+\Gamma_{(1)}\cdot\overline{B_{(1)}})\cdot(\overline{A}\cdot\lambda_{(311)}-\lambda_{(111)}\cdot\overline{C_{(1)}})\\=&(\overline{A}\cdot\Gamma_{(3)}-\Gamma_{(1)}\cdot\overline{C_{(1)}})\cdot(\overline{A}\cdot\lambda_{(211)}+\lambda_{(111)}\cdot\overline{B_{(1)}}).\end{aligned}\tag{2.26}$$

At last, if both $[L_j,\overline{L_k}]$ and $[L_1,T]$ are contained in the span of $\{L_h,\overline{L_h}\}_{1\le h\le n-1}$ and T. Then we have

$$[L_k,T]=-\big[L_1,[\overline{L_1},L_k]\big]-\big[\overline{L_1},[L_k,L_1]\big]\in\mathrm{Span}\{\{L_h,\overline{L_h}\}_{1\le h\le n-1},T\}.$$

Summarizing the above, we have proved the following:

Proposition 2.1. *Let $(M,0)$ be a $2n$-dimensional real manifold in $\mathbb{C}^{n+1}$ defined by (2.1). Suppose M is non-minimal at the CR points near the origin. Then there is an open dense subset $\mathcal{O}$ of M near 0 such that the systems (2.18), (2.20) and (2.25) are solvable over $\mathcal{O}$ with unknowns γ, $\hat{\gamma}$, τ, respectively. In particular, when M is non-minimal at the CR points near the origin, we have (2.19), (2.21), and (2.26) near the origin.*

We mention that (2.19), (2.21), and (2.26) are what we need for the proof of Theorem 1.2.

3 Derivation of three basic equations and statement of Theorem 3.1

Let $(M,0)$ be a $(2n)$-dimensional smooth real submanifold in $\mathbb{C}^{n+1}$ defined by (2.1). Suppose that M is non-minimal at its CR points near 0 and the order of $E(z,\overline{z})$ is $m(\ge 3)$. We first study three basic relations for terms in $E^{(m)}$, by making use of (2.19), (2.21) and (2.26).

By (2.8) and (2.11), we have

$$\begin{aligned}\overline{A}\cdot\lambda_{(2jk)}+\overline{B_{(k)}}\cdot\lambda_{(1jk)}&=(w_n+O(2))\cdot(-\overline{w}_n\delta_{jk}+O(2))+(w_k+O(2))\cdot(-\overline{w}_j+O(2))\\&=-(|w_n|^2\cdot\delta_{jk}+\overline{w}_jw_k)+O(3),\\A\cdot\lambda_{(6jk)}-C_{(j)}\cdot\lambda_{(4jk)}&=(\overline{w}_n+O(2))\cdot\lambda_{(6jk)}-C_{(j)}\cdot(w_k+O(2))\\&=\overline{w}_n\cdot\lambda_{(6jk)}-w_kC_{(j)}+O(m+2)\\&=\overline{w}_n\cdot(-2iw_n)\cdot(\overline{\Phi_{(j)}})_{\overline{k}}+\overline{w}_n\cdot 2iw_k(\overline{\Phi_{(j)}})_{\overline{n}}-2iw_k\cdot\overline{\Phi_{(j)}}+O(m+2)\\&=-2i\overline{\Psi_{(jk)}}+O(m+2).\end{aligned}$$

Substituting these relations to (2.19), we get

$$\begin{aligned}&(|w_n|^2\cdot\delta_{jk}+\overline{w}_jw_k)\cdot(-\overline{\Psi_{(11)}}+O(m+2))+O(m+4)\\&=(|w_n|^2+|w_1|^2)\cdot(-\overline{\Psi_{(jk)}}+O(m+2))+O(m+4).\end{aligned}\tag{3.1}$$

Hence we obtain

$$(|w_n|^2 \cdot \delta_{jk} + \overline{w}_j w_k) \cdot \overline{\Psi_{(11)}} = (|w_n|^2 + |w_1|^2) \cdot \overline{\Psi_{(jk)}}. \tag{3.2}$$

Notice that (2.21) is the same as (2.19) except that j is replaced by 1. By the same method as that used to handle (2.19), we get the following equation which is the same as (3.2) except that j is replaced by 1.

$$(|w_n|^2 \cdot \delta_{1k} + \overline{w}_1 w_k) \cdot \overline{\Psi_{(11)}} = (|w_n|^2 + |w_1|^2) \cdot \overline{\Psi_{(1k)}}. \tag{3.3}$$

We next derive an equation from (2.26). From (2.8), (2.11) and (2.23), we obtain

$$\begin{aligned}
\Gamma_{(1)} =&(\overline{w}_n + O(2)) \cdot (-2\lambda_1 + O(1)) - (\overline{w}_1 + O(2)) \cdot O(1) = -2\lambda_1 \overline{w}_n + O(2), \\
\Gamma_{(2)} =&(\overline{w}_n + O(2)) \cdot O(1) - (\overline{w}_1 + O(2)) \cdot (-2\lambda_n + O(1)) = 2\lambda_n \overline{w}_1 + O(2), \\
\frac{1}{2i}\Gamma_{(3)} =&(\overline{w}_n + O(2)) \cdot \big(- \overline{w_n}\Phi_1 + \overline{w_1}\Phi_n + O(m+1)\big)_1 \\
&- (\overline{w}_1 + O(2)) \cdot \big(- \overline{w_n}\Phi_1 + \overline{w_1}\Phi_n + O(m+1)\big)_n \\
=& - \overline{w_n}\big(\overline{w_n}\Phi_1 - \overline{w_1}\Phi_n\big)_1 + \overline{w_1}\big(\overline{w_n}\Phi_1 - \overline{w_1}\Phi_n\big)_n + O(m+1).
\end{aligned} \tag{3.4}$$

Hence we have

$$\begin{aligned}
\overline{A}\Gamma_{(2)} + \overline{B}_{(1)}\Gamma_{(1)} &= (w_n + O(2)) \cdot (2\lambda_n \overline{w}_1 + O(2)) + (w_1 + O(2)) \cdot (-2\lambda_1 \overline{w}_n + O(2)) \\
&= 2\lambda_n w_n \overline{w}_1 - 2\lambda_1 w_1 \overline{w}_n + O(3).
\end{aligned} \tag{3.5}$$

From (2.8), (2.10) and (3.4), we obtain

$$\begin{aligned}
&\frac{1}{2i}(\overline{A}\Gamma_{(3)} - \Gamma_{(1)}\overline{C_{(1)}}) \\
=&\big(w_n + O(2)\big) \cdot \Big\{ - \overline{w_n}\big(\overline{w_n}\Phi_1 - \overline{w_1}\Phi_n\big)_1 + \overline{w_1}\big(\overline{w_n}\Phi_1 - \overline{w_1}\Phi_n\big)_n + O(m+1)\Big\} \\
&- \big(- 2\lambda_1 \overline{w}_n + O(2)\big) \cdot \big(- \Phi + O(m+1)\big) \\
=& - \overline{w_n}\big(w_n\overline{w_n}\Phi_1 - w_n\overline{w_1}\Phi_n\big)_1 + \overline{w_1}\big(w_n\overline{w_n}\Phi_1 - w_n\overline{w_1}\Phi_n\big)_n \\
&- \overline{w_1}\big(\overline{w_n}\Phi_1 - \overline{w_1}\Phi_n\big) - 2\lambda_1\overline{w_n}\Phi + O(m+2) \\
=& - \overline{w_n}\big(\Psi - \overline{w_1}\Phi\big)_1 + \overline{w_1}\big(\Psi - \overline{w_1}\Phi\big)_n - \overline{w_1}\big(\overline{w_n}\Phi_1 - \overline{w_1}\Phi_n\big) - 2\lambda_1\overline{w_n}\Phi + O(m+2) \\
=& - \overline{w_n}\Psi_1 + \overline{w_1}\Psi_n + O(m+2).
\end{aligned} \tag{3.6}$$

By (2.8) and (2.11), we get

$$\begin{aligned}
\frac{1}{2i}(\overline{A}\lambda_{(311)} - \lambda_{(111)}\overline{C_{(1)}}) =&(w_n + O(2)) \cdot \big\{ - \overline{w}_n\Phi_1 + \overline{w}_1\Phi_n + O(m+1)\big\} \\
&- (-\overline{w}_1 + O(2)) \cdot \big\{ - \Phi + O(m+1)\big\} \\
=& - w_n\overline{w_n}\Phi_1 + w_n\overline{w_1}\Phi_n - \overline{w}_1\Phi + O(m+2) \\
=& - \Psi + O(m+2), \\
\overline{A}\lambda_{(211)} + \overline{B_{(1)}}\lambda_{(111)} =&(w_n + O(2)) \cdot (-\overline{w}_n + O(2)) + (w_1 + O(2)) \cdot (-\overline{w}_1 + O(2)) \\
=& - (|w_n|^2 + |w_1|^2) + O(3).
\end{aligned} \tag{3.7}$$

Substituting (3.5)-(3.7) into (2.26), we obtain

$$\begin{aligned}&-\big(|w_n|^2+|w_1|^2+O(3)\big)\cdot\big\{-\overline{w_n}\Psi_1+\overline{w_1}\Psi_n+O(m+2)\big\}\\&=\big(2\lambda_n w_n\overline{w}_1-2\lambda_1 w_1\overline{w}_n+O(3)\big)\cdot\big\{-\Psi+O(m+2)\big\}.\end{aligned}$$

Hence we get

$$\big(|w_n|^2+|w_1|^2\big)\cdot\big(\overline{w_n}\Psi_1-\overline{w_1}\Psi_n\big)+\big(2\lambda_n w_n\overline{w}_1-2\lambda_1 w_1\overline{w}_n\big)\cdot\Psi=0. \tag{3.8}$$

Now, for convenience of the reader, we put together, in the following, the equations in (3.2), (3.3) and (3.8), that will be all we need to use to prove Theorem 1.2:

$$\begin{aligned}&(|w_n|^2\cdot\delta_{1k}+\overline{w}_1 w_k)\cdot\overline{\Psi_{(11)}}=(|w_n|^2+|w_1|^2)\cdot\overline{\Psi_{(1k)}},\quad 1<k<n\\&(|w_n|^2\cdot\delta_{jk}+\overline{w}_j w_k)\cdot\overline{\Psi_{(11)}}=(|w_n|^2+|w_1|^2)\cdot\overline{\Psi_{(jk)}},\ 1,\ j,k<n\\&\big(|w_n|^2+|w_1|^2\big)\cdot\big(\overline{w_n}\Psi_1-\overline{w_1}\Psi_n\big)+\big(2\lambda_n w_n\overline{w}_1-2\lambda_1 w_1\overline{w}_n\big)\cdot\Psi=0\ \text{ with}\\&\Psi_{(jk)}=w_n\overline{w_n}(\Phi_{(j)})_k-w_n\overline{w_k}(\Phi_{(j)})_n+\overline{w_k}\cdot\Phi_{(j)},\quad\Psi=\Psi_{(11)},\ \text{where}\\&\Phi_{(j)}=w_nH_{\overline{j}}-w_jH_{\overline{n}},\ \Phi=\Phi_{(1)},\ H=E^{(m)},\quad w_l=z_l+2\lambda_l\overline{z}_l\ \text{for}\ 1\le l\le n.\end{aligned} \tag{3.9}$$

Notice that when $n=2$, the first two equations in (3.9) disappear and we only have the third one to use.

We will use (3.9) to prove the following theorem, which includes Theorem 1.2 as its special case:

Theorem 3.1. *Let $(M,0)$ be a $(2n)$-dimensional smooth real submanifold in $\mathbb{C}^{n+1}$ defined by (2.1). Suppose that there exists an $i\in[1,n]$ such that $\lambda_i\neq 1/2$. We further suppose that M is non-minimal at its CR points near 0. Then for any positive integer N, there exists a holomorphic transform of the special form $(z,w)\to(z'=z,\ w'=w+o(|z|^2,w))$ such that in the new coordinates, M is defined by an equation of the form: $w'=\rho(z',\overline{z}')$ with $\Im\rho$ vanishing at least to the order N.*

Remark 3.1. Let M be a formal $(2n)$-manifold in $\mathbb{C}^{n+1}$ near 0 defined by a formal equation of the form $w=q(z,\overline{z})+O(|z|^3)$. Here, as before, $q(z,\overline{z})=\sum_{i=1}^n(|z_i|^2+\lambda_i(z_i^2+\overline{z}_i^2))$ with $0\le\lambda_1,\cdots,\lambda_n<\infty$. Then we can similarly define the formal vector fields $\{L_1,\cdots,L_{n-1},T\}$. We call that M is formally non-minimal if (2.19), (2.21), and (2.26) hold in the formal sense. Then the exact proof for Theorem 3.1 can be used to prove the following:

Theorem 3.2. *Let $(M,0)$ be a $(2n)$-dimensional formal submanifold in $\mathbb{C}^{n+1}$ defined by (2.1). Suppose that there exists an $i\in[1,n]$ such that $\lambda_i\neq 1/2$. Further assume that M is formally non-minimal. Then for any positive integer N, there exists a holomorphic transform of the special form $(z,w)\to(z'=z,\ w'=w+o(|z|^2,w))$ such that in the new coordinates, M is defined by an equation of the form: $w'=\rho(z',\overline{z}')$ with $\Im\rho$ vanishing at least to the order N.*

As a corollary, when M is smooth near the non-degenerate CR singular point $p = 0$, defined by (2.1), and when the set O of non-minimal CR points has $p = 0$ in its closure, then one sees that (2.19), (2.21), and (2.26) hold in an open subset of $\mathbb{C}^n$ that has 0 in its boundary. We see that they must hold for all z in a neighborhood of $0 \in \mathbb{C}^n$ in the formal sense. Hence, we see that M is formally non-minimal as just defined. Thus we have the following:

Corollary 3.3. *Let $(M, 0)$ be a $(2n)$-dimensional smooth submanifold in $\mathbb{C}^{n+1}$ defined by (2.1) near 0. Suppose that there exists an $i \in [1, n]$ such that $\lambda_i \neq 1/2$. Further assume that the set of non-minimal CR points of M forms an open subset with 0 in its closure . Then for any positive integer N, there exists a holomorphic transform of the special form $(z, w) \to (z' = z,\ w' = w + o(|z|^2, w))$ such that in the new coordinates, M is defined by an equation of the form: $w' = \rho(z', \overline{z}')$ with $\Im\rho$ vanishing at least to the order N.*

We notice that under the hypothesis in Corollary 3.3, when M is real analytic, it is easy to see that M must be non-minimal at any CR point near $p = 0$. Hence, Corollary 3.3 does not give any new result in the real analytic category.

4 Formal flattening near a CR singular point: Proof of Theorem 3.1—Part I

Before reading §4-§6, the reader is suggested to read the Appendix in §8 for the proof in the special case when $n = 2, m = 3$, to see basic ideas behind all these complicated computations.

We use the notations and definitions set up so far for the proof of Theorem 3.1. Due to the complicated nature of the argument, we divided our proof into two parts. In this part, we give an initial normalization by using biholomorphic change of coordinates without involving the non-minimality at CR points.

Throughout this and the next sections, we also set up the following convention:

$$E_{(I,J)} = 0 \text{ if one of the indices in } I \text{ or } J \text{ is negative.} \tag{4.1}$$

For quantities $a, b_1, \cdots, b_t$, we write

$$a = \mathcal{F}\{b_1, \cdots, b_t\} \text{ or } a = \mathcal{F}\{(b_j)_{1\leq j\leq t}\}$$

if $a = \sum_{j=1}^{t}(c_j b_j + d_j \overline{b_j})$. Here, when $b_j's$ are complex numbers, we require that c_j, d_j are complex numbers. When a, b_j are polynomials in $(z, \overline{z})$, we require c_j, d_j are polynomials in $(z, \overline{z})$, too.

For $\S4 - \S6$, we make the range of indices $j, k \in [2, n-1]$ if $n \geq 3$. For any homogeneous polynomial $\chi(z, \overline{z})$ of degree $k \geq 1$, write

$$\chi = \sum_{\alpha\geq 0, \beta\geq 0, |\alpha|+|\beta|=k} H_{(\alpha,\beta)} z^{\alpha}\overline{z^{\beta}}.$$

Set

$$\begin{aligned}
&\xi = 2\lambda_n,\ \eta = 2\lambda_1,\ \theta = 1-\xi^2,\\
&H_{[tsrh]} = H_{(te_n+se_1, re_n+he_1)} \text{ for } t+s+r+h=m,\\
&\Phi_{[tsrh]} = \Phi_{(te_n+se_1, re_n+he_1)} \text{ for } t+s+r+h=m,\\
&\Psi_{[tsrh]} = \Psi_{(te_n+se_1, re_n+he_1)} \text{ for } t+s+r+h=m+1.
\end{aligned} \tag{4.2}$$

Here e_j is the n-tuple with its j^{th} element 1 and zero for the others. By (2.9), we have

$$\begin{aligned}
\Phi_{[tsrh]} =&\xi(h+1)H_{[ts(r-1)(h+1)]} + (h+1)H_{[(t-1)sr(h+1)]}\\
&- (r+1)H_{[t(s-1)(r+1)h]} - \eta(r+1)H_{[ts(r+1)(h-1)]}.
\end{aligned} \tag{4.3}$$

From (2.10), we obtain

$$\begin{aligned}
\Psi_{[tsrh]} =&(s+1)\big\{\xi\Phi_{[t(s+1)(r-2)h]} + (1+\xi^2)\Phi_{[(t-1)(s+1)(r-1)h]} + \xi\Phi_{[(t-2)(s+1)rh]}\big\}\\
&- \xi(t+1)\Phi_{[(t+1)s(r-1)(h-1)]} - t\Phi_{[tsr(h-1)]} - \xi\eta(t+1)\Phi_{[(t+1)(s-1)(r-1)h]}\\
&- \eta t\Phi_{[t(s-1)rh]} + \Phi_{[tsr(h-1)]} + \eta\Phi_{[t(s-1)rh]}.
\end{aligned} \tag{4.4}$$

Notice that $\Phi_{[tsrh]}$ are $\Psi_{[t's'r'h']}$ are understood as 0 if one of their indices is negative.

Collecting the coefficients of $z_n^t z_1^{s-1}\overline{z_n}^{r+3}\overline{z_1}^h$ for $t \geq 0$, $s \geq 1$, $r \geq -3$ and $h = m+1-t-s-r \geq 0$ in (3.8), we get

$$\begin{aligned}
&s\big\{\xi\Psi_{[tsrh]} + (2\xi^2+1)\Psi_{[(t-1)s(r+1)h]} + (\xi^3+2\xi)\Psi_{[(t-2)s(r+2)h]} + \xi^2\Psi_{[(t-3)s(r+3)h]}\big\}\\
&+ s\eta\big\{\Psi_{[ts(r+2)(h-2)]} + \xi\Psi_{[(t-1)s(r+3)(h-2)]}\big\} + (1+\eta^2)(s-1)\big\{\Psi_{[t(s-1)(r+2)(h-1)]}\\
&+ \xi\Psi_{[(t-1)(s-1)(r+3)(h-1)]}\big\} + (s-2)\eta\big\{\Psi_{[t(s-2)(r+2)h]} + \xi\Psi_{[(t-1)(s-2)(r+3)h]}\big\}\\
&- \big\{(t+1)\xi\Psi_{[(t+1)(s-1)(r+1)(h-1)]} + t(1+\xi^2)\Psi_{[t(s-1)(r+2)(h-1)]} + (t-1)\xi\Psi_{[(t-1)(s-1)(r+3)(h-1)]}\big\}\\
&- \eta\big\{(t+1)\xi\Psi_{[(t+1)(s-2)(r+1)h]} + t(1+\xi^2)\Psi_{[t(s-2)(r+2)h]} + (t-1)\xi\Psi_{[(t-1)(s-2)(r+3)h]}\big\}\\
&- (t+1)\big\{\eta\Psi_{[(t+1)(s-1)(r+3)(h-3)]} + (2\eta^2+1)\Psi_{[(t+1)(s-2)(r+3)(h-2)]}\\
&+ (\eta^3+2\eta)\Psi_{[(t+1)(s-3)(r+3)(h-1)]} + \eta^2\Psi_{[(t+1)(s-4)(r+3)h]}\big\}\\
&+ \xi\big\{\xi\Psi_{[t(s-1)(r+2)(h-1)]} + \Psi_{[(t-1)(s-1)(r+3)(h-1)]}\big\} + \xi\eta\big\{\xi\Psi_{[t(s-2)(r+2)h]} + \Psi_{[(t-1)(s-2)(r+3)h]}\big\}\\
&- \eta\big\{\eta\Psi_{[t(s-1)(r+2)(h-1)]} + \xi\eta\Psi_{[(t-1)(s-1)(r+3)(h-1)]} + \Psi_{[t(s-2)(r+2)h]} + \xi\Psi_{[(t-1)(s-2)(r+3)h]}\big\} = 0.
\end{aligned} \tag{4.5}$$

Notice that (4.5) takes the following form:

$$\begin{aligned}
&s\big\{\xi\Psi_{[tsrh]} + (2\xi^2+1)\Psi_{[(t-1)s(r+1)h]} + (\xi^3+2\xi)\Psi_{[(t-2)s(r+2)h]}\\
&\quad + \xi^2\Psi_{[(t-3)s(r+3)h]}\big\} + \mathcal{F}\big\{(\Psi_{[t's'r'h']})_{s'+h'\leq s+h-2, s'\leq s, h'\leq h}\big\} = 0.
\end{aligned}$$

Thus for $s \geq 1$, by keeping use this property, we can inductively get

$$\Psi_{[tsrh]} = \mathcal{F}\big\{(\Psi_{[t's'r'h']})_{s'+h'\leq s+h-2, s'\leq s, h'\leq h}\big\}. \tag{4.6}$$

Substituting (4.4) into (4.6), we get, for $s \geq 1$, the following

$$\begin{aligned}&(s+1)\big\{\xi\Phi_{[t(s+1)(r-2)h]} + (1+\xi^2)\Phi_{[(t-1)(s+1)(r-1)h]} + \xi\Phi_{[(t-2)(s+1)rh]}\big\}\\ &= \mathcal{F}\big\{(\Phi_{[t's'r'h']})_{s'+h'\leq s+h-1, s'\leq s+1, h'\leq h}\big\}.\end{aligned}$$

Hence for $s \geq 2$, we can inductively obtain

$$\Phi_{[tsrh]} = \mathcal{F}\big\{(\Phi_{[t's'r'h']})_{s'+h'\leq s+h-2, s'\leq s, h'\leq h}\big\}. \tag{4.7}$$

Substituting (4.3) into (4.7), we get, for $s \geq 2$ and $h \geq 0$, the following

$$\xi(h+1)H_{[ts(r-1)(h+1)]} + (h+1)H_{[(t-1)sr(h+1)]} = \mathcal{F}\big\{(H_{[t's'r'h']})_{s'+h'\leq s+h-1, s'\leq s, h'\leq h+1}\big\}.$$

Hence for $s \geq 2$ and $h \geq 1$, we inductively get that

$$H_{[ts(m-t-s-h)h]} = \mathcal{F}\Big\{\big(H_{[t's'(m-t'-s'-h')h']}\big)_{s'+h'\leq s+h-2, s'\leq s, h'\leq h}\Big\}.$$

Notice that $H_{[tsrh]} = \overline{H_{[rhts]}}$. Keeping applying the above until the assumption that $s \geq 2$ and $h \geq 1$ do not hold anymore, we can inductively get the following crucial formula:

$$H_{[ts(m-t-s-h)h]} = \mathcal{F}\Big\{\big(H_{[t'1(m-t'-2)1]}\big)_{1\leq t'\leq m-2}, \big(H_{[t'0(m-t'-i)i]}\big)_{i\leq \max(s,h), 0\leq t'\leq m-i}\Big\}. \tag{4.8}$$

Substituting (2.9) and (2.10) into (3.3), we get the following equation:

$$\begin{aligned}&\overline{w}_1 w_k \cdot (|w_n|^2 H_{1\overline{1}} - w_n\overline{w}_1 H_{n\overline{1}} - w_1\overline{w}_n H_{1\overline{n}} + |w_1|^2 H_{n\overline{n}})\\ &= (|w_n|^2 + |w_1|^2)\cdot(|w_n|^2 H_{1\overline{k}} - w_n\overline{w}_1 H_{n\overline{k}} - w_k\overline{w}_n H_{1\overline{n}} + w_k\overline{w}_1 H_{n\overline{n}}).\end{aligned} \tag{4.9}$$

Notice that it takes the form

$$-|w_n|^4 H_{1\overline{k}} + \sum_{i_n+j_n\leq 3} z_1^{i_1} z_k^{i_k} z_n^{i_n} \overline{z_1}^{j_1} \overline{z_k}^{j_k} \overline{z_n}^{j_n} \frac{\partial^{h_1}}{\partial z_1^{h_1}} \frac{\partial^{h_k}}{\partial z_k^{h_k}} \frac{\partial^{h_n}}{\partial z_n^{h_n}} \frac{\partial^{l_1}}{\partial \overline{z_1}^{l_1}} \frac{\partial^{l_k}}{\partial \overline{z_k}^{l_k}} \frac{\partial^{l_n}}{\partial \overline{z_1}^{l_n}} H = 0,$$

where $i_1 + i_k + i_n + j_1 + j_k + j_n - (h_1 + h_k + h_n + l_1 + l_k + l_n) = 2$. Hence we get

$$H_{(te_n+e_1+I, re_n+e_k+J)} = \mathcal{F}\big(\{H_{(t'e_n+I', r'e_n+J')}\}_{t'+r'>t+r}\big). \tag{4.10}$$

Similarly, substituting (2.9) and (2.10) into (3.2) and setting $j = k (\neq 1)$, we get the following equation:

$$\begin{aligned}&(|w_n|^2 + |w_k|^2)\cdot(|w_n|^2 H_{1\overline{1}} - w_n\overline{w}_1 H_{n\overline{1}} - w_1\overline{w}_n H_{1\overline{n}} + |w_1|^2 H_{n\overline{n}})\\ &= (|w_n|^2 + |w_1|^2)\cdot(|w_n|^2 H_{k\overline{k}} - w_n\overline{w}_k H_{n\overline{1}} - w_1\overline{w}_n H_{k\overline{n}} + |w_k|^2 H_{n\overline{n}}).\end{aligned} \tag{4.11}$$

Similar to (4.10), for any fixed $s, h \geq 1$, we get

$$\begin{aligned}&shH_{(te_n+se_k, re_n+he_k)} - H_{(te_n+(s-1)e_k+e_1, re_n+(h-1)e_k+e_1)}\\ &+ \mathcal{F}\{(H_{(t'e_n+I, r'e_n+J)})_{t'+r'>t+r}\} = 0.\end{aligned} \tag{4.12}$$

Next we prove the following lemma, which is only needed for $n \geq 3$.

Lemma 4.1. *Suppose that $n \geq 3$. For any given j with $j \geq 1$ and any given $I = (i_1, \cdots, i_n)$ with $i_1 = i_k = i_n = 0$, suppose that $H_{(te_n+se_1, re_n+he_1+I+(j'-2)e_k)} = 0$ for all $t, s, r, h \geq 0$, $j' \leq j$ and $t+s+r+h = m+2-|I|-j'$. Then*

$$
\begin{aligned}
&H_{(te_n+se_1, re_n+he_1+I+je_k)} = \mathcal{F}\{(H_{(t'e_n, r'e_n+h'e_1+I+je_k)})_{r' \geq t'}\}, \\
&\text{where } t+s+r+h = t'+r'+h' = m-|I|-j.
\end{aligned}
\tag{4.13}
$$

Proof of Lemma 4.1. Set

$$
\begin{aligned}
P^{(l)} &= \{\text{homogeneous polynomials of degree } l\} \text{ and} \\
P^{(l)}_{(1n\overline{1}\overline{n})} &= \{\text{homogeneous polynomials of degree } l \text{ in } (z_1, z_n, \overline{z_1}, \overline{z_n})\}.
\end{aligned}
$$

In (4.9), the coefficients of terms other than $(|w_n|^2 + |w_1|^2)(|w_n|^2 H_{1\overline{k}} - w_n\overline{w_1}H_{n\overline{k}})$, when projected to the space of polynomials of the form: $\overline{z}^{I+(j-1)e_k}P^{(m+3-|I|-j)}_{(1n\overline{1}\overline{n})}$, is a linear combination of $H_{(t'e_n+s'e_1, r'e_n+h'e_1+I+(j'-2)e_k)}$ with $j' \leq j$, which are 0 by our assumption. Here and in what follows we equip the space of polynomials in $(z, \overline{z})$ with $\{z^\alpha\overline{z^\beta}\}$ as an ortho-normal basis. Hence by considering terms projected to the space $\overline{z}^{I+(j-1)e_k}P^{(m+3-|I|-j)}_{(1n\overline{1}\overline{n})}$ in (4.9), we get

$$
|w_n|^2(|w_n|^2 + |w_1|^2)H_{1\overline{k}} - w_n\overline{w_1}(|w_n|^2 + |w_1|^2)H_{n\overline{k}} = 0 \bmod \left(\{\overline{z}^{I+(j-1)e_k}P^{(m+3-|I|-j)}_{(1n\overline{1}\overline{n})}\}^c\right).
$$

Here for a subspace A of the space of polynomials, we write A^c for its compliment. Namely, we have

$$
(\overline{z}_n + 2\lambda_n z_n)H_{1\overline{k}} - (\overline{z}_1 + 2\lambda_1 z_1)H_{n\overline{k}} = 0 \bmod \left(\{\overline{z}^{I+(j-1)e_k}P^{(m-|I|-j)}_{(1n\overline{1}\overline{n})}\}^c\right).
\tag{4.14}
$$

Considering the coefficients of $z_1^{s-1}\overline{z}^{I+(j-1)e_k}\overline{z}_1^h\overline{z}_n^{\,t+r+1}$ and $z_1^{s-1}z_n^t\overline{z}^{I+(j-1)e_k}\overline{z}_1^h\overline{z}_n^{\,r+1}$, respectively, with $r = m-t-s-|I|-j-h$, $t \geq 0$, $h \geq 0$, $s \geq 1$ in (4.14), we get

$$
\begin{aligned}
&sH_{(se_1, ((t+r)e_n+he_1+I+je_k)} \\
=&H_{(e_n+(s-1)e_1, (t+r+1)e_n+(h-1)e_1+I+je_k)} + 2\lambda_1 H_{(e_n+(s-2)e_1, (t+r+1)e_n+he_1+I+je_k)}, \text{ and} \\
&s\big(H_{(te_n+se_1, re_n+he_1+I+je_k)} + 2\lambda_n H_{((t-1)e_n+se_1, (r+1)e_n+he_1+I+je_k)}\big) \\
=&(t+1)\big(H_{((t+1)e_n+(s-1)e_1, (r+1)e_n+(h-1)e_1+I+je_k)} + 2\lambda_1 H_{((t+1)e_n+(s-2)e_1, (r+1)e_n+he_1+I+je_k)}\big).
\end{aligned}
\tag{4.15}
$$

Hence for $s \geq 1$, we obtain

$$
\begin{aligned}
H_{(te_n+se_1, re_n+he_1+I+je_k)} = \mathcal{F}\{&(H_{(t'e_n+(s-1)e_1, r'e_n+(h-1)e_1+I+je_k)})_{r'-t' \geq r-t}, \\
&(H_{(t'e_n+(s-2)e_1, r'e_n+he_1+I+je_k)})_{r'-t' \geq r-t}\}.
\end{aligned}
\tag{4.16}
$$

Next we prove by induction that

$$
H_{(te_n+se_1, re_n+he_1+I+je_k)} = \mathcal{F}\{(H_{(t'e_n, r'e_n+h'e_1+I+je_k)})_{r'-t' \geq r-t}\}.
\tag{4.17}
$$

In fact, the claim holds automatically for $s=0$. If $s=1$, (4.17) follows from (4.16). Now we suppose that (4.17) holds for $s<s_0$, we can get by (4.16) that

$$\begin{aligned} H_{(te_n+s_0e_1,re_n+he_1+I+je_k)} &= \mathcal{F}\big\{\big(H_{(t'e_n+(s_0-1)e_1,r'e_n+(h-1)e_1+I+je_k)}\big)_{r'-t'>r-t}, \\ &\qquad \big(H_{(t'e_n+(s_0-2)e_1,r'e_n+he_1+I+je_k)}\big)_{r'-t'\geq r-t}\big\} \\ &= \mathcal{F}\big\{\big(H_{(t'e_n,r'e_n+h'e_1+I+je_k)}\big)_{r'-t'\geq r-t}\big\}. \end{aligned} \tag{4.18}$$

The last equality follows from our assumption. Hence (4.17) also holds for s_0. This finishes the proof of (4.17).

By interchanging the role of z_1 and z_n in (4.17), we can get

$$H_{(te_n+se_1,re_n+he_1+I+je_k)} = \mathcal{F}\{(H_{(s'e_1,r'e_n+h'e_1+I+je_k)})_{h'-s'\geq h-s}\}. \tag{4.19}$$

As a special case of (4.19) or (4.17), we obtain the following:

$$\begin{aligned} H_{(t'e_n,r'e_n+h'e_1+I+je_k)} &= \mathcal{F}\{(H_{(s''e_1,r''e_n+h''e_1+I+je_k)})_{h''-s''\geq h'}\}. \\ H_{(s''e_1,r''e_n+h''e_1+I+je_k)} &= \mathcal{F}\{(H_{(t'''e_n,r'''e_n+h'''e_1+I+je_k)})_{r'''-t'''\geq r''}\}. \end{aligned} \tag{4.20}$$

Now we conclude from (4.17) and (4.20) that

$$H_{(te_n+se_1,re_n+he_1+I+je_k)} = \mathcal{F}\{(H_{(t'''e_n,r'''e_n+h'''e_1+I+je_k)})_{r'''-t'''\geq 0}\}.$$

This completes the proof of Lemma 4.1. □

For the rest of this section, for simplicity of notation, we assume that λ_n is the smallest non-parabolic Bishop invariant, namely, the smallest one that is not equal to $\frac{1}{2}$. Then we have the following normalization for $E(z,\overline{z})$. We notice that the following result holds in general even without assuming the non-minimality condition at CR points. Also, in this result, there is no need to assume that $\lambda_n \neq \frac{1}{2}$.

Theorem 4.2. *For any given $l \geq 3$, there exists a holomorphic transformation near the orign $(z,w) \to (z'=z, w'=w+o(|z|^2,w))$ such that in the new coordinates, the $E(z,\overline{z})$ defined in (2.1) satisfies*

$$E_{(I,0)} = E_{(te_n+J,se_n)} = 0 \textit{ for } t\geq s,\ |J|\neq 0,\ |I|=t+s+|J|\leq l. \tag{4.21}$$

Moreover, we have the following normalizations:
(I) When $\lambda_n=0$, we have

$$E_{(te_n,se_n)} = 0 \textit{ for } t\geq s. \tag{4.22}$$

(II) When $\lambda_n\neq 0$, for any $m_0\leq l$, the normalization is divided into the following six cases:

(II_{-3}) If $m_0 = 6\hat{m} - 3$, then we have

$$\begin{aligned} &E_{(te_n, se_n)} = 0 \text{ for } 4\hat{m} - 1 \le t \le m_0 - 1, \\ &E_{((2t+1)e_n+e_1,(m_0-2t-3)e_n+e_1)} = 0 \text{ for } 2\hat{m} - 2 \le t \le 3\hat{m} - 3. \end{aligned} \tag{4.23}$$

(II_{-2}) If $m_0 = 6\hat{m} - 2$, then we have

$$\begin{aligned} &E_{(te_n, se_n)} = 0 \text{ for } 4\hat{m} - 1 \le t \le m_0 - 1, \\ &E_{((2t+1)e_n+e_1,(m_0-2t-3)e_n+e_1)} = 0 \text{ for } 2\hat{m} - 1 \le t \le 3\hat{m} - 3, \\ &\Re E_{((4\hat{m}-3)e_n+e_1,(2\hat{m}-1)e_n+e_1)} = 0. \end{aligned} \tag{4.24}$$

(II_{-1}) If $m_0 = 6\hat{m} - 1$, then we have

$$\begin{aligned} &E_{(te_n, se_n)} = 0 \text{ for } 4\hat{m} \le t \le m_0 - 1, \\ &E_{((2t+1)e_n+e_1,(m_0-2t-3)e_n+e_1)} = 0 \text{ for } 2\hat{m} - 1 \le t \le 3\hat{m} - 2. \end{aligned} \tag{4.25}$$

(II_0) If $m_0 = 6\hat{m}$, then we have

$$\begin{aligned} &E_{(te_n, se_n)} = 0 \text{ for } 4\hat{m} + 1 \le t \le m_0 - 1, \\ &E_{((2t+1)e_n+e_1,(m_0-2t-3)e_n+e_1)} = 0 \text{ for } 2\hat{m} - 1 \le t \le 3\hat{m} - 2, \\ &\Re E_{(4\hat{m}e_n, 2\hat{m}e_n)} = 0. \end{aligned} \tag{4.26}$$

(II_1) If $m_0 = 6\hat{m} + 1$, then we have

$$\begin{aligned} &E_{(te_n, se_n)} = 0 \text{ for } 4\hat{m} + 1 \le t \le m_0 - 1, \\ &E_{((2t+1)e_n+e_1,(m_0-2t-3)e_n+e_1)} = 0 \text{ for } 2\hat{m} \le t \le 3\hat{m} - 1. \end{aligned} \tag{4.27}$$

(II_2) If $m_0 = 6\hat{m} + 2$, then we have

$$\begin{aligned} &E_{(te_n, se_n)} = 0 \text{ for } 4\hat{m} + 2 \le t \le m_0 - 1, \\ &E_{((2t+1)e_n+e_1,(m_0-2t-3)e_n+e_1)} = 0 \text{ for } 2\hat{m} \le t \le 3\hat{m} - 1, \\ &\Re E_{((4\hat{m}+1)e_n, (2\hat{m}+1)e_n)} = 0. \end{aligned} \tag{4.28}$$

Proof of Theorem 4.2. Suppose that $z' = z$, $w' = w + B(z,w)$ transforms $w = q(z,\overline{z}) + p(z,\overline{z}) + iE(z,\overline{z})$ to $w' = q(z',\overline{z}') + p'(z',\overline{z}') + iE'(z',\overline{z}')$, where $p'(z',\overline{z}')$ and $E'(z',\overline{z}')$ are real valued and both of their orders are at least three. Then

$$q(z,\overline{z}) + p(z,\overline{z}) + iE(z,\overline{z}) + B(z,w) = q(z,\overline{z}) + p'(z,\overline{z}) + iE'(z,\overline{z}). \tag{4.29}$$

Hence we get

$$\Im(B(z,w)) = E'(z,\overline{z}) - E(z,\overline{z}). \tag{4.30}$$

Set

$$B^{(m_0)}(z,w) = \sum_{|I|+2j=m_0} b_{(Ij)} z^I w^j.$$

We further normalize $B(z,\overline{z})$ such that $\mathrm{Re}(b_{(0\frac{m_0}{2})}) = 0$ if m_0 is even. Then the real dimension of $B^{(m_0)}$ is

$$
\begin{aligned}
&2\cdot \sharp\big\{(i_1,\cdots,i_n,j)\in\mathbb{R}^{n+1}:\ i_1,\cdots,i_n,j\geq 0,\ i_1+\cdots+i_n+2j=m_0,\ 2j\neq m_0\big\}+\delta_{m_0}\\
=&2\cdot \sharp\big\{(i_1,\cdots,i_n,j)\in\mathbb{R}^{n+1}:\ I'\neq 0,\ i_1+\cdots+i_n+2j=m_0\big\}+2[\frac{m_0+1}{2}]+\delta_{m_0}.
\end{aligned}
\tag{4.31}
$$

Here $\delta_{m_0}=1$ when m_0 is even and 0, otherwise.
The dimension of the term $\sum\limits_{I'\neq 0,|I|+2j=m_0} a_{(Ij)}(z')^{I'}z_n^{i_n}|z_n|^{2j}$ is

$$
2\cdot \sharp\big\{(i_1,\cdots,i_n,j)\in\mathbb{R}^{n+1}:\ i_1,\cdots,i_n,j\geq 0,\ I'\neq 0,\ i_1+\cdots+i_n+2j=m_0\big\}. \tag{4.32}
$$

(I) Assume that $\lambda_n=0$. Set

$$
\hat{P}^{(m_0)}=\big\{\text{polynomials of the form } 2\Re\sum_{|I|+2j=m_0} a_{(Ij)}z^I|z_n|^{2j},\ \Im(a_{0,[m_0/2]})=0 \text{ for } m_0 \text{ even}\big\}.
$$

To get the normalization condition (4.21) and (4.22), we only need to prove that

$$
\Im\big(B^{(m_0)}(z,q(z,\overline{z}))\big)\big|_{\hat{P}^{(m_0)}}=Q^{(m_0)}(z,\overline{z}) \tag{4.33}
$$

is solvable for any $Q^{(m_0)}(z,\overline{z})\in\hat{P}^{(m_0)}$. Notice that $\hat{P}^{(m_0)}$ and the space $\{B^{(m_0)}(z,q(z,\overline{z}))\}$ have the same dimension. Here, we recall that for a polynomial A and a subspace of polynomials, we write $A|_P$ for the projection of A to P. Hence to prove (4.33), we need to show that

$$
\Im\big(B^{(m_0)}(z,q(z,\overline{z}))\big)\big|_{\hat{P}^{(m_0)}}=0 \text{ and } \Re(b_{(0\frac{m_0}{2})})=0 \text{ for } m_0 \text{ even} \Longleftrightarrow B=0.
$$

By considering the coefficients of terms involving only z_n and $\overline{z}_n$, we get

$$
\Im\big(\sum\nolimits_{i+2j=m_0} b_{(0i_nj)}z_n^{i_n}|z_n|^{2j}\big)=0.
$$

Thus we get $b_{(0i_nj)}=0$. Suppose that $b_{(I'i_nj)}=0$ for $|I'|\leq k_0$. Considering terms of the form: $z^{I'}z_n^{i_n}|z_n|^j$ with $|I'|=k_0+1$, we get

$$
\sum\nolimits_{|I'|=k_0+1,i+2j=m} b_{(I'i_nj)}z'^{I'}z_n^{i_n}|z_n|^{2j}=0,
$$

from which it follows that $b_{(I'i_nj)}=0$. Thus we get $B^{(m_0)}(z,\overline{z})=0$.

(II) Assume that $\lambda_n \neq 0$. Write $\widetilde{P}$ for the space of polynomials of the form: $|z_1|^2 P_1(z_n, \overline{z}_n) + P_2(z_n, \overline{z}_n)$. Then

$$
\begin{aligned}
\Im B(z, q(z,\overline{z}))\big|_{\widetilde{P}} =& \Im B(0, z_n, |z_n|^2 + \lambda_n z_n^2 + \lambda_n \overline{z}_n^2 + |z_1|^2)\big|_{\widetilde{P}} \\
=& \sum_{h+2j=m_0} \frac{1}{2i}(b_{(0hj)} z_n^h - \overline{b_{(0hj)}}\overline{z}_n^h)(|z_n|^2 + \lambda_n z_n^2 + \lambda_n \overline{z}_n^2 + |z_1|^2)^j\big|_{\widetilde{P}} \\
=& \sum_{h+2j=m_0} \frac{1}{2i} b_{(0hj)} z_n^h (|z_n|^2 + \lambda_n z_n^2 + \lambda_n \overline{z}_n^2 + |z_1|^2)^j\big|_{\widetilde{P}} \\
&- \sum_{h+2j=m_0} \frac{1}{2i}\overline{b_{(0hj)}}\overline{z_n}^h (|z_n|^2 + \lambda_n z_n^2 + \lambda_n \overline{z}_n^2 + |z_1|^2)^j\big|_{\widetilde{P}} \\
:=& \widetilde{I}\big|_{\widetilde{P}} - \widetilde{J}\big|_{\widetilde{P}}.
\end{aligned}
\tag{4.34}
$$

Here, for a subspace A of the space of polynomials and for a polynomial X, we write $X|_A$ for the projection of X to A. Write

$$
I_{kl} = \sum_{h+2j=m_0, j\geq k+l} \frac{1}{2i} b_{(0hj)} \tbinom{j}{j-k-l} \lambda_n^{j-k} \tbinom{k+l}{l},\ J_{kl} = \sum_{h+2j=m_0, j\geq k+l} \frac{1}{2i}\overline{b_{(0hj)}} \tbinom{j}{j-k-l} \lambda_n^{j-k} \tbinom{k+l}{l}. \tag{4.35}
$$

Then we have

$$
J_{kl} = -\overline{I_{kl}},\ I_{kl} = \tbinom{k+l}{k} \lambda_n^l I_{k+l,0}. \tag{4.36}
$$

A direct computation shows that

$$
\begin{aligned}
\widetilde{I}\big|_{\widetilde{P}} =& \sum_{h+2j=m_0} \frac{1}{2i} b_{(0hj)} z_n^h \sum_{0\leq k+l\leq j} \tbinom{j}{j-k-l} (\lambda_n z_n^2)^{j-k-l} \tbinom{k+l}{l} (\lambda_n \overline{z}_n^2)^l |z_n|^{2k} \\
&+ \sum_{h+2j=m_0} \frac{1}{2i} b_{(0hj)} z_n^h \sum_{0\leq k+l\leq j} \tbinom{j}{j-k-l} (\lambda_n z_n^2)^{j-k-l} \tbinom{k+l}{l} (\lambda_n \overline{z}_n^2)^l |z_n|^{2(k-1)} \cdot k|z_1|^2 \\
=& \sum_{0\leq k+2l\leq m_0, k+l\leq \frac{m_0}{2}} I_{kl} z_n^{m_0-k-2l} \overline{z}_n^{k+2l} + \sum_{\substack{k\geq 1, 0\leq k+2l\leq m_0,\\ k+l\leq \frac{m_0}{2}}} k I_{kl} |z_1|^2 z_n^{m_0-k-2l-1} \overline{z}_n^{k+2l-1}.
\end{aligned}
\tag{4.37}
$$

Similarly, we have

$$
\widetilde{J}\big|_{\widetilde{P}} = \sum_{\substack{0\leq k+2l\leq m\\ k+l\leq \frac{m_0}{2}}} J_{kl} z_n^{k+2l} \overline{z_n}^{m_0-k-2l} + \sum_{\substack{k\geq 1, 0\leq k+2l\leq m_0\\ k+l\leq \frac{m_0}{2}}} k J_{kl} |z_1|^2 z_n^{k+2l-1} \overline{z_n}^{m_0-k-2l-1}. \tag{4.38}
$$

Hence the coefficients of $z_n^t \overline{z}_n^s (t \geq s, t+s = m_0)$ and $z_n^{t-1} \overline{z}_n^{s-1} |z_1|^2 (t \geq s, t+s = m_0)$ in $\mathrm{Im}(B(z, q(z,\overline{z})))$ are, respectively, the following:

$$
\sum_{k+2l=s} I_{kl} - \sum_{\substack{k+2l=t\\ k+l\leq m_0/2}} J_{kl} \quad \text{and} \quad \sum_{k+2l=s} k I_{kl} - \sum_{\substack{k+2l=t\\ k+l\leq m_0/2}} k J_{kl}. \tag{4.39}
$$

$\mathbf{(II_{-3})}$: In this case, we have set $m_0 = 6\hat{m} - 3$. Write

$$\hat{P}^{(m_0)}_{-3} = \big\{\text{polynomials of the form} \sum_{t\ge 4\hat{m}-1} a_t z_n^t \overline{z}_n^{m_0-t} + \sum_{t\ge 2\hat{m}-2} b_t z_n^{2t+1}\overline{z}_n^{m_0-2t-3}|z_1|^2 + \sum_{I'\neq 0} C_{I't}(z')^{I'} z_n^{m-|I'|-2t}|z_n|^{2t}\big\}.$$

To get the normalization condition (4.21) and (4.23), we only need to prove that

$$\Im\big(B^{(m_0)}(z,q(z,\overline{z}))\big)\big|_{\hat{P}^{(m_0)}_{-3}} = Q^{(m_0)}(z,\overline{z}) \tag{4.40}$$

is solvable for any real valued polynomial $Q^{(m_0)}(z,\overline{z}) \in \hat{P}^{(m_0)}_{-3}$. Notice that the dimension of the space of polynomials of the form: $\sum\limits_{t\ge 4\hat{m}-1} a_t z_n^t \overline{z}_n^{m_0-t} + \sum\limits_{t\ge 2\hat{m}-2} b_t z_n^{2t+1}\overline{z}_n^{m_0-2t-3}|z_1|^2$ is

$$2\big(6\hat{m}-3-(4\hat{m}-2)\big) + 2\Big(\frac{6\hat{m}-6}{2} - (2\hat{m}-3)\Big) = 6\hat{m}-2 = 2\Big[\frac{6\hat{m}-2}{2}\Big]. \tag{4.41}$$

Combining this with (4.31) and (4.32), we know that the space $\{B^{(m_0)}(z,q(z,\overline{z}))\}$ and the space $\hat{P}^{(m_0)}_{-3}$ have the same dimension. Hence to prove (4.40), we need to show that $B^{(m_0)} = 0$ if

$$\Im\big(B^{(m_0)}(z,q(z,\overline{z}))\big)\big|_{\hat{P}^{(m_0)}_{-3}} = 0. \tag{4.42}$$

By (4.39), the condition (4.23) gives that

$$\begin{gathered}\sum_{k+2l=2t-1} kI_{kl} = \sum_{\substack{k+2l=m_0-2t+1\\ k+l\le m_0/2}} kJ_{kl}, \quad \sum_{k+2l=2t-1} I_{kl} = \sum_{\substack{k+2l=m_0-2t+1\\ k+l\le m_0/2}} J_{kl},\\ \sum_{k+2l=2t} I_{kl} = \sum_{\substack{k+2l=m_0-2t\\ k+l\le m_0/2}} J_{kl} \text{ for } 1\le t\le \hat{m}-1,\end{gathered} \tag{4.43}$$

and

$$\sum_{k+2l=2\hat{m}-1} kI_{kl} = \sum_{\substack{k+2l=m_0-2\hat{m}+1\\ k+l\le m_0/2}} kJ_{kl}. \tag{4.44}$$

Next we prove by induction that, for $1\le t\le \hat{m}-1$, we have

$$I_{2t-1,0} = I_{2t,0} = J_{0,3\hat{m}-1-t} = 0 \tag{4.45}$$

Setting $t=1$ in (4.43), we get

$$I_{10} = 0, \; I_{1,0} = J_{0,3\hat{m}-2}, \; I_{20} + I_{01} = J_{1,3\hat{m}-3}.$$

Together with (4.36), we obtain $I_{10} = I_{20} = J_{0,3\hat{m}-2} = 0$.

Suppose that (4.45) holds for $t \le t_0 \in [1, \hat{m} - 2]$. Setting $t = t_0 + 1$ in (4.43) and making use of the relations above, we get

$$I_{2t_0+1,0} = 0,\ I_{2t_0+1,0} = J_{0,3\hat{m}-t_0-2},\ I_{2t_0+2,0} + I_{2t_0,1} = J_{1,3\hat{m}-t_0-3}.$$

Combining this with (4.36), we obtain (4.45) for $t = t_0+1$. Hence (4.45) holds for $1 \le t \le \hat{m}-1$. Substituting the relations in (4.45) to (4.44), we get $I_{2\hat{m}-1,0} = 0$. Thus we get

$$I_{t,0} = J_{0,t'} = 0 \text{ for } 1 \le t \le 2\hat{m} - 1,\ 2\hat{m} \le t' \le 3\hat{m} - 2. \tag{4.46}$$

Namely, we have $I_{t,0} = 0$ for $1 \le t \le 3\hat{m} - 2$. By (4.35), we know that

$$I_{k0} = \frac{1}{2i} b_{(0(m_0-2k)k)} + \mathcal{F}\{(b_{(0(m_0-2t)t)})_{t>k}\}. \tag{4.47}$$

In particular, we have $b_{(01(3\hat{m}-2))} = 2iI_{(3\hat{m}-2)0}$. Combining this with (4.46) and (4.47), we inductively get $b_{(0(m_0-2k)k)} = 0$.

Suppose that $b_{(I'i_nj)} = 0$ for $|I'| \le k_0$. Next we will prove $b_{(I'i_nj)} = 0$ for $|I'| = k_0 + 1$. Considering all terms of forms $z^{I'} z_n^h |z_n|^{2j}$ with $|I'| = k_0+1$, $|I'| + h + 2j = m_0$ in (4.42), we get

$$\sum\nolimits_{|I'|=k_0+1,h+2j=m_0-|I'|} b_{(I'hj)} z'^{I'} z_n^h \big(|z_n|^2 + \lambda_n z_n^2 + \lambda_n \overline{z}_n^2\big)^j \big|_{\hat{P}_{-3}^{(m_0)}} = 0, \tag{4.48}$$

Write

$$\hat{I}_{kl} = \sum_{h+2j=m_0-|I'|, j \ge k+l} b_{(I'hj)} \tbinom{j}{j-k-l} \lambda_n^{j-k} \tbinom{k+l}{l}. \tag{4.49}$$

Then we have

$$\hat{I}_{kl} = \tbinom{k+l}{k} \lambda_n^l \hat{I}_{k+l,0}. \tag{4.50}$$

A direct computation shows that

$$\begin{aligned}
&\sum\nolimits_{|I'|=k_0+1,h+2j=m_0-|I'|} b_{(I'hj)} z'^{I'} z_n^h \big(|z_n|^2 + \lambda_n z_n^2 + \lambda_n \overline{z}_n^2\big)^j \big|_{\hat{P}_{-3}^{(m_0)}} \\
=&\sum\nolimits_{|I'|=k_0+1,h+2j=m_0-|I'|} b_{(I'hj)} z'^{I'} z_n^h \sum_{0 \le k+l \le j} \tbinom{j}{j-k-l} (\lambda_n z_n^2)^{j-k-l} \tbinom{k+l}{l} (\lambda_n \overline{z}_n^2)^l |z_n|^{2k} \big|_{\hat{P}_{-3}^{(m_0)}} \\
=& \sum_{\substack{|I'|=k_0+1 \\ 0 \le k+2l \le \frac{m_0-|I'|}{2}}} \hat{I}_{kl} z'^{I'} z_n^{m_0-|I'|-k-2l} \overline{z}_n^{k+2l}.
\end{aligned}$$

Thus (4.48) is equivalent to

$$\sum_{k+2l=\check{m}} \hat{I}_{kl} = 0 \text{ for } 1 \le \check{m} \le \big[\frac{m_0 - |I'|}{2}\big]. \tag{4.51}$$

Setting $\check{m}=1$ in (4.51), we get $\hat{I}_{10}=0$. Combining this with (4.51) and (4.50), we inductively obtain $\hat{I}_{t,0}=0$ for $1\le t\le [\frac{m_0-|I'|}{2}]$. From (4.49), we know that

$$\hat{I}_{k0}=b_{(I'(m_0-|I'|-2k)k)}+\mathcal{F}\{(b_{(I'(m_0-|I'|-2t)t)})_{t>k}\}. \tag{4.52}$$

In particular, we have

$$b_{\left(I'(m_0-|I'|-2[\frac{m_0-|I'|}{2}])[\frac{m_0-|I'|}{2}]\right)}=\hat{I}_{[\frac{m_0-|I'|}{2}],0}.$$

Combining this with (4.52), we inductively get $b_{(I'hj)}=0$ for $|I'|=k_0+1, h+2j=m_0-|I'|$. Thus we get $B^{(m_0)}(z,\overline{z})=0$.

(II$_{-2}$) In this case, we have set $m_0=6\hat{m}-2$. Write

$$\begin{aligned}\hat{P}^{(m_0)}_{-2}=\Big\{&\text{polynomials of the form }\sum_{t\ge 4\hat{m}-1}a_t z_n^t\overline{z}_n^{m_0-t}+\sum_{t\ge 2\hat{m}-1}b_t z_n^{2t+1}\overline{z}_n^{m_0-2t-3}|z_1|^2\\ &+2\Re\big(b_{4\hat{m}-3}z_n^{4\hat{m}-3}\overline{z}_n^{2\hat{m}-1}|z_1|^2\big)+\sum_{I'\neq 0,0\le 2t\le m_0-|I'|}C_{I't}(z')^{I'}z_n^{m-|I'|-2t}|z_n|^{2t}\Big\}.\end{aligned}$$

To get the normalization condition (4.21) and (4.24), we only need to prove that

$$\Im\big(B^{(m_0)}(z,q(z,\overline{z}))\big)\big|_{\hat{P}^{(m_0)}_{-2}}=Q^{(m_0)}(z,\overline{z}) \tag{4.53}$$

is solvable for any real valued formal power series $Q^{(m_0)}(z,\overline{z})\in\hat{P}^{(m_0)}_{-2}$.

The dimension of

$$\sum_{t\ge 4\hat{m}-1}a_t z_n^t\overline{z}_n^{m_0-t}+\sum_{t\ge 2\hat{m}-1}b_t z_n^{2t+1}\overline{z}_n^{m_0-2t-3}|z_1|^2+2\Re\big(b_{4\hat{m}-3}z_n^{4\hat{m}-3}\overline{z}_n^{2\hat{m}-1}|z_1|^2\big)$$

is

$$2\big(6\hat{m}-2-(4\hat{m}-2)\big)+2\Big(\frac{6\hat{m}-6}{2}-(2\hat{m}-2)\Big)+1=6\hat{m}-1=2\Big[\frac{6\hat{m}-1}{2}\Big]+1. \tag{4.54}$$

Combining this with (4.31) and (4.32), we know that $B^{(m_0)}(z,\overline{z})$ and $\hat{P}^{(m_0)}_{-2}$ have the same dimension. Hence to prove (4.53), now we only need to prove that $B^{(m_0)}\equiv 0$ if

$$\Im\big(B^{(m_0)}(z,q(z,\overline{z}))\big)\big|_{\hat{P}^{(m_0)}_{-2}}=0,\ \Re\big(b_{0(3\hat{m}-1)}\big)=0. \tag{4.55}$$

By (4.39), the condition (4.24) means that

$$\begin{aligned}&\sum_{k+2l=2t-1}I_{kl}=\sum_{\substack{k+2l=m_0-2t+1\\ k+l\le m_0/2}}J_{kl},\ \sum_{k+2l=2t}kI_{kl}=\sum_{\substack{k+2l=m_0-2t\\ k+l\le m_0/2}}kJ_{kl},\\ &\sum_{k+2l=2t}I_{kl}=\sum_{\substack{k+2l=m_0-2t\\ k+l\le m_0/2}}J_{kl}\text{ for }1\le t\le \hat{m}-1,\end{aligned} \tag{4.56}$$

and

$$\sum_{k+2l=2\hat{m}-1} I_{kl} = \sum_{\substack{k+2l=m_0-2\hat{m}+1 \\ k+l\le m_0/2}} J_{kl}, \quad \sum_{k+2l=2\hat{m}} \Re\big(kI_{kl}\big) = \sum_{\substack{k+2l=m_0-2\hat{m} \\ k+l\le m_0/2}} \Re\big(kJ_{kl}\big). \tag{4.57}$$

Next we prove by induction that the following hods for $1 \le t \le \hat{m}-1$:

$$\begin{aligned} &I_{j0} = \tbinom{3\hat{m}-1}{j}\lambda_n^{3\hat{m}-1-j} J_{3\hat{m}-1,0} \text{ for } 1 \le j \le 2t, \\ &J_{j0} = \tbinom{3\hat{m}-1}{j}\lambda_n^{3\hat{m}-1-j} J_{3\hat{m}-1,0} \text{ for } 3\hat{m}-1-t \le j \le 3\hat{m}-2. \end{aligned} \tag{4.58}$$

Setting $t=1$ in (4.56), we get

$$I_{10} = J_{1,3\hat{m}-2}, \ 2I_{2,0} = 2J_{2,3\hat{m}-3}, \ I_{20}+I_{01} = J_{2,3\hat{m}-3} + J_{0,3\hat{m}-2}.$$

Hence we obtain

$$\begin{aligned} I_{10} = (3\hat{m}-1)\lambda_n^{3\hat{m}-2} J_{3\hat{m}-1,0}, \ I_{20} = \tbinom{3\hat{m}-1}{2}\lambda_n^{3\hat{m}-3} J_{3\hat{m}-1,0}, \\ J_{3\hat{m}-2,0} = \lambda_n^{-3\hat{m}+2} I_{01} = \lambda_n^{-3\hat{m}+2}\lambda_n(3\hat{m}-1)\lambda_n^{3\hat{m}-2} J_{3\hat{m}-1,0} = \tbinom{3\hat{m}-1}{3\hat{m}-2}\lambda_n J_{3\hat{m}-1,0}. \end{aligned} \tag{4.59}$$

This proves (4.58) for $t=1$.

Suppose that (4.58) holds for some $t \le t_0 \in [1, \hat{m}-2]$. Next we will prove it also holds for $t = t_0+1$.

By our assumption, we get, for $k+l \le 2t_0$ and $l \le t_0$, the following

$$\begin{aligned} I_{kl} &= \tbinom{k+l}{k}\lambda_n^l I_{k+l,0} = \tbinom{k+l}{k}\lambda_n^l \cdot \tbinom{3\hat{m}-1}{k+l}\lambda_n^{3\hat{m}-1-k-l} J_{3\hat{m}-1,0} \\ &= \tbinom{3\hat{m}-1}{k}\tbinom{3\hat{m}-1-k}{l}\lambda_n^{3\hat{m}-1-k} J_{3\hat{m}-1,0}, \\ J_{k,3\hat{m}-1-k-l} &= \tbinom{3\hat{m}-1-l}{k}\lambda_n^{3\hat{m}-1-k-l} J_{3\hat{m}-1-l,0} = \tbinom{3\hat{m}-1-l}{k}\lambda_n^{3\hat{m}-1-k-l} \cdot \tbinom{3\hat{m}-1}{3\hat{m}-1-l}\lambda_n^l J_{3\hat{m}-1,0} \\ &= \tbinom{3\hat{m}-1}{k}\tbinom{3\hat{m}-1-k}{l}\lambda_n^{3\hat{m}-1-k} J_{3\hat{m}-1,0}. \end{aligned}$$

Hence we get, for $k+l \le 2t_0$ and $l \le t_0$, the following

$$I_{kl} = J_{k,3\hat{m}-1-k-l}. \tag{4.60}$$

Setting $t = t_0+1$ in (4.56) and making use of (4.60), we obtain

$$\begin{aligned} &I_{2t_0+1,0} = J_{2t_0+1,3\hat{m}-2t_0-2}, \\ &(2t_0+2)I_{2t_0+2,0} + 2t_0 I_{2t_0,1} = (2t_0+2)J_{2t_0+2,3\hat{m}-2t_0-3} + 2t_0 J_{2t_0,3\hat{m}-2t_0-2}, \\ &I_{2t_0+2,0} + I_{2t_0,1} + I_{0,t_0+1} = J_{2t_0+2,3\hat{m}-2t_0-3} + J_{2t_0,3\hat{m}-2t_0-2} + J_{0,3\hat{m}-t_0-2}. \end{aligned} \tag{4.61}$$

From the first equation, we get

$$I_{2t_0+1,0} = \tbinom{3\hat{m}-1}{2t_0+1}\lambda_n^{3\hat{m}-2t_0-2} J_{3\hat{m}-1,0}.$$

Then (4.60) holds for $k+l = 2t_0+1, l \leq t_0$. Namely, we obtain $I_{2t_0,1} = J_{2t_0,3\hat{m}-2t_0-2}$. Hence we have

$$\begin{aligned} &I_{2t_0+2,0} = J_{2t_0+2,3\hat{m}-2t_0-3} = \binom{3\hat{m}-1}{2t_0+2}\lambda_n^{3\hat{m}-2t_0-3} J_{3\hat{m}-1,0}, \\ &J_{3\hat{m}-t_0-2,0} = \lambda_n^{-3\hat{m}+t_0+2} J_{0,3\hat{m}-t_0-2} = \lambda_n^{-3\hat{m}+t_0+2} I_{0,t_0+1} = \binom{3\hat{m}-1}{3\hat{m}-t_0-2}\lambda_n^{t_0+1} J_{3\hat{m}-1,0}. \end{aligned} \tag{4.62}$$

This proves (4.58) for $t = t_0+1$. Hence we get

$$\begin{aligned} I_{j0} &= \binom{3\hat{m}-1}{j}\lambda_n^{3\hat{m}-1-j} I_{3\hat{m}-1,0} \text{ for } 1 \leq j \leq 2\hat{m}-2, \\ J_{j0} &= \binom{3\hat{m}-1}{j}\lambda_n^{3\hat{m}-1-j} J_{3\hat{m}-1,0} \text{ for } 2\hat{m} \leq j \leq 3\hat{m}-2. \end{aligned} \tag{4.63}$$

Notice that now (4.60) holds for $k+l \leq 2\hat{m}-2, l \leq \hat{m}-1$. Substituting these relations to (4.57) and making use of (4.60) for $k+l \leq 2\hat{m}-2, l \leq \hat{m}-1$, we get

$$\begin{aligned} &I_{2\hat{m}-1,0} = J_{2\hat{m}-1,\hat{m}}, \\ &\Re\big(2\hat{m} I_{2\hat{m},0} + (2\hat{m}-2) I_{2\hat{m}-2,1}\big) = \Re\big(2\hat{m} J_{2\hat{m},\hat{m}-1} + (2\hat{m}-2) J_{2\hat{m}-2,\hat{m}}\big). \end{aligned}$$

From (4.63) and the first equation above, we get

$$I_{2\hat{m}-1,0} = \binom{3\hat{m}-1}{2\hat{m}-1}\lambda^{\hat{m}} J_{3\hat{m}-1,0}. \tag{4.64}$$

Combining this with (4.36), we obtain $I_{2\hat{m}-2,1} = J_{2\hat{m}-2,\hat{m}}$. Thus we obtain

$$\Re\big(I_{2\hat{m},0} - 2\hat{m} J_{2\hat{m},\hat{m}-1}\big) = 0.$$

Since $b_{(0,3\hat{m}-1)}$ is purely imaginary, we know $I_{3\hat{m}-1,0} = \frac{1}{2i} b_{(0,3\hat{m}-1)}$ is real. Hence

$$I_{2\hat{m},0} = -\overline{J_{2\hat{m},0}} = -\binom{3\hat{m}-1}{2\hat{m}}\lambda^{\hat{m}-1} J_{3\hat{m}-1,0}, \ \ J_{2\hat{m},\hat{m}-1} = \binom{3\hat{m}-1}{\hat{m}-1}\lambda^{\hat{m}-1} J_{3\hat{m}-1,0}.$$

Thus we obtain $J_{3\hat{m}-1,0} = 0$. Combining this with (4.63) and (4.64), we get $I_{k,0} = 0$ for $1 \leq k \leq 3\hat{m}-1$. By (4.35), we know that

$$I_{k0} = \frac{1}{2i} b_{(0(m_0-2k)k)} + \mathcal{F}\{(b_{(0(m_0-2t)t)})_{t>k}\}. \tag{4.65}$$

In particular, we have $b_{(00(3\hat{m}-1))} = 2i I_{3\hat{m}-1,0}$. Hence we inductively get $b_{(0(m_0-2k)k)} = 0$.

By a similar induction argument as that used in the (II_{-3}) case, we get $b_{(Ij)} = 0$. Hence we obtain $B^{(m_0)}(z,\overline{z}) = 0$.

The cases (II_{-1}) and (II_1) can be similarly done as for (II_{-3}), while the cases (II_0) and (II_2) can be similarly done as in the case (II_{-2}). This completes the proof of Theorem 4.2. □

5 Proof of Theorem 3.1— Part II

We continue our proof of Theorem 3.1. In this part, we assume that M is non-minimal at its CR points near the origin. We will prove $H \equiv 0$ when it satisfies the normalization in Theorem 4.2.

The crucial step is to prove the following proposition, which is more or less the content of Theorem 3.1 when $n = 3$:

Proposition 5.1. *Suppose that $\lambda_n \neq 1/2$. Then for $t, r, s \geq 0$ with $t + r + s \leq m$, we have*

$$H_{(te_n+re_1,(m-t-r-s)e_n+se_1)} = 0. \tag{5.1}$$

Proof. The proof of Proposition 5.1 is carried out in three steps, according to $\lambda_1 = \lambda_n = 0$ or $\lambda_n = 0, \lambda_1 \neq 0$ or $\lambda_n \neq 0, \lambda_1 \neq 0$. We notice that when $\lambda_n \neq 0$, it must hold that $\lambda_1 \neq 0$ by our choice of λ_n.

Step I: In this case, we assume that $\lambda_n = \lambda_1 = 0$. Then (3.8) has the following form:

$$\overline{z}_n \Psi_1 = \overline{z_1} \Psi_n. \tag{5.2}$$

By considering the coefficients of $z_n^t z^{s-1} \overline{z_n}^{r+1} \overline{z_1}^h$ for $t \geq 0$, $s \geq 1$, $r \geq 0$ and $h = m+1-t-s-r \geq 0$ in (5.2), we get

$$s\Psi_{[tsrh]} = (t+1)\Psi_{[(t+1)(s-1)(r+1)(h-1)]}. \tag{5.3}$$

Setting $h = 0$ in (5.3), we get $\Psi_{[tsr0]} = 0$ for $s \geq 1$. Combining this with (5.3), we inductively get $\Psi_{[tsrh]} = 0$ for $s \geq h + 1$. Together with (4.4), we obtain:

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)h]} = (t-1)\Phi_{[tsr(h-1)]} \text{ for } s \geq h+1. \tag{5.4}$$

Setting $h = 0$ in (5.4), we get $\Phi_{[tsr0]} = 0$ for $s \geq 2$. Combining this with (5.4), we inductively get $\Phi_{[tsrh]} = 0$ for $s \geq h + 2$. Together with (4.3), we get

$$(h+1)H_{[(t-1)sr(h+1)]} = (r+1)H_{[t(s-1)(r+1)h]} \text{ for } s \geq h+2. \tag{5.5}$$

Setting $t = 0$, we get $H_{[0srh]} = 0$ for $s \geq h + 1,\ r \geq 1$. Then we inductively get $H_{[tsrh]} = 0$ for $s \geq h + 1,\ r \geq t + 1$. When $s \geq h + 1, r \leq t$, from (5.5), we inductively get $H_{[tsrh]} = \mathcal{F}\{(H_{[t's'r'0]})_{t' \geq r'}\}$, which is 0 by our normalization in (4.21). Thus we have proved

$$H_{[tsrh]} = 0 \text{ for } s \geq h+1. \tag{5.6}$$

Next we will prove that $H_{[tsrs]} = 0$. Setting $s = h \geq 1$, $t \geq 0$ and $r = -1$ in (5.3), we get $\Psi_{[ts0s]} = 0$ for $t \geq 1$. Substituting it back to (5.3), we inductively get

$$\Psi_{[tsrs]} = 0 \text{ for } t \geq r+1.$$

Substituting (4.4) into this equation, we get

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)s]} = (t-1)\Phi_{[tsr(s-1)]} \text{ for } t \geq r+1. \tag{5.7}$$

Setting $s = 0$, we get $\Phi_{[t1r0]} = 0$ for $t \geq r+1$. Substituting this back to (5.7), we get $\Phi_{[t(s+1)rs]} = 0$ for $t \geq r+1$. Together with (4.3), we get

$$(s+1)H_{[(t-1)(s+1)r(s+1)]} = (r+1)H_{[ts(r+1)s]} \text{ for } t \geq r+1.$$

Notice that $H_{[t0r0]} = 0$ by our normalization. Hence we inductively get

$$H_{[tsrs]} = 0 \text{ for } t \geq r. \tag{5.8}$$

Since $H_{[tsrh]} = \overline{H_{[rhts]}}$, (5.6) and (5.8) imply (5.1) for the case $\lambda_n = \lambda_1 = 0$.

Step II: In this step, we assume that $\lambda_n = 0$ and $\lambda_1 \neq 0$. Proposition 5.1 is an immediate consequence of the following lemma:

Lemma 5.2. *Suppose that $\lambda_n = 0$ and $\lambda_1 \neq 0$. Assume that there exists an $h_0 \geq -1$ such that*

$$\Psi_{[tsrh]} = \Phi_{[tsrh]} = 0 \textit{ for } h \leq h_0, \ H_{[tsrh]} = 0 \textit{ for } \max(s,h) \leq h_0 + 1. \tag{5.9}$$

Then we have

$$\Psi_{[tsrh]} = \Phi_{[tsrh]} = 0 \textit{ for } h \leq h_0 + 1, \ H_{[tsrh]} = 0 \textit{ for } \max(s,h) \leq h_0 + 2. \tag{5.10}$$

Once we have Lemma 5.2 at our disposal, since (5.9) holds for $h_0 = -1$ by our normalization, hence (5.10) holds for $h_0 = -1$. Then by an induction, we see that (5.10) holds for all $h_0 \leq m-2$. This will complete the proof of Proposition 5.1 in this setting.

Proof of Lemma 5.2. Setting $\xi = 0$ in (4.5) and making use of the assumptions in Lemma 5.2, we get:

$$\begin{aligned} &s\Psi_{[(t-1)s(r+1)(h_0+1)]} + (s-2)\eta\Psi_{[t(s-2)(r+2)(h_0+1)]} - t\eta\Psi_{[t(s-2)(r+2)(h_0+1)]} \\ &- (t+1)\eta^2\Psi_{[(t+1)(s-4)(r+3)(h_0+1)]} - \eta\Psi_{[t(s-2)(r+2)(h_0+1)]} = 0. \end{aligned}$$

Namely, we have

$$s\Psi_{[(t-1)s(r+1)(h_0+1)]} = (t+3-s)\eta\Psi_{[t(s-2)(r+2)(h_0+1)]} + (t+1)\eta^2\Psi_{[(t+1)(s-4)(r+3)(h_0+1)]}. \tag{5.11}$$

By setting $r = -3$ in (5.11), we get $\Psi_{[ts0(h_0+1)]} = 0$ for $t \geq 1$. Substituting this back to (5.11), we inductively get that $\Psi_{[tsr(h_0+1)]} = 0$ for $t \geq r+1$. Combining this with (4.4) and (5.9), we obtain

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)(h_0+1)]} = (t-1)\eta\Phi_{[t(s-1)r(h_0+1)]} \text{ for } t \geq r+1. \tag{5.12}$$

Setting $r = 0$ in (5.12), we get $\Phi_{[ts0(h_0+1)]} = 0$ for $t \geq 2$. Hence we inductively get $\Phi_{[tsr(h_0+1)]} = 0$ for $t \geq r+2$. In particular, we have $\Phi_{[t0r(h_0+1)]} = 0$ for $t \geq r+2$. Combining this with (4.3), (5.9) and $\lambda_n = 0$, we get $(h_0+2)H_{[(t-1)0r(h_0+2)]} = 0$ for $t \geq r+2$. Namely, we obtain $H_{[t0r(h_0+2)]} = 0$ for $t \geq r+1$. Together with our normalization (4.21), we obtain:

$$H_{[t0r(h_0+2)]} = 0. \tag{5.13}$$

Case I of Step II: When $h_0 = -1$, setting $s = 0$ in (5.12), we get $\Phi_{[(t-1)1(r-1)0]} = 0$ for $t \geq r+1$. Together with (4.3) and (4.22), we get $H_{[(t-1)1r1]} = (r+1)H_{[t0(r+1)0]} = 0$ for $t \geq r+1$. Namely, we obtain $H_{[t1r1]} = 0$ for $t \geq r$. By the reality of H, we get

$$H_{[t1r1]} = 0. \tag{5.14}$$

From (4.3), (5.13) and (5.14), we obtain $\Phi_{[t0r0]} = \Phi_{[t1r0]} = 0$. Together with (4.4), we see that $\Psi_{[t0r0]} = 0$.

Setting $h = 0$ in (4.6) and making use of $\Psi_{[t0r0]} = 0$, we first get $\Psi_{[t1r0]} = \Psi_{[t2r0]} = 0$, then inductively get $\Psi_{[tsr0]} = 0$. Combining this with (4.4), we get

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)0]} = (t-1)\eta\Phi_{[t(s-1)r0]}, \tag{5.15}$$

Setting $s = 0$ in (5.15), we obtain $\Phi_{[t1r0]} = 0$. By an induction argument, we get $\Phi_{[tsr0]} = 0$. This proves (5.10) for the case $h_0 = -1$.

Case II of Step II: When $h_0 \geq 0$, from (4.8),(5.13) and (5.14), we inductively get $H_{[tsr(h_0+2)]} = 0$ for $s \leq h_0+2$. Combining this with (4.3) and (5.14), we get $\Phi_{[t0r(h_0+1)]} = \Phi_{[t1r(h_0+1)]} = 0$. Substituting this back to (4.4), we obtain $\Psi_{[t0r(h_0+1)]} = 0$. Together with (4.6), we inductively get $\Psi_{[tsr(h_0+1)]} = 0$. Combining this with (4.4), we obtain

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)(h_0+1)]} = (t-1)\eta\Phi_{[t(s-1)r(h_0+1)]}.$$

As in Case I, we inductively get $\Phi_{[tsr(h_0+1)]} = 0$. This proves (5.10) for the case $h_0 \geq 0$ and thus completes the proof of Lemma 5.2. □

Step III: In this step, we assume that $\lambda_n \neq 0$ and $\lambda_1 \neq 0$. Similar to the situation in Step II, Proposition 5.1 in this setting follows from the following lemma:

Lemma 5.3. *Suppose that $\lambda_n \neq 0$ and $\lambda_1 \neq 0$. Then we have the following:*

(I)

$$H_{(te_n+e_1,(m-t-2)e_n+e_1)} = H_{(te_n,(m-t)e_n)} = 0. \tag{5.16}$$

(II) Assume that there exists an $h_0 \geq -1$ such that

$$\Psi_{[tsrh]} = \Phi_{[tsrh]} = 0 \text{ for } h \leq h_0,\ H_{[tsrh]} = 0 \text{ for } \max(s,h) \leq h_0+1. \tag{5.17}$$

Then we have

$$\Psi_{[tsrh]} = \Phi_{[tsrh]} = 0 \text{ for } h \leq h_0+1,\ H_{[tsrh]} = 0 \text{ for } \max(s,h) \leq h_0+2. \tag{5.18}$$

Before proving Lemma 5.3, we first make the needed preparations. For any fixed $s, h \geq 0$ and nonnegative integer k, set

$$\begin{aligned}\Psi_{[sh]}^{(k)} &= \sum\nolimits_{t=k}^{m+1-s-h} (-\xi)^{m+1-t-s-h} \tbinom{t}{k} \Psi_{[ts(m+1-t-s-h)h]},\\ \Phi_{[sh]}^{(k)} &= \sum\nolimits_{t=k}^{m-s-h} (-\xi)^{m-t-s-h} \tbinom{t}{k} \Phi_{[ts(m-t-s-h)h]},\\ H_{[sh]}^{(k)} &= \sum\nolimits_{t=k}^{m-s-h} (-\xi)^{m-t-s-h} \tbinom{t}{k} H_{[ts(m-t-s-h)h]}.\end{aligned} \tag{5.19}$$

Next we would like to transfer the relations among Ψ, Φ and H into the relations among $\Psi_{[s(h_0+1)]}^{(k)}$, $\Phi_{[s(h_0+1)]}^{(k)}$ and $H_{[s(h_0+2)]}^{(k)}$.

Lemma 5.4. *Assume that there exists an $h_0 \geq -1$ such that*

$$\Psi_{[tsrh]} = \Phi_{[tsrh]} = 0 \text{ for } h \leq h_0, \ H_{[tsrh]} = 0 \text{ for } \max(s,h) \leq h_0. \tag{5.20}$$

Then we have

$$\begin{aligned}\Phi_{[s(h_0+1)]}^{(k)} =& (h_0+2)\theta H_{[s(h_0+2)]}^{(k)} + (h_0+2) H_{[s(h_0+2)]}^{(k-1)}\\ &+ \frac{1}{\xi}\Big((m-s-h_0-k) H_{[(s-1)(h_0+1)]}^{(k)} - (k+1) H_{[(s-1)(h_0+1)]}^{(k+1)}\Big),\end{aligned} \tag{5.21}$$

$$\begin{aligned}\Psi_{[s(h_0+1)]}^{(k)} =& (s+1)\Big(\xi\theta\Phi_{[(s+1)(h_0+1)]}^{(k-1)} + \xi\Phi_{[(s+1)(h_0+1)]}^{(k-2)}\Big)\\ &- \eta(k+1)\theta\Phi_{[(s-1)(h_0+1)]}^{(k+1)} - \eta(k-1)\Phi_{[(s-1)(h_0+1)]}^{(k)}.\end{aligned} \tag{5.22}$$

Moreover, $\Psi_{[s(h_0+1)]}^{(k)}$ satisfies the following equation:

$$\begin{aligned}s\xi^2\theta\Psi_{[s(h_0+1)]}^{(k-2)} + s\xi^2\Psi_{[s(h_0+1)]}^{(k-3)} =& \xi\eta\Big\{(k-1)\theta\Psi_{[(s-2)(h_0+1)]}^{(k)} + (k+1-s)\Psi_{[(s-2)(h_0+1)]}^{(k-1)}\Big\}\\ &+ (k+1)\eta^2\Psi_{[(s-4)(h_0+1)]}^{(k+1)}.\end{aligned} \tag{5.23}$$

Proof of Lemma 5.4. Under the assumption in (5.20), we easily conclude that (4.3) and (4.4) have the following expressions:

$$\Phi_{[tsr(h_0+1)]} = (h_0+2)(\xi H_{[ts(r-1)(h_0+2)]} + H_{[(t-1)sr(h_0+2)]}) - (r+1) H_{[t(s-1)(r+1)(h_0+1)]}. \tag{5.24}$$

$$\begin{aligned}\Psi_{[tsr(h_0+1)]} =& (s+1)\big\{\xi\Phi_{[t(s+1)(r-2)(h_0+1)]} + (1+\xi^2)\Phi_{[(t-1)(s+1)(r-1)(h_0+1)]}\\ &+ \xi\Phi_{[(t-2)(s+1)r(h_0+1)]}\big\} - \big\{\xi\eta(t+1)\Phi_{[(t+1)(s-1)(r-1)(h_0+1)]}\\ &+ \eta(t-1)\Phi_{[t(s-1)r(h_0+1)]}\big\}.\end{aligned} \tag{5.25}$$

For fixed $s \geq 0$, a direct computation shows that

$$
\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0-1}(-\xi)^{m-t-s-h_0-1}\tbinom{t}{k}H_{[ts(m-t-s-h_0-2)(h_0+2)]}\\
=&(-\xi)\sum\nolimits_{t=k}^{m-s-h_0-1}(-\xi)^{m-t-s-h_0-2}\tbinom{t}{k}H_{[ts(m-t-s-h_0-2)(h_0+2)]}\\
=&-\xi H^{(k)}_{[s(h_0+2)]},\\
&\sum\nolimits_{t=k}^{m-s-h_0-1}(-\xi)^{m-t-s-h_0-1}\tbinom{t}{k}H_{[(t-1)s(m-t-s-h_0-1)(h_0+2)]}\\
=&\sum\nolimits_{t=k}^{m-s-h_0-1}(-\xi)^{m-t-s-h_0-1}\big(\tbinom{t-1}{k}+\tbinom{t-1}{k-1}\big)H_{[(t-1)s(m-t-s-h_0-1)(h_0+2)]}\\
=&H^{(k)}_{[s(h_0+2)]}+H^{(k-1)}_{[s(h_0+2)]}.
\end{aligned}
\tag{5.26}
$$

Notice that

$$
(m-t-s-h_0)\tbinom{t}{k}=\big((m-s-k-h_0)-(t-k)\big)\tbinom{t}{k}=(m-s-k-h_0)\tbinom{t}{k}-(k+1)\tbinom{t}{k+1}.
$$

Thus we have

$$
\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0-1}(-\xi)^{m-t-s-h_0-1}\tbinom{t}{k}(m-t-s-h_0)H_{[t(s-1)(m-t-s-h_0)(h_0+1)]}\\
=&\frac{1}{-\xi}\sum\nolimits_{t=k}^{m-s-h_0}(-\xi)^{m-t-s-h_0}\big((m-s-k-h_0)\tbinom{t}{k}-(k+1)\big)\tbinom{t}{k+1}\big)H_{[t(s-1)(m-t-s-h_0)(h_0+1)]}\\
=&\frac{1}{-\xi}\big((m-s-k-h_0)H^{(k)}_{[(s-1)(h_0+1)]}-(k+1)H^{(k+1)}_{[(s-1)(h_0+1)]}\big).
\end{aligned}
\tag{5.27}
$$

Substituting (5.26) and (5.27) into (5.24), we get (5.21).

Next we prove (5.22). A direct computation shows that

$$
\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0}(-\xi)^{m-t-s-h_0}\tbinom{t}{k}\big\{\xi\Phi_{[t(s+1)(m-t-s-h_0-2)(h_0+1)]}+(1+\xi^2)\Phi_{[(t-1)(s+1)(m-t-s-h_0-1)(h_0+1)]}\\
&\qquad+\xi\Phi_{[(t-2)(s+1)(m-t-s-h_0)(h_0+1)]}\big\}\\
=&\sum\nolimits_{t=k-2}^{m-s-h_0-2}(-\xi)^{m-t-s-h_0-2}\big\{\xi\cdot\xi^2\tbinom{t}{k}-\xi(1+\xi^2)\tbinom{t+1}{k}+\xi\tbinom{t+2}{k}\big\}\Phi_{[t(s+1)(m-t-s-h_0-2)(h_0+1)]}\\
=&\sum\nolimits_{t=k-2}^{m-s-h_0-2}(-\xi)^{m-t-s-h_0-2}\big\{\xi\theta\tbinom{t}{k-1}+\xi\tbinom{t}{k-2}\big\}\Phi_{[t(s+1)(m-t-s-h_0-2)(h_0+1)]}\\
=&\xi\theta\Phi^{(k-1)}_{[(s+1)(h_0+1)]}+\xi\Phi^{(k-2)}_{[(s+1)(h_0+1)]}.
\end{aligned}
\tag{5.28}
$$

We also have

$$\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0}(-\xi)^{m-t-s-h_0}\tbinom{t}{k}\big\{\xi(t+1)\Phi_{[(t+1)(s-1)(m-t-s-h_0-1)(h_0+1)]}\\
&\qquad\qquad+(t-1)\Phi_{[t(s-1)(m-t-s-h_0)(h_0+1)]}\big\}\\
=&-\xi^2\sum\nolimits_{t=k}^{m-s-h_0}(-\xi)^{m-t-s-h_0-1}(k+1)\tbinom{t+1}{k+1}\Phi_{[(t+1)(s-1)(m-t-s-h_0-1)(h_0+1)]}\\
&+\sum\nolimits_{t=k}^{m-s-h}(-\xi)^{m-t-s-h_0}\big\{(k+1)\tbinom{t}{k+1}+(k-1)\tbinom{t}{k}\big\}\Phi_{[t(s-1)(m-t-s-h_0)(h_0+1)]}\\
=&-\xi^2(k+1)\Phi^{(k+1)}_{[(s-1)(h_0+1)]}+(k+1)\Phi^{(k+1)}_{[(s-1)(h_0+1)]}+(k-1)\Phi^{(k)}_{[(s-1)(h_0+1)]}\\
=&\theta(k+1)\Phi^{(k+1)}_{[(s-1)(h_0+1)]}+(k-1)\Phi^{(k)}_{[(s-1)(h_0+1)]}.
\end{aligned}\tag{5.29}$$

Substituting (5.28) and (5.29) into (5.25), we get (5.22).

Now we turn to the proof of (5.23). Under the assumption (5.20), (4.5) has the following form:

$$\begin{aligned}
&s\big\{\xi\Psi_{[tsr(h_0+1)]}+(2\xi^2+1)\Psi_{[(t-1)s(r+1)(h_0+1)]}+(\xi^3+2\xi)\Psi_{[(t-2)s(r+2)(h_0+1)]}+\xi^2\Psi_{[(t-3)s(r+3)(h_0+1)]}\big\}\\
&+\eta\big\{(s-2)\Psi_{[t(s-2)(r+2)(h_0+1)]}+\xi(s-2)\Psi_{[(t-1)(s-2)(r+3)(h_0+1)]}\big\}\\
&-\eta\big\{(t+1)\xi\Psi_{[(t+1)(s-2)(r+1)(h_0+1)]}+t(1+\xi^2)\Psi_{[t(s-2)(r+2)(h_0+1)]}+(t-1)\xi\Psi_{[(t-1)(s-2)(r+3)(h_0+1)]}\big\}\\
&-(t+1)\eta^2\Psi_{[(t+1)(s-4)(r+3)(h_0+1)]}-\eta\theta\Psi_{[t(s-2)(r+2)(h_0+1)]}=0.
\end{aligned}\tag{5.30}$$

Notice that

$$\begin{aligned}
&(-\xi)^3\xi\tbinom{t}{k}+(-\xi)^2(2\xi^2+1)\tbinom{t+1}{k}+(-\xi)(\xi^3+2\xi)\tbinom{t+2}{k}+\xi^2\tbinom{t+3}{k}\\
=&\xi^2\big\{\tbinom{t+3}{k}-2\tbinom{t+2}{k}+\tbinom{t+1}{k}\big\}-\xi^4\big\{\tbinom{t+2}{k}-2\tbinom{t+1}{k}+\tbinom{t}{k}\big\}\\
=&\xi^2\big\{\tbinom{t}{k-2}+\tbinom{t}{k-3}\big\}-\xi^4\tbinom{t}{k-2}=\xi^2\theta\tbinom{t}{k-2}+\xi^2\tbinom{t}{k-3}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0+3}(-\xi)^{m-t-s-h_0+3}\tbinom{t}{k}\big\{\xi\Psi_{[ts(m-t-s-h_0)(h_0+1)]}+(2\xi^2+1)\Psi_{[(t-1)s(m-t-s-h_0+1)(h_0+1)]}\\
&+(\xi^3+2\xi)\Psi_{[(t-2)s(m-t-s-h_0+2)(h_0+1)]}+\xi^2\Psi_{[(t-3)s(m-t-s-h_0+3)(h_0+1)]}\big\}\\
=&\sum\nolimits_{t=k-3}^{m-s-h_0}(-\xi)^{m-t-s-h_0}\big\{(-\xi)^3\xi\tbinom{t}{k}+(-\xi)^2(2\xi^2+1)\tbinom{t+1}{k}\\
&+(-\xi)(\xi^3+2\xi)\tbinom{t+2}{k}+\xi^2\tbinom{t+3}{k}\big\}\Psi_{[ts(m-t-s-h_0)(h_0+1)]}\\
=&\sum\nolimits_{t=k-3}^{m-s-h_0}(-\xi)^{m-t-s-h_0}\big\{\xi^2\theta\tbinom{t}{k-2}+\xi^2\tbinom{t}{k-3}\big\}\Psi_{[ts(m-t-s-h_0)(h_0+1)]}\\
=&\xi^2\theta\Psi^{(k-2)}_{[s(h_0+1)]}+\xi^2\Psi^{(k-3)}_{[s(h_0+1)]}.
\end{aligned}\tag{5.31}$$

A direct computation shows that

$$\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0+3}(-\xi)^{m-t-s-h_0+3}\tbinom{t}{k}\Big\{\Psi_{[t(s-2)(m-t-s-h_0+2)(h_0+1)]}+\xi\Psi_{[(t-1)(s-2)(m-t-s-h_0-3)(h_0+1)]}\Big\}\\
=&\sum\nolimits_{t=k-1}^{m-s-h_0+2}(-\xi)^{m-t-s-h_0+2}\Big\{-\xi\tbinom{t}{k}+\xi\tbinom{t+1}{k}\Big\}\Psi_{[t(s-2)(m-t-s-h_0+2)(h_0+1)]}\\
=&\sum\nolimits_{t=k-1}^{m-s-h_0+2}(-\xi)^{m-t-s-h_0+2}\xi\tbinom{t}{k-1}\Psi_{[t(s-2)(m-t-s-h_0+2)(h_0+1)]}\\
=&\xi\Psi^{(k-1)}_{[(s-2)(h_0+1)]}.
\end{aligned}\tag{5.32}$$

We also obtain the following formulas:

$$\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0+3}(-\xi)^{m-t-s-h_0+3}\tbinom{t}{k}\Big\{(t+1)\xi\Psi_{[(t+1)(s-2)(m-t-s-h_0+1)(h_0+1)]}\\
&+t(1+\xi^2)\Psi_{[t(s-2)(m-t-s-h_0+2)(h_0+1)]}+(t-1)\xi\Psi_{[(t-1)(s-2)(m-t-s-h_0+3)(h_0+1)]}\Big\}\\
=&\sum\nolimits_{t=k-1}^{m-s-h_0+2}(-\xi)^{m-t-s-h_0+2}\Big\{(-\xi)^2t\xi\tbinom{t-1}{k}-\xi(1+\xi^2)t\tbinom{t}{k}+t\xi\tbinom{t+1}{k}\Big\}\Psi_{[t(s-2)(m-t-s-h_0+2)(h_0+1)]}\\
=&\sum\nolimits_{t=k-1}^{m-s-h_0+2}(-\xi)^{m-t-s-h_0+2}\Big\{k\xi\theta\tbinom{t}{k}+(k-1)\xi\tbinom{t}{k-1}\Big\}\Psi_{[t(s-2)(m-t-s-h_0+2)(h_0+1)]}\\
=&k\xi\theta\Psi^{(k)}_{[(s-2)(h_0+1)]}+(k-1)\xi\Psi^{(k-1)}_{[(s-2)(h_0+1)]}.
\end{aligned}\tag{5.33}$$

$$\begin{aligned}
&\sum\nolimits_{t=k}^{m-s-h_0+3}(-\xi)^{m-t-s-h_0+3}\tbinom{t}{k}(t+1)\Psi_{[(t+1)(s-4)(m-t-s-h_0+3)(h_0+1)]}=(k+1)\Psi^{[(k+1)}_{[(s-4)(h_0+1)]},\\
&\sum\nolimits_{t=k}^{m-s-h_0+3}(-\xi)^{m-t-s-h_0+3}\tbinom{t}{k}\Psi_{[t(s-2)(m-t-s-h_0+2)(h_0+1)]}=-\xi\Psi^{(k)}_{[(s-2)(h_0+1)]}.
\end{aligned}\tag{5.34}$$

Combining (5.31)-(5.34) with (5.30), we obtain

$$\begin{aligned}
&s\xi^2\theta\Psi^{(k-2)}_{[s(h_0+1)]}+s\xi^2\Psi^{(k-3)}_{[s(h_0+1)]}+(s-2)\xi\eta\Psi^{(k-1)}_{[(s-2)(h_0+1)]}-\eta\big(k\xi\theta\Psi^{(k)}_{[(s-2)(h_0+1)]}+(k-1)\xi\Psi^{(k-1)}_{[(s-2)(h_0+1)]}\big)\\
&-(k+1)\eta^2\Psi^{(k+1)}_{[(s-4)(h_0+1)]}+\xi\eta\theta\Psi^{(k)}_{[(s-2)(h_0+1)]}=0.
\end{aligned}$$

This finishes the proof of (5.23). $\square$

Now we are in a position to prove Lemma 5.3.

Proof of Lemma 5.3. (I) Setting $h_0=-1$, $k=0$ and $h_0=-1$, $k=1$ in (5.23), respectively, we get

$$\xi\eta(-1)\theta\Psi^{(0)}_{[s-2,0]}+\eta^2\Psi^{(1)}_{[s-4,0]}=0,\ \ \xi\eta(2-s)\Psi^{(0)}_{[s-2,0]}+2\eta^2\Psi^{(2)}_{[s-4,0]}=0.$$

Namely, we have

$$\xi\theta\Psi^{(0)}_{[s-2,0]}=\eta\Psi^{(1)}_{[s-4,0]},\ \ (s-2)\xi\Psi^{(0)}_{[s-2,0]}=2\eta\Psi^{(2)}_{[s-4,0]}.\tag{5.35}$$

Next we prove that for $2s \leq m+2$, $2k \leq m+2-2s$, we have

$$2s\xi\theta\Psi^{(2k)}_{[2s,0]} = (2s+2k)\eta\Psi^{(2k+1)}_{[2s-2,0]},\ 2s\xi\Psi^{(2k)}_{[2s,0]} = (2k+2)\eta\Psi^{(2k+2)}_{[2s-2,0]}. \tag{5.36}$$

Notice that $\Psi^{(k)}_{[ij]} = 0$ for $i+j+k \geq m+2$. Hence all the terms in (5.36) are 0 when $2k > m+2-2s$. Thus (5.36) holds for all $k \geq 0$. Also notice that (5.36) implies

$$(2s+2k)\Psi^{(2k+1)}_{[2s-2,0]} = (2k+2)\theta\Psi^{(2k+2)}_{[2s-2,0]}. \tag{5.37}$$

We prove (5.36) by induction on s.

If $m = 2\hat{m}$ then the largest possible s is $s = \hat{m}+1$. In this case, (5.36) is has only one nontrivial equation: $\Psi^{(1)}_{[2\hat{m},0]} = 0$. This can be got by setting $s = 2\hat{m}+4$ and $k = 0$ in the first equation of (5.35).

If $m = 2\widetilde{m}+1$, then the largest possible s is $s = \widetilde{m}+1$. In this case, we must have $k = 0$. Hence (5.36) is the same as (5.35).

Suppose that (5.36) holds for $s \geq s_0 (\geq 2)$. Since (5.36) holds for $s = s_0 - 1$ and $k = 0$, we can further suppose that (5.36) holds for $s = s_0 - 1$ and $k \leq k_0$. Next we will prove (5.36) for $s = s_0 - 1$ and $k = k_0 + 1$.

Setting $h_0 = -1$, $s = 2s_0$, $k = 2k_0 + 2$ in (5.23), we get

$$\begin{aligned}&2s_0\xi^2\theta\Psi^{(2k_0)}_{[2s_0,0]} + 2s_0\xi^2\Psi^{(2k_0-1)}_{[2s_0,0]}\\ =&\xi\eta\big\{(2k_0+1)\theta\Psi^{(2k_0+2)}_{[2s_0-2,0]} + (2k_0+3-2s_0)\Psi^{(2k_0+1)}_{[2s_0-2,0]}\big\} + \eta^2(2k_0+3)\Psi^{(2k_0+3)}_{[2s_0-4,0]}.\end{aligned} \tag{5.38}$$

By our assumption, (5.37) holds for $s = s_0+1,\ k = k_0 - 1$ and $s = s_0,\ k = k_0$, respectively. Hence we get:

$$(2s_0+2k_0)\Psi^{(2k_0-1)}_{[2s_0,0]} = 2k_0\theta\Psi^{(2k_0)}_{[2s_0,0]},\ (2s_0+2k_0)\Psi^{(2k_0+1)}_{[2s_0-2,0]} = (2k_0+2)\theta\Psi^{(2k_0+2)}_{[2s_0-2,0]}. \tag{5.39}$$

By (5.36) with $s = s_0, k = k_0$ and (5.39), we obtain

$$\begin{aligned}2s_0\xi^2\theta\Psi^{(2k_0)}_{[2s_0,0]} + 2s_0\xi^2\Psi^{(2k_0-1)}_{[2s_0,0]} =&\big(2s_0\xi^2\theta + 2s_0\xi^2\frac{2k_0}{2s_0+2k_0}\theta\big)\Psi^{(2k_0)}_{[2s_0,0]}\\ =&\big(1+\frac{2k_0}{2s_0+2k_0}\big)2s_0\xi^2\theta\Psi^{(2k_0)}_{[2s_0,0]}\\ =&\big(1+\frac{2k_0}{2s_0+2k_0}\big)(2k_0+2)\xi\eta\theta\Psi^{(2k_0+2)}_{[2s_0-2,0]}.\end{aligned} \tag{5.40}$$

By (5.39), we have

$$\begin{aligned}&\xi\eta\big\{(2k_0+1)\theta\Psi^{(2k_0+2)}_{[2s_0-2,0]} + (2k_0+3-2s_0)\Psi^{(2k_0+1)}_{[2s_0-2,0]}\big\} + \eta^2(2k_0+3)B^{(2k_0+3)}_{[2s_0-4,0]}\\ =&\xi\eta\big\{2k_0+1+(2k_0+3-2s_0)\frac{2k_0+2}{2s_0+2k_0}\big\}\theta\Psi^{(2k_0+2)}_{[2s_0-2,0]} + (2k_0+3)\eta^2\Psi^{(2k_0+3)}_{[2s_0-4,0]}.\end{aligned} \tag{5.41}$$

Substituting (5.40)-(5.41) back into (5.38), we get

$$\begin{aligned}\eta^2(2k_0+3)\Psi^{(2k_0+3)}_{[2s_0-4,0]} &= \big(1+\frac{2k_0+2}{2s_0+2k_0}(2s_0-3)\big)\xi\eta\theta\Psi^{(2k_0+2)}_{[2s_0-2,0]}\\ &= \frac{(2k_0+3)(2s_0-2)}{2s_0+2k_0}\xi\eta\theta\Psi^{(2k_0+2)}_{[2s_0-2,0]}\end{aligned}$$

This is just the first equation of (5.36).

Setting $h_0=-1$, $s=2s_0$ and $k=2k_0+3$ in (5.23), we get

$$\begin{aligned}&2s_0\xi^2\theta\Psi^{(2k_0+1)}_{[2s_0,0]}+2s_0\xi^2\Psi^{(2k_0)}_{[2s_0,0]}\\ =&\xi\eta\big\{(2k_0+2)\theta\Psi^{(2k_0+3)}_{[2s_0-2,0]}+(2k_0+4-2s_0)\Psi^{(2k_0+2)}_{[2s_0-2,0]}\big\}+\eta^2(2k_0+4)\Psi^{(2k_0+4)}_{[2s_0-4,0]}.\end{aligned} \tag{5.42}$$

By our assumption, (the second equation in) (5.36) holds for $s=s_0$, $k=k_0+1$ and for $s=s_0$, $k=k_0$. Namely, we have

$$2s_0\xi\Psi^{(2k_0+2)}_{[2s_0,0]}=(2k_0+4)\eta\Psi^{(2k_0+4)}_{[2s_0-2,0]},\ 2s_0\xi\Psi^{(2k_0)}_{[2s_0,0]}=(2k_0+2)\eta\Psi^{(2k_0+2)}_{[2s_0-2,0]}. \tag{5.43}$$

By our assumption, (5.37) holds for $s=s_0+1,\ k=k_0$ and $s=s_0,\ k=k_0+1$, respectively. Hence we get:

$$(2s_0+2k_0+2)\Psi^{(2k_0+1)}_{[2s_0,0]}=(2k_0+2)\theta\Psi^{(2k_0+2)}_{[2s_0,0]},\ (2s_0+2k_0+2)\Psi^{(2k_0+3)}_{[2s_0-2,0]}=(2k_0+4)\theta\Psi^{(2k_0+4)}_{[2s_0-2,0]}.$$

Combining this with (5.43), we get

$$\begin{aligned}2s_0\xi^2\Psi^{(2k_0+1)}_{[2s_0,0]} &= \frac{2s_0}{2s_0+2k_0+2}(2k_0+2)\xi^2\theta\Psi^{(2k_0+2)}_{[2s_0,0]}\\ &= \frac{2k_0+2}{2s_0+2k_0+2}\xi\theta(2k_0+4)\eta\Psi^{(2k_0+4)}_{[2s_0-2,0]}\\ &= (2k_0+2)\xi\eta\Psi^{(2k_0+3)}_{[2s_0-2,0]}.\end{aligned} \tag{5.44}$$

Substituting (5.43) and (5.44) into (5.42), we get

$$\xi(2s_0-2)\Psi^{(2k_0+2)}_{[2s_0-2,0]}=\eta(2k_0+4)\Psi^{(2k_0+4)}_{[2s_0-4,0]}.$$

This completes the proof of (5.36).

Setting $s=1$ in (5.37), we get

$$\theta\Psi^{(2k+2)}_{[0,0]}=\Psi^{(2k+1)}_{[0,0]}\text{ for }0\le 2k\le m. \tag{5.45}$$

By (5.22), we have $\Psi^{(k)}_{[0,0]}=\xi\theta\Phi^{(k-1)}_{[1,0]}+\xi\Phi^{(k-2)}_{[1,0]}$. Hence (5.45) is equivalent to

$$\theta\big(\xi\theta\Phi^{(2k+1)}_{[1,0]}+\xi\Phi^{(2k)}_{[1,0]}\big)=\xi\theta\Phi^{(2k)}_{[1,0]}+\xi\Phi^{(2k-1)}_{[1,0]}\text{ for }0\le 2k\le m. \tag{5.46}$$

Namely, we have $\theta^2\Phi^{(2k+1)}_{[1,0]} = \Phi^{(2k-1)}_{[1,0]}$. Setting $k=0$ in this equation, we get $\Phi^{(1)}_{[1,0]} = 0$. Thus we inductively get $\Phi^{(2k-1)}_{[1,0]} = 0$. Together with (5.21), we obtain

$$(-\xi)\big(\theta H^{(2k-1)}_{[11]} + H^{(2k-2)}_{[11]}\big) - (m+1-2k)H^{(2k-1)}_{[00]} + 2kH^{(2k)}_{[00]} = 0. \tag{5.47}$$

(II$_{-3}$): In this case, we have $m = 6\hat{m}-3$.

First, we prove by induction that

$$H_{[t1(m-t-2)1]} = 0 \text{ for} t \geq 4\hat{m}-3. \tag{5.48}$$

In fact, from (4.23), we get

$$H^{(t)}_{[00]} = 0 \text{ for } t \geq 4\hat{m}-1 \text{ and } H^{(6\hat{m}-5)}_{[11]} = 0. \tag{5.49}$$

Setting $2k = 6\hat{m}-4$ in (5.47), we get $H^{(6\hat{m}-6)}_{[11]} = 0$. Together with $H_{[(6\hat{m}-5)101]} = 0$, we obtain $H_{[(6\hat{m}-6)111]} = 0$.

Suppose that we have obtained $H^{(2t)}_{[11]} = 0$ for $t \geq t_0 (\geq 2\hat{m})$. Then $H^{(2t_0-1)}_{[11]} = 0$. Setting $2k = 2t_0$ in (5.47), we get $H^{(2t_0-2)}_{[11]} = 0$. Since $H_{[t1(m-t-2)1]} = 0$ for $t \geq 2t_0-1$. Hence we obtain $H_{[(2t_0-2)1(m-2t_0)1]} = 0$. This completes the proof of (5.48).

Now (5.47) takes the following form

$$\begin{aligned}
&(-\xi)\sum_{2\hat{m}-1\leq t\leq 4\hat{m}-4}\Big\{\theta\tbinom{t}{2k-1}(-\xi)^{m-t-2}H_{[t1(m-t-2)1]} + \tbinom{t}{2k-2}(-\xi)^{m-t-2}H_{[t1(m-t-2)1]}\Big\}\\
&-\sum_{2\hat{m}-1\leq t\leq 4\hat{m}-2}\Big\{(m+1-2k)\tbinom{t}{2k-1}(-\xi)^{m-t}H_{[t0(m-t)0]} - 2k\tbinom{t}{2k}(-\xi)^{m-t}H_{[t0(m-t)0]}\Big\} = 0.
\end{aligned} \tag{5.50}$$

Notice that

$$\begin{aligned}
&\theta\tbinom{t}{2k-1} + \tbinom{t}{2k-2} = \tbinom{t+1}{2k-1} - \tbinom{t}{2k-1}\xi^2,\\
&(m+1-2k)\tbinom{t}{2k-1} - 2k\tbinom{t}{2k} = (m+1-2k)\tbinom{t}{2k-1} - (t-2k+1)\tbinom{t}{2k-1} = (m-t)\tbinom{t}{2k-1}.
\end{aligned}$$

Hence (5.50) takes the following form

$$\begin{aligned}
&(-\xi)\sum_{2\hat{m}-1\leq t\leq 4\hat{m}-4}\Big\{\Big(\tbinom{t+1}{2k-1} - \tbinom{t}{2k-1}\xi^2\Big)(-\xi)^{m-t-2}H_{[t1(m-t-2)1]}\Big\}\\
&-\sum_{2\hat{m}-1\leq t\leq 4\hat{m}-2}\Big\{(m-t)\tbinom{t}{2k-1}(-\xi)^{m-t}H_{[t0(m-t)0]}\Big\} = 0.
\end{aligned} \tag{5.51}$$

Recall that $H_{IJ}=\overline{H_{JI}}$. By considering the real part and the imaginary part in (5.51), respectively, we obtain:

$$
\begin{aligned}
&(-\xi)\sum_{3\hat m-2\le t\le 4\hat m-4}\Big(\tbinom{t+1}{2k-1}-\tbinom{t}{2k-1}\xi^2\pm(-\xi)^{2t+2-m}\big(\tbinom{m-t-1}{2k-1}-\tbinom{m-t-2}{2k-1}\xi^2\big)\Big)(-\xi)^{m-t-2}\hat H^{\pm}_{[t1]}\\
&-\sum_{3\hat m-1\le t\le 4\hat m-2}\Big((m-t)\tbinom{t}{2k-1}\pm(-\xi)^{2t-m}t\tbinom{m-t}{2k-1}\Big)(-\xi)^{m-t}\hat H^{\pm}_{[t0]}=0.
\end{aligned}
\tag{5.52}
$$

Here we write

$$
\begin{aligned}
\hat H^{+}_{[t1]}&=\Re(H_{[t1(m-t-2)1]}),\ \hat H^{-}_{[t1]}=\Im(H_{[t1(m-t-2)1]}),\\
\hat H^{+}_{[t0]}&=\Re(H_{[t0(m-t)0]}),\ \hat H^{-}_{[t0]}=\Im(H_{[t0(m-t)0]}).
\end{aligned}
\tag{5.53}
$$

Then to prove that $\hat H^{\pm}_{[t1]}=\hat H^{\pm}_{[t0]}=0$, we only need to prove that the matrices $(R^{\pm}_{ij})_{1\le i,j\le 2\hat m-1}$ are invertible, where $R^{\pm}_{ij}$ are defined as follows:

$$
R^{\pm}_{ij}(\xi)=\begin{cases}
\tbinom{4\hat m-2-j}{2i-1}-\tbinom{4\hat m-3-j}{2i-1}\xi^2\pm(-\xi)^{2\hat m-1-2j}\big(\tbinom{2\hat m-1+j}{2i-1}-\tbinom{2\hat m-2+j}{2i-1}\xi^2\big)\\
\qquad\qquad \text{for } 1\le j\le \hat m-1,\\
(\hat m-1+j)\tbinom{5\hat m-2-j}{2i-1}\pm(-\xi)^{4\hat m-1-2j}(5\hat m-2-j)\tbinom{\hat m-1+j}{2i-1}\\
\qquad\qquad \text{for } \hat m\le j\le 2\hat m-1.
\end{cases}
\tag{5.54}
$$

This will be done in Lemma 6.2 of the next section. The proof for the Case (II_{-3}) is complete.

($\mathbf{II}_{-2}$): In this case, we have $m=6\hat m-2$. First, we prove by induction that

$$
H_{t1(m-t-2)1}=0 \text{ for } t\ge 4\hat m-2. \tag{5.55}
$$

In fact, from (4.24), we get

$$
H^{(t)}_{[00]}=0 \text{ for } t\ge 4\hat m-1. \tag{5.56}
$$

Setting $2k=6\hat m-2$ in (5.47) and noticing that $H^{(6\hat m-3)}_{[11]}=0$, we get $H^{(6\hat m-4)}_{[11]}=0$, which gives that $H_{[(6\hat m-4)101]}=0$.

Suppose that we know that $H^{(2t)}_{[11]}=0$ for $t\ge t_0(\ge 2\hat m)$. Then $H^{(2t_0-1)}_{[11]}=0$. Setting $2k=2t_0$ in (5.47), we get $H^{(2t_0-2)}_{[11]}=0$. Since $H_{[t1(m-t-2)1]}=0$ for $t\ge 2t_0-1$. Hence we obtain $H_{[(2t_0-2)1(m-2t_0)1]}=0$. This proves (5.55).

Now (5.47) takes the form:

$$
\begin{aligned}
&(-\xi)\sum_{2\hat m-1\le t\le 4\hat m-3}\Big\{\theta\tbinom{t}{2k-1}(-\xi)^{m-t-2}H_{[t1(m-t-2)1]}+\tbinom{t}{2k-2}(-\xi)^{m-t-2}H_{[t1(m-t-2)1]}\Big\}\\
&-\sum_{2\hat m\le t\le 4\hat m-2}\Big\{(m+1-2k)\tbinom{t}{2k-1}(-\xi)^{m-t}H_{[t0(m-t)0]}-2k\tbinom{t}{2k}(-\xi)^{m-t}H_{[t0(m-t)0]}\Big\}=0.
\end{aligned}
\tag{5.57}
$$

As for (5.50), it takes the form:

$$
\begin{aligned}
&(-\xi) \sum_{2\hat{m}-1\le t\le 4\hat{m}-3} \Big\{ \big(\tbinom{t+1}{2k-1} - \tbinom{t}{2k-1}\xi^2\big)(-\xi)^{m-t-2} H_{[t1(m-t-2)1]} \Big\} \\
&- \sum_{2\hat{m}\le t\le 4\hat{m}-2} \Big\{ (m-t)\tbinom{t}{2k-1}(-\xi)^{m-t} H_{[t0(m-t)0]} \Big\} = 0.
\end{aligned} \tag{5.58}
$$

Recall that $H_{IJ} = \overline{H_{JI}}$. By considering the real part and imaginary part in (5.58), respectively, we obtain:

$$
\begin{aligned}
&(-\xi) \sum_{3\hat{m}-1\le t\le 4\hat{m}-4} \Big(\tbinom{t+1}{2k-1} - \tbinom{t}{2k-1}\xi^2 + (-\xi)^{2t+2-m}\big(\tbinom{m-t-1}{2k-1} - \tbinom{m-t-2}{2k-1}\xi^2\big)\Big)(-\xi)^{m-t-2}\hat{H}^+_{[t1]} \\
&+ (-\xi)(-\xi)^{m-(3\hat{m}-2)-2}\big(\tbinom{3\hat{m}-1}{2k-1} - \tbinom{3\hat{m}-2}{2k-1}\xi^2\big)\hat{H}^+_{[(3\hat{m}-2)1]} \\
&- \sum_{3\hat{m}\le t\le 4\hat{m}-2} \Big((m-t)\tbinom{t}{2k-1} + (-\xi)^{2t-m} t \tbinom{m-t}{2k-1}\Big)(-\xi)^{m-t}\hat{H}^+_{[t0]} \\
&- (3\hat{m}-1)\tbinom{3\hat{m}-1}{2t-1}(-\xi)^{3\hat{m}-1}\hat{H}^+_{[(3\hat{m}-1)0]} = 0
\end{aligned} \tag{5.59}
$$

and

$$
\begin{aligned}
&(-\xi) \sum_{3\hat{m}-1\le t\le 4\hat{m}-3} \Big(\tbinom{t+1}{2k-1} - \tbinom{t}{2k-1}\xi^2 - (-\xi)^{2t+2-m}\big(\tbinom{m-t-1}{2k-1} - \tbinom{m-t-2}{2k-1}\xi^2\big)\Big)(-\xi)^{m-t-2}\hat{H}^-_{[t1]} \\
&- \sum_{3\hat{m}\le t\le 4\hat{m}-2} \Big((m-t)\tbinom{t}{2k-1} - (-\xi)^{2t-m} t \tbinom{m-t}{2k-1}\Big)(-\xi)^{m-t}\hat{H}^-_{[t0]} = 0.
\end{aligned} \tag{5.60}
$$

To prove that $\hat{H}_{[t1]} = \hat{H}_{[t0]} = 0$, we only need to prove that the matrices $(N_{ij})_{1\le i,j\le 2\hat{m}-1}$ $(T_{ij})_{1\le i,j\le 2\hat{m}-2}$ are nonsingular, where N_{ij} and T_{ij} are defined by:

$$
\begin{aligned}
N_{ij} &= \begin{cases} \tbinom{4\hat{m}-2-j}{2i-1} - \tbinom{4\hat{m}-3-j}{2i-1}\xi^2 + \xi^{2\hat{m}-2-2j}\big(\tbinom{2\hat{m}+j}{2i-1} - \tbinom{2\hat{m}-1+j}{2i-1}\xi^2\big) \text{ for } 1\le j\le \hat{m}-2, \\ \tbinom{3\hat{m}-1}{2i-1} - \tbinom{3\hat{m}-2}{2i-1}\xi^2 \text{ for } j=\hat{m}-1, \\ (\hat{m}+j)\tbinom{5\hat{m}-2-j}{2i-1} + \xi^{4\hat{m}-2-2j}(5\hat{m}-2-j)\tbinom{\hat{m}+j}{2i-1} \text{ for } \hat{m}\le j\le 2\hat{m}-2, \\ (3\hat{m}-1)\tbinom{3\hat{m}-1}{2i-1} \text{ for } j=2\hat{m}-1, \end{cases} \\
T_{ij} &= \begin{cases} \tbinom{4\hat{m}-1-j}{2i+1} - \tbinom{4\hat{m}-2-j}{2i+1}\xi^2 - \xi^{2\hat{m}-2j}\big(\tbinom{2\hat{m}-1+j}{2i+1} - \tbinom{2\hat{m}-2+j}{2i+1}\xi^2\big) \text{ for } 1\le j\le \hat{m}-1, \\ (\hat{m}+j)\tbinom{5\hat{m}-2-j}{2i+1} - \xi^{4\hat{m}-2-2j}(5\hat{m}-2-j)\tbinom{\hat{m}+j}{2i+1} \text{ for } \hat{m}\le j\le 2\hat{m}-2. \end{cases}
\end{aligned} \tag{5.61}
$$

These will be the content of Lemma 6.3 of the next section. Hence we complete the proof of Lemma 5.3 for the case (II_{-2}).

Case(II$_{-1}$) and Case (II$_1$) can be done in a similar way as that for Case (II$_{-3}$), while Case (II$_0$) and case (II$_2$) can be done in a similar way as that for Case (II$_{-2}$). Hence we see the proof of Part (I) of Lemma 5.3.

Now we turn to the proof of Part (II) of Lemma 5.3. From (4.5), we can conclude that

$$\Psi_{[t(2s+1)r(h_0+1)]} = \mathcal{F}\{(\Psi_{[t'(2s'+1)r'(h_0+1)]})_{s'<s}, (\Psi_{[t''s''r''h'']})_{h''\leq h_0}\} \text{ for } s \geq 0. \tag{5.62}$$

In particular, we obtain

$$\Psi_{[t1r(h_0+1)]} = \mathcal{F}\{(\Psi_{[t's'r'h']})_{h'\leq h_0}\} = 0.$$

The last equality follows from the assumptions in (5.17). By an induction argument, we obtain

$$\Psi_{[t(2s+1)r(h_0+1)]} = 0 \text{ for } 1 \leq 2s+1 \leq m - h_0. \tag{5.63}$$

Combining this with (5.22), we get

$$(2s+2)\xi\big(\theta\Phi^{(k-1)}_{[2s+2,h_0+1]} + \Phi^{(k-2)}_{[2s+2,h_0+1]}\big) = \eta(k+1)\theta\Phi^{(k+1)}_{[2s,h_0+1]} + \eta(k-1)\Phi^{(k)}_{[2s,h_0+1]}. \tag{5.64}$$

Setting $k = 0$ in (5.64), we obtain

$$\theta\Phi^{(1)}_{[2s,h_0+1]} = \Phi^{(0)}_{[2s,h_0+1]} \text{ for } 0 \leq 2s \leq m - h_0 - 1. \tag{5.65}$$

Next we prove by induction that

$$\theta\Phi^{(2k+1)}_{[2s,h_0+1]} = \Phi^{(2k)}_{[2s,h_0+1]} \text{ for } 0 \leq 2s \leq m - h_0 - 1,\ 0 \leq 2k \leq m - 2s - h_0 - 1. \tag{5.66}$$

Notice that (5.66) also holds for $2k > m - 2s - h_0 - 1$, in which case, all the terms in (5.66) are 0.

When $m - h_0 - 1 = 2\hat{m}$ ($2\widetilde{m} + 1$, respectively), then the largest possible s is $s = \hat{m}$ ($\widetilde{m}$, respectively). In this case, $k = 0$, (5.66) reduces to (5.65).

Suppose that we already have (5.66) for $s \geq s_0$. By (5.65), we see that (5.66) also holds for $k = 0, s = s_0 - 1$. Hence we can suppose that (5.66) holds for $k \in [0, k_0]$, $s = s_0 - 1$, Next we will prove that (5.66) also holds for $s = s_0 - 1$, $k = k_0 + 1$.

Setting $s = s_0$, $k = 2k_0, 2k_0+1, 2k_0+2$ in (5.64), respectively, we get

$$2s_0\xi\big(\theta\Phi^{(2k_0-1)}_{[2s_0,h_0+1]} + \Phi^{(2k_0-2)}_{[2s_0,h_0+1]}\big) = \eta(2k_0+1)\theta\Phi^{(2k_0+1)}_{[2s_0-2,h_0+1]} + \eta(2k_0-1)\Phi^{(2k_0)}_{[2s_0-2,h_0+1]}, \tag{5.67}$$

$$2s_0\xi\big(\theta\Phi^{(2k_0)}_{[2s_0,h_0+1]} + \Phi^{(2k_0-1)}_{2s_0,h_0+1}\big) = \eta(2k_0+2)\theta\Phi^{(2k_0+2)}_{[2s_0-2,h_0+1]} + \eta 2k_0\Phi^{(2k_0+1)}_{[2s_0-2,h_0+1]}, \tag{5.68}$$

$$2s_0\xi\big(\theta\Phi^{(2k_0+1)}_{[2s_0,h_0+1]} + \Phi^{(2k_0)}_{[2s_0,h_0+1]}\big) = \eta(2k_0+3)\theta\Phi^{(2k_0+3)}_{[2s_0-2,h_0+1]} + \eta(2k_0+1)\Phi^{(2k_0+2)}_{[2s_0-2,h_0+1]}. \tag{5.69}$$

Since (5.66) holds for $s = s_0$, $k = k_0 - 1, k_0, k_0 + 1$, respectively, we conclude that the left hand side of (5.67) $- 2\theta$(5.68) $+ \theta^2$(5.69) can be written as follows:

$$2s_0\xi\big\{2\theta\Phi^{(2k_0-1)}_{[2s_0,h_0+1]} - 2\theta\big(\theta\Phi^{(2k_0)}_{[2s_0,h_0+1]} + \Phi^{(2k_0-1)}_{[2s_0,h_0+1]}\big) + 2\theta^2\Phi^{(2k_0)}_{[2s_0,h_0+1]}\big\} = 0. \tag{5.70}$$

Setting $s = s_0 - 1$, $k = k_0$ in (5.66), we get $\theta\Phi^{(2k_0+1)}_{[2s_0-2,h_0+1]} = \Phi^{(2k_0)}_{[2s_0-2,h_0+1]}$. Thus we obtain:

$$(2k_0+1)\theta\Phi^{(2k_0+1)}_{[2s_0-2,h_0+1]} + (2k_0-1)\Phi^{(2k_0)}_{[2s_0-2,h_0+1]} - 2\theta\cdot 2k_0\Phi^{(2k_0+1)}_{[2s_0-2,h_0+1]} = 0. \tag{5.71}$$

Hence by calculating (5.67) $- 2\theta$(5.68) $+ \theta^2$(5.69) and making use of (5.70)-(5.71), we obtain:

$$\theta^2\big\{(2k_0+3)\theta\Phi^{(2k_0+3)}_{[2s_0-2,h_0+1]} + (2k_0+1)\Phi^{(2k_0+2)}_{[2s_0-2,h_0+1]}\big\} - 2\theta\cdot(2k_0+2)\theta\Phi^{(2k_0+2)}_{[2s_0-2,h_0+1]} = 0. \tag{5.72}$$

Thus we get

$$\theta\Phi^{(2k_0+3)}_{[2s_0-2,h_0+1]} = \Phi^{(2k_0+2)}_{[2s_0-2,h_0+1]}.$$

This completes the proof of (5.66).

Setting $s = 0$ in (5.66), we get

$$\theta\Phi^{(2k+1)}_{[0,h_0+1]} = \Phi^{(2k)}_{[0,h_0+1]} \text{ for } 0 \le 2k \le m - h_0 - 1. \tag{5.73}$$

Substituting (5.21) into (5.73) and making use of the assumptions in (5.9), we get

$$\theta(\theta H^{(2k+1)}_{[0,h_0+2]} + H^{(2k)}_{[0,h_0+2]}) = \theta H^{(2k)}_{[0,h_0+2]} + H^{(2k-1)}_{[0,h_0+2]}.$$

Hence we get

$$\theta^2 H^{(2k+1)}_{[0,h_0+2]} = H^{(2k-1)}_{[0,h_0+2]} \text{ for } 0 \le 2k \le m - h_0 - 1. \tag{5.74}$$

Setting $k = 0$ in (5.74), we get $H^{(1)}_{[0,h_0+2]} = 0$. By an induction, we get

$$H^{(2k+1)}_{[0,h_0+2]} = 0 \text{ for } 0 \le 2k \le m - h_0 - 3. \tag{5.75}$$

Next, we will use the just obtained (5.75) to show that $H_{[t0(m-t-h_0-2)(h_0+2)]} = 0$. We will proceed in terms of the even or odd property of $m - h_0 - 1$.

(1) In this case, we assume $m - h_0 - 1 = 2\hat{m}$. By $H^{(2\hat{m}-1)}_{[0(h_0+2)]} = 0$, we get $H_{[(2\hat{m}-1)00(h_0+2)]} = 0$. By our normalization (4.21), we have $H_{[t0r(h_0+2)]} = 0$ for $t \le \hat{m} - 1$. Hence (5.75) with $0 \le k \le \hat{m} - 2$ takes the following form:

$$\sum_{j=1}^{\hat{m}-1} S^{(2\hat{m}-2)}_{ij}(-\xi)^{j-1} H_{[(2\hat{m}-1-j)0j(h_0+2)]} = 0.$$

Here we have set

$$S = (S^{(2\hat{m}-2)}_{ij}) = \left(\binom{2\hat{m}-1-j}{2i-1}\right)_{1\le i,j\le \hat{m}-1}. \tag{5.76}$$

By Lemma 6.1, $S = (S^{(2\hat{m}-2)}_{ij})$ is nonsingular. Hence we have $H_{[(2\hat{m}-1-j)0j(h_0+2)]} = 0$.

(2) In this case, we assume $m - h_0 - 1 = 2\widetilde{m} + 1$. By our normalization (4.21), we have $H_{[t0r(h_0+2)]} = 0$ for $t \le \widetilde{m}$. Hence (5.75) has the following form:

$$\sum_{j=1}^{\widetilde{m}} S_{ij}(-\xi)^{j-1} H_{[(2\widetilde{m}+1-j)0(j-1)(h_0+2)]} = 0 \text{ for } 1 \le i \le \widetilde{m}.$$

Here $S = (S_{ij}) = \left(\binom{2\widetilde{m}+1-j}{2i-1}\right)_{1\le i,j\le \widetilde{m}}$. Now, by Lemma 6.1, we conclude that $H_{[t0r(h_0+2)]} = 0$.

Thus we got $H_{[t0r(h_0+2)]} = 0$. By (4.8), we get $H_{[tsr(h_0+2)]} = 0$ for $s \le h_0 + 2$. Combining this with (5.24), we get

$$\Phi_{[t0r(h_0+2)]} = \Phi_{[t1r(h_0+2)]} = 0. \tag{5.77}$$

Substituting this back to (4.4), we obtain $\Psi_{[t0r(h_0+2)]} = 0$. By (4.6), we inductively get $\Psi_{[tsr(h_0+2)]} = 0$. Combining (4.7) with (5.77), we inductively get $\Phi_{[tsr(h_0+2)]} = 0$. This proves (5.18) for the case $h_0 \ge 0$ and completes the proof of Lemma 5.3. This also finishes the proof of Proposition 5.1. □

Finally, we are in a position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. By an induction argument, we need only to that $H \equiv 0$.

When $n = 3$, Theorem 3.1 is the content of Proposition 5.1. Next we suppose that $n > 3$. We prove $H = 0$ by induction according to the order of z_n in H. By Lemma 4.1 and Proposition 5.1, we have $H_{(te_n+se_k, re_n+he_n)} = 0$ for $t + s = m$ or $m - 1$. Suppose that

$$H_{(te_n+I, re_n+J)} = 0 \text{ for } t + r \ge m_0 \ (m_0 \le m - 1). \tag{5.78}$$

Next we will prove that $H_{(te_n+I, re_n+J)} = 0$ for $t + r \ge m_0 - 1$. The terms of H can be divided into the following four types:

$$H_{(te_n, re_n+I)}, \quad H_{(te_n+se_1, re_n+he_1)}, \quad H_{(te_n+se_k, re_n+he_k)} \text{ with } s, h \ge 1, \quad H_{(te_n+e_j+I, re_n+e_k+J)}.$$

By Lemma 4.1 and Proposition 5.1, terms of the first two types are 0. $H_{(te_n+se_k, re_n+he_k)} = 0$ follows from (4.12) and (5.78), while $H_{(te_n+e_j+I, re_n+e_k+J)} = 0$ follows from (4.10) and (5.78). Thus we get $H \equiv 0$. This completes the proof of Theorem 3.1. □

6 Computation of determinants

In this section, we will prove that the matrices $S^{(m)}$, $R^{\pm(m)}$, $N^{(m)}$ and $T^{(m)}$ in the previous section are nonsingular when $\lambda_n \ne 0, 1/2$.

Lemma 6.1. *The matrices* $D^{(2\hat{m})} = \left(\binom{2\hat{m}-j}{2i-2}\right)_{1\le i,j\le \hat{m}}$ *and* $S^{(2\hat{m})} = \left(\binom{2\hat{m}+1-j}{2i-1}\right)_{1\le i,j\le \hat{m}}$ *are nonsingular.*

Proof of Lemma 6.1. Set

$$\begin{aligned} \widetilde{S}_{ij}^{(2\hat{m})} &= \frac{(2i-1)}{2\hat{m}-j+1} S_{ij}^{(2\hat{m})} \text{ for } 1 \le i \le \hat{m},\ 1 \le j \le \hat{m}, \\ \check{S}_{ij}^{(2\hat{m})} &= \widetilde{S}_{ij}^{(2\hat{m})} - \widetilde{S}_{i,j+1}^{(2\hat{m})} \text{ for } 1 \le i \le \hat{m},\ 1 \le j \le \hat{m}-1, \\ \check{S}_{i,\hat{m}}^{(2\hat{m})} &= \widetilde{S}_{i,\hat{m}}^{(2\hat{m})} \text{for } 1 \le i \le \check{m}. \end{aligned} \tag{6.1}$$

Then we have

$$\begin{aligned} \widetilde{S}_{ij}^{(2\hat{m})} &= \frac{2i-1}{2\hat{m}-j+1} \tbinom{2\hat{m}-j+1}{2i-1} = \tbinom{2\hat{m}-j}{2i-2} = D_{ij}^{(2\hat{m})} \text{ for } 1 \le i \le \hat{m}. \\ \check{S}_{1,j}^{(2\hat{m})} &= 0 \text{ for } 1 \le j \le \hat{m}-1,\ \check{S}_{1,\check{m}}^{(2\hat{m})} = 1, \\ \check{S}_{ij}^{(2\hat{m})} &= \tbinom{2\hat{m}-j}{2i-2} - \tbinom{2\hat{m}-j-1}{2i-2} = \tbinom{2\hat{m}-j-1}{2i-3} = S_{i-1,j}^{(2\hat{m}-2)} \text{ for } 2 \le i \le \hat{m}. \end{aligned} \tag{6.2}$$

For $1 \le i, j \le \hat{m}$, we write $\widetilde{S}_{ij}^{(2\hat{m})} = (\widetilde{S}_{ij}^{(2\hat{m})})$, $\check{S}_{ij}^{(2\hat{m})} = (\check{S}_{ij}^{(2\hat{m})})$. By (6.2), we obtain

$$\check{S}^{(2\hat{m})} = \begin{pmatrix} 0 & 1 \\ (S_{i-1,j}^{(2\hat{m}-2)})_{2 \le i \le \hat{m}, 1 \le j \le \hat{m}-1} & * \end{pmatrix}.$$

Hence we have

$$\det(\check{S}^{(2\hat{m})}) = (-1)^{\hat{m}+1} \det S^{(2\hat{m}-2)}. \tag{6.3}$$

By (6.1)-(6.3), we get

$$\begin{aligned} \det(S^{(2\hat{m})}) &= \prod_{i,j=1}^{\hat{m}} (2\hat{m}-j+1)(2i-1)^{-1} \cdot \det(\widetilde{S}^{(2\hat{m})}) \\ &= \prod_{i,j=1}^{\hat{m}} (2\hat{m}-j+1)(2i-1)^{-1} (-1)^{\hat{m}+1} \det(S^{(2\hat{m}-2)}). \end{aligned} \tag{6.4}$$

Notice that $\det(S^{(2)}) = \binom{2}{1} = 2$. Thus $S^{(2\hat{m})}$ is nonsingular. Combining this with the first equation in (6.2), we also conclude that $D^{(2\hat{m})}$ is nonsingular. □

Lemma 6.2. *Assume that $\xi \neq 0, \frac{1}{2}$. Then the matrices $R^{\pm}(\xi)$ defined by (5.54) are nonsingular.*

Proof. Set

$$R_{ij}^{[1]} = (2i-1) R_{ij}^{+} \text{ for } j \le \hat{m}-1;\ R_{ij}^{[1]} = \frac{2i-1}{(5\hat{m}-2-j)(\hat{m}-1+j)} R_{ij}^{+} \text{ for } \hat{m} \le j \le 2\hat{m}-1.$$

Notice that $(2i-1)\binom{t}{2i-1} = t\binom{t-1}{2i-2}$. Thus for $1 \le j \le \hat{m}-1$, we get

$$\begin{aligned} R_{ij}^{[1]} =& (2i-1)\tbinom{4\hat{m}-2-j}{2i-1} - (2i-1)\tbinom{4\hat{m}-3-j}{2i-1}\xi^2 \\ &+ (-\xi)^{2\hat{m}-1-2j} \big\{ (2i-1)\tbinom{2\hat{m}-1+j}{2i-1} - (2i-1)\tbinom{2\hat{m}-2+j}{2i-1} \big\} \\ =& (4\hat{m}-2-j)\tbinom{4\hat{m}-3-j}{2i-2} - (4\hat{m}-3-j)\tbinom{4\hat{m}-4-j}{2i-2}\xi^2 \\ &+ (-\xi)^{2\hat{m}-1-2j} \big\{ (2\hat{m}-1+j)\tbinom{2\hat{m}-2+j}{2i-2} - (2\hat{m}-2+j)\tbinom{2\hat{m}-3+j}{2i-2}\xi^2 \big\}. \end{aligned} \tag{6.5}$$

For $\hat{m} \le j \le 2\hat{m}-1$, we obtain:

$$\begin{aligned} R_{ij}^{[1]} =& \frac{2i-1}{5\hat{m}-2-j}\tbinom{5\hat{m}-2-j}{2i-1} + (-\xi)^{4\hat{m}-1-2j}\frac{2i-1}{\hat{m}-1+j}\tbinom{\hat{m}-1+j}{2i-1} \\ =& \tbinom{5\hat{m}-3-j}{2i-2} + (-\xi)^{4\hat{m}-1-2j}\tbinom{\hat{m}-2+j}{2i-2}. \end{aligned} \tag{6.6}$$

For $1 \le i \le 2\hat{m}-1$, write

$$\begin{aligned} R_{ij}^{[2]} =& R_{ij}^{[1]} - (4\hat{m}-2-j)R_{i,j+\hat{m}}^{[1]} + (4\hat{m}-3-j)R_{i,j+\hat{m}+1}^{[1]}\xi^2 \text{ for } 1 \le j \le \hat{m}-2, \\ R_{i,\hat{m}-1}^{[2]} =& R_{i,\hat{m}-1}^{[1]} - \big(3\hat{m}-1+(3\hat{m}-2)\xi\big)R_{i,2\hat{m}-1}^{[1]}, \\ R_{ij}^{[2]} =& (\hat{m}-1+j)R_{ij}^{[1]} + (4\hat{m}-3-2j)R_{i,j+1}^{[1]} \\ & - (5\hat{m}-4-j)R_{i,j+2}^{[1]}\xi^2 + R_{i,j-\hat{m}+1}^{[2]} \text{ for } \hat{m} \le j \le 2\hat{m}-3, \\ R_{i,2\hat{m}-2}^{[2]} =& (3\hat{m}-3)R_{i,2\hat{m}-2}^{[1]} + \big(1+(3\hat{m}-2)\xi\big)R_{i,2\hat{m}-1}^{[1]} + R_{i,\hat{m}-1}^{[2]},\ R_{i,2\hat{m}-1}^{[2]} = R_{i,2\hat{m}-1}^{[1]}. \end{aligned} \tag{6.7}$$

Then for $1 \le i \le 2\hat{m}-1$ and $1 \le j \le \hat{m}-2$, we have

$$\begin{aligned} R_{ij}^{[2]} =& (4\hat{m}-2-j)\tbinom{4\hat{m}-3-j}{2i-2} - (4\hat{m}-3-j)\tbinom{4\hat{m}-4-j}{2i-2}\xi^2 \\ & + (-\xi)^{2\hat{m}-1-2j}\big\{(2\hat{m}-1+j)\tbinom{2\hat{m}-2+j}{2i-2} - (2\hat{m}-2+j)\tbinom{2\hat{m}-3+j}{2i-2}\xi^2\big\} \\ & - (4\hat{m}-2-j)\tbinom{4\hat{m}-3-j}{2i-2} - (4\hat{m}-2-j)(-\xi)^{4\hat{m}-1-2(\hat{m}+j)}\tbinom{\hat{m}-2+\hat{m}+j}{2i-2} \\ & + (4\hat{m}-3-j)\tbinom{4\hat{m}-4-j}{2i-2}\xi^2 + (4\hat{m}-3-j)(-\xi)^{4\hat{m}-1-2(\hat{m}+j+1)}\tbinom{\hat{m}-2+\hat{m}+j+1}{2i-2}\xi^2 \\ =& (-\xi)^{2\hat{m}-1-2j}\Big\{(4\hat{m}-3-j)\tbinom{2\hat{m}-1+j}{2i-2} - (2\hat{m}-1-2j)\tbinom{2\hat{m}-2+j}{2i-2} \\ & - (2\hat{m}-2+j)\tbinom{2\hat{m}-3+j}{2i-2}\xi^2\Big\}. \end{aligned} \tag{6.8}$$

When $j = \hat{m}-1$, we obtain

$$\begin{aligned} R_{i,\hat{m}-1}^{[2]} =& (4\hat{m}-2-\hat{m}+1)\tbinom{4\hat{m}-3-\hat{m}+1}{2i-2} - (4\hat{m}-3-\hat{m}+1)\tbinom{4\hat{m}-4-\hat{m}+1}{2i-2}\xi^2 \\ & + (-\xi)^{2\hat{m}-1-2\hat{m}+2}\big\{(2\hat{m}-1+\hat{m}-1)\tbinom{2\hat{m}-2+\hat{m}-1}{2i-2} - (2\hat{m}-2+\hat{m}-1)\tbinom{2\hat{m}-3+\hat{m}-1}{2i-2}\xi^2\big\} \\ & - \big(3\hat{m}-1+(3\hat{m}-2)\xi\big)\big\{\tbinom{5\hat{m}-3-2\hat{m}+1}{2i-2} + (-\xi)^{4\hat{m}-1-4\hat{m}+2}\tbinom{\hat{m}-2+2\hat{m}-1}{2i-2}\big\} \\ =& (-\xi)\big\{(3\hat{m}-2)\tbinom{3\hat{m}-2}{2i-2} - \tbinom{3\hat{m}-3}{2i-2} - (3\hat{m}-3)\tbinom{3\hat{m}-4}{2i-2}\xi^2\big\}. \end{aligned} \tag{6.9}$$

For $1 \le i \le 2\hat{m}-1$ and $\hat{m} \le j \le 2\hat{m}-3$, we get

$$\begin{aligned} R_{ij}^{[2]} =& (\hat{m}-1+j)\big\{\tbinom{5\hat{m}-3-j}{2i-2} + (-\xi)^{4\hat{m}-1-2j}\tbinom{\hat{m}-2+j}{2i-2}\big\} \\ & + (4\hat{m}-3-2j)\cdot\big\{\tbinom{5\hat{m}-4-j}{2i-2} + (-\xi)^{4\hat{m}-3-2j}\tbinom{\hat{m}-1+j}{2i-2}\big\} \\ & - (5\hat{m}-4-j)\xi^2\big\{\tbinom{5\hat{m}-5-j}{2i-2} + (-\xi)^{4\hat{m}-5-2j}\tbinom{\hat{m}+j}{2i-2}\big\} \\ & + (-\xi)^{2\hat{m}-1-2j+2\hat{m}-2}\big\{(4\hat{m}-3-j+\hat{m}-1)\tbinom{2\hat{m}-1+j-\hat{m}+1}{2i-2} \\ & - (2\hat{m}-1-2j+2\hat{m}-2)\tbinom{2\hat{m}-2+j-\hat{m}+1}{2i-2} - (2\hat{m}-2+j-\hat{m}+1)\tbinom{2\hat{m}-3+j-\hat{m}+1}{2i-2}\big\} \\ =& (\hat{m}-1+j)\tbinom{5\hat{m}-3-j}{2i-2} + (4\hat{m}-3-2j)\tbinom{5\hat{m}-4-j}{2i-2} - (5\hat{m}-4-j)\tbinom{5\hat{m}-5-j}{2i-2}\xi^2. \end{aligned} \tag{6.10}$$

When $j = 2\hat{m} - 2$, we obtain

$$
\begin{aligned}
R^{[2]}_{i,2\hat{m}-2} =&(3\hat{m}-3)\left\{\tbinom{5\hat{m}-3-2\hat{m}+2}{2i-2} + (-\xi)^{4\hat{m}-1-4\hat{m}+4}\tbinom{\hat{m}-2+2\hat{m}-2}{2i-2}\right\} \\
&+ \big(1+(3\hat{m}-2)\xi\big)\left\{\tbinom{5\hat{m}-3-2\hat{m}+1}{2i-2} + (-\xi)^{4\hat{m}-1-4\hat{m}+2}\tbinom{\hat{m}-2+2\hat{m}-1}{2i-2}\right\} \\
&+ (-\xi)\left\{(3\hat{m}-2)\tbinom{3\hat{m}-2}{2i-2} - \tbinom{3\hat{m}-3}{2i-2} - (3\hat{m}-3)\tbinom{3\hat{m}-4}{2i-2}\xi^2\right\} \\
=&(3\hat{m}-3)\tbinom{3\hat{m}-1}{2i-2} + \tbinom{3\hat{m}-2}{2i-2} - (3\hat{m}-2)\tbinom{3\hat{m}-3}{2i-2}\xi^2.
\end{aligned} \tag{6.11}
$$

Set

$$
\begin{aligned}
&R^{[3]}_{ij} = R^{[2]}_{i,\hat{m}-1+j} \text{ for } 1 \le j \le \hat{m}-1, \\
&R^{[3]}_{ij} = \frac{-1}{\xi^{2j+1-2\hat{m}}} R^{[2]}_{i,2\hat{m}-1-j} \text{ for } \hat{m} \le j \le 2\hat{m}-2,\ R^{[3]}_{i,2\hat{m}-1} = R^{[2]}_{i,2\hat{m}-1}.
\end{aligned} \tag{6.12}
$$

Then for $1 \le i \le 2\hat{m}-1$, $1 \le j \le \hat{m}-1$, we get

$$
\begin{aligned}
R^{[3]}_{ij} =&(\hat{m}-1+\hat{m}-1+j)\tbinom{5\hat{m}-3-\hat{m}+1-j}{2i-2} + (4\hat{m}-3-2\hat{m}+2-2j)\tbinom{5\hat{m}-4-\hat{m}+1-j}{2i-2} \\
&- (5\hat{m}-4-\hat{m}+1-j)\tbinom{5\hat{m}-5-\hat{m}+1-j}{2i-2}\xi^2 \\
=&(2\hat{m}-2+j)\tbinom{4\hat{m}-2-j}{2i-2} + (2\hat{m}-1-2j)\tbinom{4\hat{m}-3-j}{2i-2} - (4\hat{m}-3-j)\tbinom{4\hat{m}-4-j}{2i-2}\xi^2.
\end{aligned} \tag{6.13}
$$

For $1 \le i \le 2\hat{m}-1$, $\hat{m} \le j \le 2\hat{m}-2$, we get

$$
\begin{aligned}
R^{[3]}_{ij} =&(4\hat{m}-3-2\hat{m}+1+j)\tbinom{2\hat{m}-1+2\hat{m}-1-j}{2i-2} - (2\hat{m}-1-4\hat{m}+2+2j)\tbinom{2\hat{m}-2+2\hat{m}-1-j}{2i-2} \\
&- (2\hat{m}-2+2\hat{m}-1-j)\tbinom{2\hat{m}-3+2\hat{m}-1-j}{2i-2}\xi^2 \\
=&(2\hat{m}-2+j)\tbinom{4\hat{m}-2-j}{2i-2} + (2\hat{m}-1-2j)\tbinom{4\hat{m}-3-j}{2i-2} - (4\hat{m}-3-j)\tbinom{4\hat{m}-4-j}{2i-2}\xi^2.
\end{aligned} \tag{6.14}
$$

Thus for $1 \le i \le 2\hat{m}-1$, $1 \le j \le 2\hat{m}-2$, we get

$$
\begin{aligned}
&R^{[3]}_{ij} = (2\hat{m}-2+j)\tbinom{4\hat{m}-2-j}{2i-2} + (2\hat{m}-1-2j)\tbinom{4\hat{m}-3-j}{2i-2} - (4\hat{m}-3-j)\tbinom{4\hat{m}-4-j}{2i-2}\xi^2, \\
&R^{[3]}_{i,\hat{m}-1} = \tbinom{3\hat{m}-2}{2i-2} - \tbinom{3\hat{m}-3}{2i-2}\xi.
\end{aligned} \tag{6.15}
$$

Thus we get

$$
\det(R^+) = C_1(-\xi)^{C_0} \det \begin{pmatrix} R^{[3]}_{11} & \cdots & R^{[3]}_{1,2\hat{m}-2} & \tbinom{3\hat{m}-2}{0} - \xi\tbinom{3\hat{m}-3}{0} \\ \vdots & \ddots & \vdots & \vdots \\ R^{[3]}_{2\hat{m}-1,1} & \cdots & R^{[3]}_{2\hat{m}-1,2\hat{m}-2} & \tbinom{3\hat{m}-2}{4\hat{m}-4} - \xi\tbinom{3\hat{m}-3}{4\hat{m}-4} \end{pmatrix}. \tag{6.16}
$$

Here $C_0 = \sum_{i=1}^{\hat{m}-1}(2i-1)$. Set

$$
R^{[4]}_{i,j} = \frac{1}{4\hat{m}-3-j} R^{[3]}_{ij}, R^{[4]}_{i,2\hat{m}-1} = R^{[3]}_{i,2\hat{m}-1} \text{ for } 1 \le i \le 2\hat{m}-1,\ 1 \le j \le 2\hat{m}-2.
$$

Then for $2 \le i \le 2\hat{m}-1$ and $1 \le j \le 2\hat{m}-2$, we have

$$\begin{aligned}
R^{[4]}_{i,j} =& \frac{2\hat{m}-2+j}{4\hat{m}-3-j}\left(\tbinom{4\hat{m}-2-j}{2i-2} - \tbinom{4\hat{m}-3-j}{2i-2}\right) + \tbinom{4\hat{m}-3-j}{2i-2} - \tbinom{4\hat{m}-4-j}{2i-2}\xi^2 \\
=& \frac{2\hat{m}-2+j}{4\hat{m}-3-j}\tbinom{4\hat{m}-3-j}{2i-3} + \tbinom{4\hat{m}-4-j}{2i-3} + \tbinom{4\hat{m}-4-j}{2i-2}\theta \\
=& \frac{2\hat{m}-2+j}{2i-3}\tbinom{4\hat{m}-4-j}{2i-4} + \frac{4\hat{m}-4-j-2i+4}{2i-3}\tbinom{4\hat{m}-4-j}{2i-4} + \tbinom{4\hat{m}-4-j}{2i-2}\theta \\
=& \frac{6\hat{m}-2-2i}{2i-3}\tbinom{4\hat{m}-4-j}{2i-4} + \tbinom{4\hat{m}-4-j}{2i-2}\theta.
\end{aligned} \tag{6.17}$$

Set

$$R^{[5]}_{2\hat{m}-1,j} = R^{[4]}_{2\hat{m}-1,j},\ R^{[5]}_{i,j} = R^{[4]}_{i,j} - \frac{(2i-1)\theta}{6\hat{m}-2i-4}R^{[5]}_{i+1,j} \text{ for } 1 \le i \le 2\hat{m}-2. \tag{6.18}$$

Then

$$R^{[5]}_{1,j} = 0, R^{[5]}_{i,j} = \frac{6\hat{m}-2-2i}{2i-3}\tbinom{4\hat{m}-4-j}{2i-4} \text{ for } 2 \le i \le 2\hat{m}-1,\ 1 \le j \le 2\hat{m}-2. \tag{6.19}$$

Hence we obtain

$$\begin{aligned}
\det(R^+) &= C_2\xi^{C_0}\det\begin{pmatrix} 0 & R^{[5]}_{1,2\hat{m}-1} \\ \left(\frac{6\hat{m}-2i-4}{2i-1}\tbinom{4\hat{m}-4-j}{2i-2}\right)_{1\le i,j\le 2\hat{m}-2} & * \end{pmatrix} \\
&= C_3\xi^{C_0}\det\left(\tbinom{4\hat{m}-4-j}{2i-2}_{1\le i,j\le 2\hat{m}-2}\right)R^{[4]}_{1,2\hat{m}-1}.
\end{aligned} \tag{6.20}$$

By Lemma 6.1, $\left(\tbinom{4\hat{m}-4-j}{2i-2}_{1\le i,j\le 2\hat{m}-2}\right)$ is nonsingular. Now we only need to prove that $R^{[5]}_{1,2\hat{m}-1} \ne 0$ for $\xi \ne 0, 1$, which follows from the following claim:

$$R^{[5]}_{1,2\hat{m}-1} = \alpha^{3\hat{m}-2} \text{ with } \alpha = (1-\xi)/2.$$

Notice that $\tbinom{3\hat{m}-2}{2k-2} = 0$ when $k \ge [\frac{3\hat{m}}{2}] + 1$. By (6.19), we inductively get

$$R^{[5]}_{1,2\hat{m}-1} = R^{[4]}_{1,2\hat{m}-1} + \sum_{k=2}^{[\frac{3\hat{m}}{2}]}\frac{(-1)^{k-1}\prod_{j=2}^{k}(2j-3)\theta^{k-1}}{\prod_{j=1}^{k-1}(6\hat{m}-4-2j)}R^{[4]}_{k,2\hat{m}-1}.$$

Recall that $\theta = 1 - \xi^2 = (1-\xi)(1+\xi) = 2^2\alpha(1-\alpha)$. Hence we have

$$\begin{aligned}
&\frac{(-1)^{k-1}\prod_{j=2}^{k}(2j-3)\theta^{k-1}}{\prod_{j=1}^{k-1}(6\hat{m}-4-2j)}R^{[4]}_{k,2\hat{m}-1}\\
=&\frac{(-1)^{k-1}2^{2k-2}\alpha^{k-1}(1-\alpha)^{k-1}}{2^{k-1}(k-1)!\binom{3\hat{m}-3}{k-1}}\frac{(2k-3)!}{2^{k-2}(k-2)!}\big(\tbinom{3\hat{m}-3}{2k-3}+2\alpha\tbinom{3\hat{m}-3}{2k-2}\big)\\
=&2\alpha^{k-1}(\alpha-1)^{k-1}\frac{\binom{2k-3}{k-1}}{\binom{3\hat{m}-3}{k-1}}\tbinom{3\hat{m}-3}{2k-3}\big(1+\frac{3\hat{m}-2k}{k-1}\alpha\big)\\
=&2(\alpha-1)^{k-1}\alpha^{k-1}\tbinom{3\hat{m}-2-k}{k-2}\big(1+\frac{3\hat{m}-2k}{k-1}\alpha\big)\\
=&2(\alpha-1)^{k-1}\alpha^{k-1}\big(\tbinom{3\hat{m}-2-k}{k-2}+\tbinom{3\hat{m}-2-k}{k-1}\alpha\big).
\end{aligned}\tag{6.21}$$

Hence we get

$$R^{[5]}_{1,2\hat{m}-1} = 2\alpha + \sum_{k=1}^{[\frac{3\hat{m}}{2}]-1} 2(\alpha-1)^k\alpha^k\big(\tbinom{3\hat{m}-3-k}{k-1}+\tbinom{3\hat{m}-3-k}{k}\alpha\big).$$

Next we prove by induction the following:

$$\begin{aligned}
&\alpha^{3\hat{m}-2}-\alpha-\sum_{k=1}^{k_0}(\alpha-1)^k\alpha^k\big(\tbinom{3\hat{m}-3-k}{k-1}+\tbinom{3\hat{m}-3-k}{k}\alpha\big)\\
&\qquad=(\alpha-1)^{k_0+1}\alpha^{k_0+1}\sum_{t=k_0}^{3\hat{m}-4-k_0}\tbinom{t}{k_0}\alpha^{3\hat{m}-4-k_0-t}.
\end{aligned}\tag{6.22}$$

Notice that $\alpha^{3\hat{m}-2}-\alpha = \alpha(\alpha-1)\cdot\sum_{i=0}^{3\hat{m}-4}\alpha^i$. This proves (6.22) for $k_0 = 0$. Suppose that (6.22) holds for k_0, then

$$\begin{aligned}
&\alpha^{3\hat{m}-2}-\alpha-\sum_{k=1}^{k_0+1}(\alpha-1)^k\alpha^k\big(\tbinom{3\hat{m}-3-k}{k-1}+\tbinom{3\hat{m}-3-k}{k}\alpha\big)\\
=&(\alpha-1)^{k_0+1}\alpha^{k_0+1}\Big\{\sum_{t=k_0}^{3\hat{m}-4-k_0}\tbinom{t}{k_0}\alpha^{3\hat{m}-4-k_0-t}-\big(\tbinom{3\hat{m}-4-k_0}{k_0}+\tbinom{3\hat{m}-4-k_0}{k_0+1}\alpha\big)\Big\}\\
=&(\alpha-1)^{k_0+1}\alpha^{k_0+1}\Big\{\sum_{t=k_0}^{3\hat{m}-5-k_0}\tbinom{t}{k_0}\alpha^{3\hat{m}-4-k_0-t}-\tbinom{3\hat{m}-4-k_0}{k_0+1}\alpha\Big\}.
\end{aligned}$$

Notice that $\binom{3\hat{m}-4-k_0}{k_0+1} = \sum\limits_{t=k_0}^{3\hat{m}-5-k_0} \binom{t}{k_0}$. Hence we have

$$
\begin{aligned}
&\alpha^{3\hat{m}-2} - \alpha - \sum_{k=1}^{k_0+1} (\alpha-1)^k \alpha^k \big(\tbinom{3\hat{m}-3-k}{k-1} + \tbinom{3\hat{m}-3-k}{k}\alpha\big) \\
=&(\alpha-1)^{k_0+1}\alpha^{k_0+1} \sum_{t=k_0}^{3\hat{m}-5-k_0} \tbinom{t}{k_0}\big(\alpha^{3\hat{m}-4-k_0-t} - \alpha\big) \\
=&(\alpha-1)^{k_0+1}\alpha^{k_0+1} \sum_{t=k_0}^{3\hat{m}-5-k_0} \tbinom{t}{k_0}(\alpha-1)\alpha \sum_{i=0}^{3\hat{m}-5-k_0-t} \alpha^i \\
=&(\alpha-1)^{k_0+2}\alpha^{k_0+2} \sum_{t=k_0+1}^{3\hat{m}-5-k_0} \tbinom{t}{k_0+1}\alpha^{3\hat{m}-5-k_0-t}.
\end{aligned}
\tag{6.23}
$$

This proves (6.22) for $k = k_0 + 1$ and completes the proof of (6.22). Setting $k_0 = [\frac{3\hat{m}}{2}]$, we obtain $R^{[5]}_{1,2\hat{m}-1} = 2\alpha^{3\hat{m}-2}$. Hence $R^{\pm} = C_1\xi^{C_0}(1 \mp \xi)^{3\hat{m}-2}$ for some $C_1 \neq 0$. This finishes the proof of Lemma 6.2. □

Lemma 6.3. *Assume that $\xi \neq 0$. Then the matrices N and T defined by (5.61) are nonsingular.*

Proof. For $1 \leq t \leq \hat{m} - 2$ and $\hat{m} \leq t' \leq 2\hat{m} - 2$, we set

$$
\begin{aligned}
N^{[1]}_{i,2\hat{m}-1} &= \frac{1}{3\hat{m}-1} N_{i,2\hat{m}-1} = \tbinom{3\hat{m}-1}{2i-1}, \\
N^{[1]}_{i,\hat{m}-1} &= -\frac{1}{\xi^2}\big(N_{i,\hat{m}-1} - N^{[1]}_{i,2\hat{m}-1}\big) = \tbinom{3\hat{m}-2}{2i-1}, \\
N^{[1]}_{it} &= -\frac{1}{\xi^{2(\hat{m}-t)}}\big(N_{it} - N^{[1]}_{i,t+\hat{m}} + N^{[1]}_{i,t+\hat{m}+1}\xi^2 - \xi^{2\hat{m}-2t-2}N^{[1]}_{i,t+1}\big) = \tbinom{2\hat{m}-1+t}{2i-1}, \\
N^{[1]}_{it'} &= -\frac{1}{t'+\hat{m}}\big(N_{it'} - (5\hat{m}-2-t')N^{[1]}_{i,t'-m+1}\xi^{4\hat{m}-2-2t'}\big) = \tbinom{5\hat{m}-2-t'}{2i-1}.
\end{aligned}
\tag{6.24}
$$

Set

$$
N^{[2]}_t = N^{[1]}_{t+\hat{m}-1} \text{ for } 1 \leq t \leq \hat{m},\ N^{[2]}_t = N^{[1]}_{2\hat{m}-t} \text{ for } \hat{m}+1 \leq t \leq 2\hat{m}-1. \tag{6.25}
$$

Then $N^{[2]}_{ij} = \binom{4\hat{m}-1-j}{2i-1}$. By Lemma 6.1, the matrix $\big(\binom{2(2\hat{m}-1)+1-t}{2i-1}\big)_{1\leq t\leq 2\hat{m}-1}$ is nonsingular.

Next we calculate the determination of the matrix T, which is done by a similar argument as that for $R^{\pm}$ (And the proof now, in fact, is much simpler). For the convenience of the reader, we include the following details.

Set

$$
T^{[1]}_{ij} = (2i+1)T_{ij} \text{ for } 1 \leq j \leq \hat{m}-1,\ T^{[1]}_{ij} = \frac{2i+1}{(5\hat{m}-2-j)(\hat{m}+j)} T_{ij} \text{ for } \hat{m} \leq j \leq 2\hat{m}-2.
$$

Corresponding to (6.5) and (6.6), we get

$$
\begin{aligned}
T_{ij}^{[1]} =& (4\hat{m}-1-j)\tbinom{4\hat{m}-2-j}{2i} - (4\hat{m}-2-j)\tbinom{4\hat{m}-3-j}{2i}\xi^2 \\
& - \xi^{2\hat{m}-2j}\big\{(2\hat{m}-1+j)\tbinom{2\hat{m}-2+j}{2i} - (2\hat{m}-2+j)\tbinom{2\hat{m}-3+j}{2i}\xi^2\big\} \text{ for } 1 \le j \le \hat{m}-1, \\
T_{ij}^{[1]} =& \tbinom{5\hat{m}-3-j}{2i} - (-\xi)^{4\hat{m}-2-2j}\tbinom{\hat{m}-1+j}{2i} \text{ for } \hat{m} \le j \le 2\hat{m}-2.
\end{aligned}
\tag{6.26}
$$

Set

$$
\begin{aligned}
T_{ij}^{[2]} =& \frac{1}{-\xi^{2\hat{m}-2j}}\big(T_{ij}^{[1]} - (4\hat{m}-1-j)T_{i,\hat{m}-1+j}^{[1]} + (4\hat{m}-j-2)T_{i,\hat{m}+j}^{[1]}\xi^2\big) \text{ for } 1 \le i \le \hat{m}-2, \\
T_{\hat{m}-1}^{[2]} =& \frac{1}{-\xi^2}\big(T_{i,\hat{m}-1}^{[1]} - 3\hat{m}T_{i,2\hat{m}-2}^{[1]}\big), \\
T_{ij}^{[2]} =& (j+\hat{m})T_{ij}^{[1]} + (4\hat{m}-2j-4)T_{j+1}^{[1]} - (5\hat{m}-j-4)T_{i,j+2}^{[1]}\xi^2 - \xi^{4\hat{m}-2j-4}T_{i,j-\hat{m}+2}^{[2]} \\
& \text{for } \hat{m} \le j \le 2\hat{m}-4, \\
T_{i,2\hat{m}-3}^{[2]} =& (3\hat{m}-3)T_{i,2\hat{m}-3}^{[1]} + 2T_{i,2\hat{m}-2}^{[1]},\ T_{i,2\hat{m}-2}^{[2]} = (3\hat{m}-2)T_{i,2\hat{m}-2}^{[1]}.
\end{aligned}
\tag{6.27}
$$

By exactly the same argument as that in (6.8)-(6.11), we get

$$
\begin{aligned}
T_{ij}^{[2]} =& (4\hat{m}-j-2)\tbinom{2\hat{m}-1+j}{2i} - (2\hat{m}-2j)\tbinom{2\hat{m}+j-2}{2i} - (2\hat{m}-2+j)\tbinom{2\hat{m}+j-3}{2i}\xi^2 \\
& \text{for } 1 \le i \le \hat{m}-1, \\
T_{i,\hat{m}-1}^{[2]} =& (3\hat{m}-1)\tbinom{3\hat{m}-2}{2i} - 2\tbinom{3\hat{m}-3}{2i} - (3\hat{m}-3)\tbinom{3\hat{m}-4}{2i}\xi^2, \\
T_{ij}^{[2]} =& (j+\hat{m})\tbinom{5\hat{m}-3-j}{2i} + (4\hat{m}-4-2j)\tbinom{5\hat{m}-4-j}{2i} - (5\hat{m}-4-j)\tbinom{5\hat{m}-5-j}{2i}\xi^2 \\
& \text{for } \hat{m} \le j \le 2\hat{m}-4, \\
T_{i,2\hat{m}-3}^{[2]} =& (3\hat{m}-3)\tbinom{3\hat{m}}{2i} + 2\tbinom{3\hat{m}-1}{2i} - (3\hat{m}-1)\tbinom{3\hat{m}-2}{2i}\xi^2, \\
T_{i,2\hat{m}-2}^{[2]} =& (3\hat{m}-2)\tbinom{3\hat{m}-1}{2i} - (3\hat{m}-2)\tbinom{3\hat{m}-3}{2i}\xi^2.
\end{aligned}
\tag{6.28}
$$

Set

$$
T_{ij}^{[3]} = T_{i,\hat{m}-1+j}^{[2]} \text{ for } 1 \le j \le \hat{m}-1,\ N_{ij}^{[3]} = N_{i,2\hat{m}-1-j}^{[2]} \text{ for } \hat{m} \le j \le 2\hat{m}-2. \tag{6.29}
$$

Then for $1 \le i \le 2\hat{m}-2$, corresponding to (6.15), we have

$$
T_{ij}^{[3]} = (2\hat{m}+j-1)\tbinom{4\hat{m}-2-j}{2i} - (2j+2-2\hat{m})\tbinom{4\hat{m}-j-3}{2i} - (4\hat{m}-3-j)\tbinom{4\hat{m}-j-4}{2i}\xi^2. \tag{6.30}
$$

By the same computation as that used in (6.17), we obtain

$$
T_{ij}^{[4]} := \frac{1}{4\hat{m}-3-j}T_{ij}^{[3]} = \frac{6\hat{m}-3-2i}{2i-1}\tbinom{4\hat{m}-4-j}{2i-2} + \tbinom{4\hat{m}-4-j}{2i}\theta. \tag{6.31}
$$

Set

$$T^{[5]}_{(2\hat{m}-2)j} = \frac{4\hat{m}-5}{2\hat{m}+1} T^{[4]}_{(2\hat{m}-2)j},\ T^{[5]}_{ij} = \frac{2i-1}{6\hat{m}-3-2i} \big(T^{[4]}_{ij} - \theta T^{[5]}_{(i+1)j} \big) \text{ for } 1 \le i \le 2\hat{m}-3.$$

Then $T^{[5]}_{ij} = \left(\binom{4\hat{m}-4-j}{2i-2}\right)_{1\le i,j\le 2\hat{m}-2}$. By lemma 6.1, $T^{[5]}$ is non-singular. Thus T is non-singular. This completes the proof of Lemma 6.3. □

7 Holomorphic flattening, proofs of Theorems 1.3, 1.4

Our proof of Theorem 1.3 is fundamentally based on Theorem 3.1 and the two dimensional results in Kenig-Webster [KW1] and Huang-Krantz [HK]. First, as we observed already in §1, when M can be holomorphically flattened near $p = 0$, all CR points of M near 0 must be non-minimal. Hence, in Theorem 1.3, we need only to prove the converse. The proof of Theorem 1.3 is an immediate consequence of Theorem 3.1 and the following result:

Theorem 7.1. *Let M be a real analytic hypersurface with a CR singularity at 0. Suppose that for any $N \ge 3$, there is a holomorphic change of coordinates of the special form $(z', w') := (z, w + O(|zw| + |w|^2 + |z|^3))$ such that M in the new coordinates (which for simplicity we still write as (z, w)) is defined by an equation of the form:*

$$w = G(z,\overline{z}) + iE(z,\overline{z}) = O(|z|^2),\ G(z_1,0,\overline{z_1},0) = |z_1|^2 + \lambda_1(z_1^2 + \overline{z_1}^2) + o(|z_1|^2), E(z,\overline{z}) = O(|z|^N). \tag{7.1}$$

Here the constant λ_1 is such that $0 \le \lambda_1 < \frac{1}{2}$ and the real analytic functions G, E are real-valued. Then M can be holomorphically flattened near 0,

Proof. We now proceed to the proof of Theorem 7.1. The special form for the change of coordinates in the theorem suggests us to slice M along the $t := (z_2, \cdots, z_n) = const$–direction and apply the two dimensional result in [HK]. By the stability of the elliptic tangency (see [For] for instance), we get a family of elliptic Bishop surfaces parametrized by t. By the work in Kenig-Webster [KW1] and Huang-Krantz [HK], each surface bounds a three dimensional real-analytic Levi-flat manifold. Putting these manifolds together and tracing the construction of these manifolds through the Bishop disks, we will obtain a real-analytic hypersurface $\widehat{M_N}$. A major feature for $\widehat{M_N}$ is that it has an order $O(N)$ of vanishing for its Levi-form at 0. Now, the crucial point is that the assumption in the theorem and the uniqueness in Kenig-Webster [KW1] assures that $\widehat{M_N}$ will be biholomorphically transformed to each other near 0 when making N larger and larger. Hence, we see that $\widehat{M_N}$ is a real-analytic hypersurface with its Levi-form vanishing to the infinite order at 0. Thus the Levi-form of $\widehat{M_N}$ vanishes everywhere. Hence $\widehat{M_N}$ is Levi-flat. This then completes the proof of the theorem. We next give the details on these.

In the following, we write $t = (z_2, \cdots, z_n) =$ and write $u = \Re w,\ v = \Im w$. For $|t|$ small, define $M_t = \{(z, w) \in M : (z_2, \cdots, z_n) = t\}$. Then M_t is a small deformation of the original M_0, which has a unique elliptic complex tangent at $z_1 = 0$ for $|z_1| < \epsilon_0 << 1$. Since a small deformation of the surface will only move the complex tangent point to a nearby point and elliptic complex tangency is stable under small deformation, intuitively, M_t must have an elliptic complex tangent near $z_1 \approx 0$, which is completely determined by the equation:

$$\frac{\partial w}{\partial \overline{z_1}} = 2\lambda_1 \overline{z_1} + z_1 + \frac{\partial (p + iE)}{\partial \overline{z_1}}(z_1, t, \overline{z_1}, \overline{t}) = 0.$$

Here, we also write $p(z, \overline{z}) = G(z, \overline{z}) - (|z_1|^2 + \lambda_1(z_1^2 + \overline{z_1}^2))$. By the implicit function theorem, one can solve uniquely $z_1 = a(t, \overline{t}) = O(|t|)$, which is C^ω in t. Then

$$P(t) = \left(a(t, \overline{t}), t, (G + \sqrt{-1}E)(a, t, \overline{a}, \overline{t})\right)$$

is the elliptic complex tangent point over M_t obtained by deforming the 0 on M_0 to M_t. Next, we expand (7.1) at $(a(t, \overline{t}), t)$:

$$\begin{aligned} w = {} & w_0(t, \overline{t}) + b(t, \overline{t})(z_1 - a(t, \overline{t})) + 2\Re\left(c(t, \overline{t})(z_1 - a(t, \overline{t}))^2\right) + d(t, \overline{t})|z_1 - a(t, \overline{t})|^2 + \\ & h^*(z_1 - a(t, \overline{t}), t, \overline{z_1 - a(t, \overline{t})}, \overline{t}) + \sqrt{-1}G^*\left(z_1 - a(t, \overline{t}), t, \overline{z_1 - a(t, \overline{t})}, \overline{t}\right) \end{aligned} \tag{7.2}$$

Here, all functions appeared above depend C^ω-smoothly on their variables with $w_0(0, 0) = 0$, $d(0, 0) = 1$, $b(0, 0) = 0$, $c(0, 0) = \lambda_1$. Moreover, $h^*(\eta, t, \overline{\eta}, \overline{t}) = O(|\eta|^3)$, $G^*(\eta, t, \overline{\eta}, \overline{t}) = O(|\eta|^2) \cap O(|\eta|^N + |t|^N)$ and $d(t, \overline{t})$ are all real-valued. By continuity, for $|t|$ small, we have $A(\eta, \overline{\eta}, t, \overline{t}) := 2\Re\left(c(t, \overline{t})\eta^2\right) + d(t, \overline{t})|\eta|^2 \geq C|\eta|^2$ for a certain positive constant C independent of $|t|$. Hence, for $|t|$ small and for a real number r with $|r| << 1$, the following defines a simply connected (convex) domain D_t in $\mathbb{C}$ with a real analytic boundary:

$$D_t := \{\eta \in \mathbb{C} : \quad 2\Re\left(c(t, \overline{t})\eta^2\right) + d(t, \overline{t})|\eta|^2 + r^{-2}h^*(r\eta, r\overline{\eta}, t, \overline{t}) \leq 1\}.$$

Let $\sigma(\xi, t, \overline{t}, r)$ be the Riemann mapping from the unit disk to D_t preserving the origin. By [Lemma 2.1, Hu1], $\sigma(\xi, t, \overline{t}, r)$ depends C^ω on its variables and is holomorphic in ξ in a fixed neighborhood of $\overline{\Delta}$. (See also [Lemma 4.1, Hu2] for a detailed proof on this.)

Now, we construct a family of holomorphic disks with parameter (t, r) for $|t|, |r| << 1$ attached to M, which takes the following form:

$$\begin{aligned} & z_1(\xi, t, \overline{t}, r) = a(t, \overline{t}) + r\sigma(\xi, t, \overline{t}, r)(1 + \psi_1(\xi, t, \overline{t}, r)), \\ & (z_2, \cdots, z_n) = t, \\ & w(\xi, t, \overline{t}, r) = w_0(t, \overline{t}) + b_1(t, \overline{t}) \cdot r\sigma(\xi, t, \overline{t}, r)(1 + \psi_1(\xi, t, \overline{t}, r)) + r^2(1 + \psi_2(\xi, t, \overline{t}, r)), \\ & \Re\psi_1(0, t, \overline{t}, r) = 0, \quad \Im\psi_2(0, t, \overline{t}, r) = 0, \\ & \psi = (z_1(\xi, t, \overline{t}, r), t, w(\xi, t, \overline{t}, r)) \end{aligned} \tag{7.3}$$

Here ψ_1, ψ_2 are holomorphic functions in $\xi \in \Delta$, and are C^ω on (ξ, t, r) over $\overline{\Delta} \times \{t \in \mathbb{C}^{n-2} : |t| < \epsilon_0\} \times \{r \in \mathbb{R} : |r| < \epsilon_0\}$. Substituting (7.3) into (7.2) with $|\xi| = 1$, we get the following:

$$\psi_2(\xi, t, \overline{t}, r) = \Omega_1 + \Omega_2 + \sqrt{-1}\Omega_3. \tag{7.4}$$

Here $\Omega_1 = 2\Re\left(\{\frac{\partial A}{\partial\eta}(\sigma,t,\overline{\sigma},\overline{t})\sigma + \sigma r^{-1}\frac{\partial h^*}{\partial\eta}(r\sigma,t,r\overline{\sigma},\overline{t})\}\psi_1\right)$, $\Omega_2 = O(|\psi_1|^2)$, and $\Omega_3 = O(|t|^{N-2} + |r|^{N-2})$ are all real-valued. Moreover, Ω_j $(j = 1,2,3)$ depend C^ω on there variables (ψ_1, t, r) in a certain suitable Banach space defined in [§5, Hu1]. Write $g(\xi,\overline{\xi},t,\overline{t},r) = 2\sigma\{\frac{\partial A}{\partial\eta}(\sigma,t,\overline{\sigma},\overline{t})\sigma + r^{-1}\frac{\partial h^*}{\partial\eta}(r\sigma,t,r\overline{\sigma},\overline{t})\}$. Then we similarly have $\Re g > 0$, which makes results in [Lemma 5.1, Hu1] applicable in our setting. Write $\mathcal{H}$ for the standard Hilbert transform, we obtain the following singular Bishop equation:

$$\Re\{g(\xi,\overline{\xi},t,\overline{t},r)\psi_1\} + \Omega_2(\psi_1,\overline{\psi_1},t,\overline{t},r) = -\mathcal{H}(\Omega_3). \tag{7.5}$$

Now, write $\psi_1 = U(\xi,\overline{\xi},t,\overline{t},r) + \sqrt{-1}\mathcal{H}(U(\xi,\overline{\xi},t,\overline{t},r))$ for $|\xi| = 1$. By the argument in [§5, Hu1], from (7.5), one can uniquely solve $U(\xi,\overline{\xi},t,\overline{t},r)$ for $|t|, |r| << 1$. Moreover, $U(\xi,\overline{\xi},t,\overline{t},r)$ depends C^ω on $(\xi,\overline{\xi},t,\overline{t},r)$ and $U(\xi,\overline{\xi},t,\overline{t},r) = O(|t|^{N-2} + |r|^{N-2})$. Hence $U(\xi,\overline{\xi},t,\overline{t},r) + \sqrt{-1}\mathcal{H}(U(\xi,\overline{\xi},t,\overline{t},r))$ extends to a holomorphic function in ξ which also depends C^ω on its variables $(\xi,\overline{\xi},t,\overline{t},r)$ with $|\xi| \le 1$. Moreover, we have the estimates

$$\psi_1, \psi_2 = O(|t|^{N-2} + |r|^{N-2}). \tag{7.6}$$

Next, we let $\widehat{M_N} = \bigcup_{0\le r<<1,\ |t|<<1, \xi\in\overline{\Delta}} \psi(\xi,t,\overline{t},r)$. Let $\widetilde{M} = \pi(\widehat{M_N})$ where π is the projection from $\mathbb{C}^{n+1}$ into the (z,u)-space. By the results in Kenig-Webster [KW1] and Huang-Krantz [HK], for each fixed t, $\widehat{M_{N,t}} = \widehat{M_N} \cap \{z' = t\} \cap B_{P(t,\overline{t})}(r_0)$ must be the local hull of holomorphic of M_t, that is a manifold C^ω-regular up to the boundary M_t. Here $B_{a(t,\overline{t})}(r_0)$ is the ball centered at $P(t,\overline{t})$ with a certain fixed radius $r_0 > 0$. Also, since $v = G(z_1,t,\overline{z_1},\overline{t})$ defines a strongly pseudoconvex hypersurface in $\mathbb{C}^2$ for each fixed t, we see that $\pi(\widehat{M_{N,t}}) \subset \widetilde{M^*}_t$, where $\widetilde{M^*} := \{(z,w) : u \ge G(z,\overline{z})\}$ and $\widetilde{M_t^*} = \widetilde{M^*} \cap \{(z_2,\cdots,z_n) = t\}$. Indeed, π, when restricted to $\widehat{M_{N,t}}$ is a C^ω-diffeomorphism to $\widetilde{M_t^*}$ in the intersection of $\widehat{M_{N,t}}$ with the ball centered at $P(t)$ with a certain fixed radius $1 >> r_0 > 0$. To see this, by our normalization presented in the previous section or by the Kenig-Webster [KW], we have a change of variables in (z_1, w):

$$z_1' = z_1 - a(t,\overline{t}), \quad w' = w_0(t,\overline{t}) + \sum_{j=1}^{m} b_j(t,\overline{t})(w - w_0(t,\overline{t}))^j,$$

where $w_0,\ b_j$ depend smoothly on t and takes values 0 at 0. In this coordinates, M_t is mapped to M_t' that is flattened to order m at 0. Hence, for $m >> 1$, the holomorphic hull of M_t' near 0 now is tangent to (z_1', u')-space (See [KW] [HK]), in particular, must be transversal to the $v'-$axis. Since the hull is a biholomorphic invariant, we see that $\widehat{M_{N,t}}$ has to be transversal to the v-axis when $|t|$ is small. Hence, π is a one to one and onto map from $\widehat{M_N}$ to $\widetilde{M^*}$ near 0. Write the inverse map of π as $v(z_1,\overline{z_1},t,\overline{t})$, which is defined over $\widetilde{M^*}$ near 0. Notice that it is the graph function of $\widehat{M_N}$ near 0 and has to be C^ω-regular for each fixed t.

Next, we solve $v(z_1,\overline{z_1},t,\overline{t})$ from (7.3). For this, we use the computation in [HK]. First, we let

$$\begin{aligned} z_1' &= z_1 - a(t,\overline{t}) \\ w' &= w - \big(w_0(t,\overline{t}) + b_1(t,\overline{t})(z_1 - a(t,\overline{t}))\big) \end{aligned} \tag{7.7}$$

Then (7.3) can be rewritten as

$$\begin{aligned}
&z_1'(\xi,t,\bar t,r)=r\sigma(\xi,t,\bar t,r)(1+\psi_1(\xi,t,\bar t,r)),\\
&(z_2,\cdots,z_n)=t,\\
&w'(\xi,t,\bar t,r)=r^2(1+\psi_2(\xi,t,\bar t,r)).
\end{aligned}\tag{7.8}$$

Write $w'=u'+\sqrt{-1}v'$. Now, by the proof in [HK, pp 225], we see that for each (z_1',u',t), there is a unique v' satisfying (7.8). Moreover v', as a function in (z_1',u',t), has the following generalized Puiseux expansion:

$$v'=\sum_{i,j,s,\alpha,\beta\ge 0}S_{ijs\alpha\beta}u'^{\frac{i-j-s}{2}}z_1'^j\overline{z_1'}^s t^\alpha\overline{t^\beta},$$

where $|S_{ijk\alpha\beta}|\lesssim C^{i+j+k}$ for some positive constant C. By the regularity of $\widehat{M_{N,t}}$ as mentioned above, we know that $\frac{\partial^{j+s}v'}{\partial z_1'^j\partial\overline{z_1'}^s}|_{z_1=0}$ must be smooth for each $|t|$ small and $u'\ge 0$. This shows that $S_{ijs\alpha\beta}=0$ when $\frac{i-j-s}{2}$ is not a positive integer. As in [HK, pp 227], we see that v' is a real analytic function in (z_1',u',t) near 0. By (7.7), we see that v is analytic function in (z_1,u,t). Hence, we proved that $\widehat{M_N}$ is a real analytic manifold, which can be represented as a graph over $\widetilde{M^*}$ in (z,u)-space. Moreover the analytic graph function $v=\rho=O(|t|^{N-2}+|z_1|^{N/2})$. To see this, by (7.3) (7.6), we need only to explain that $\Im(w_0)=O(|t|^N)$ and $b_1(t,\bar t)=O(|t|^{N-1})$. Indeed, $\Im(w_0)=E(a(t,\bar t),\overline{a(t,\bar t)},t,\bar t)=O(|t|^N)$ and

$$b_1=\frac{\partial(G+\sqrt{-1}E)}{\partial z_1}(a(t,\bar t),\overline{a(t,\bar t)},t,\bar t).$$

Since

$$\frac{\partial(G+\sqrt{-1}E)}{\partial\overline{z_1}}(a(t,\bar t),\overline{a(t,\bar t)},t,\bar t)=0,$$

we get

$$b_1=2\sqrt{-1}\frac{\partial E}{\partial z_1}(a(t,\bar t),\overline{a(t,\bar t)},t,\bar t)=O(|t|^{N-1}).$$

Still let $v=\rho$ be the defining function as mentioned above. Since ρ is real analytic, we can extend $\widehat{M_N}$ to a real analytic hypersurface $M_N^\#$ near the origin by using the graph of ρ. Now, we let

$$\theta=\sqrt{-1}\partial\Big(-\frac{w-\overline{w}}{2\sqrt{-1}}+\rho\Big),L_j=\frac{\partial}{\partial z_j}+\frac{2\sqrt{-1}\rho_{z_j}}{1-2\sqrt{-1}\rho_w}\frac{\partial}{\partial w}.$$

Then θ is a contact form along $M_N^\#$ and $\{L_j\}_{j=1}^n$ forms a basis of real analytic tangent vector fields of type $(1,0)$ along $M_N^\#$ near 0. With respect to such a contact form and a basis of tangent vector fields of type $(1,0)$, we obtain the following Levi-matrix, which is a real analytic $n \times n$-matrix near 0:

$$\mathcal{L}_N = \left((\sqrt{-1}d\theta, L_j \wedge \overline{L_k})\right)_{1\le j,k\le n}.$$

Since $\rho = O(|z|^{N/2})$, we see that $\mathcal{L} = O(|z|^{N/2-3})$ as $z \to 0$. Now, for $N' > N$, by the existence of the special change of coordinates as in the hypothesis of the theorem, we have a transformation of the form $z' = z, w' = w + h(z,w) = w + o(1)$, which further flattens M to M' near 0 to the order of N'. For M', we similarly have $\widehat{M'_{N'}}$, which, by the special property that $z' = z$ of our transformation, can be seen to be precisely the image of $\widehat{M_N}$, near 0, under the transformation. (Here, it suffices to use the two dimensional uniqueness result of Kenig-Webster [KW]). Next, we can similarly define θ', L'_j, as well as, the Levi matrix $\mathcal{L}'_{N'}$. Now $\mathcal{L}'_{N'} = O(|z|^{N'/2-3})$. By the transformation formula of the Levi -form, we see that there is an invertible real analytic matrix B near 0 and a positive real analytic function κ near 0 such that

$$\mathcal{L}_N = \kappa A \mathcal{L}'_{N'} \overline{A^t}.$$

Hence, $\mathcal{L}_N = O(|z|^{N'/2-3})$ for any $N' > N$. By the analyticity of $\mathcal{L}_N$. We see that $\mathcal{L}_N \equiv 0$ and thus $\widehat{M_N}$ is Levi-flat. Next, by the classical theorem of Cartan, we see that $\widehat{M_N}$ can be hiholomorphically mapped to an open piece in $\mathbb{C}^n \times \mathbb{R}$. This completes the proof of the theorem. $\square$

Proof of Theorem 1.3 and Theorem 1.4: The proof of Theorem 1.3 and Theorem 1.4 is an immediate consequence of Theorem 3.1 and Theorem 7.1. Here, we only need to mention that when M is already flattened, it is obvious that the $\widehat{M_N}$ constructed is the local hull of holomorphy of M near 0. By the invariant property of holomorphic hull, we conclude that this is also the case when M is not flattened yet. ∎

Example 7.2. Define $M \subset \mathbb{C}^3$ by the following equation near 0:

$$w = q(z,\overline{z}) + p(z,\overline{z}) + iE(z,\overline{z}).$$

Here as before $q = |z_1|^2 + \lambda_1(z_1^2 + \overline{z_1^2}) + |z_2|^2 + \lambda_2(z_2^2 + \overline{z_2^2})$ with $0 \le \lambda_1, \lambda_2 < \infty$, and $p, E = O(|z|^3)$ are real-valued. Also $G(z,\overline{z}) := q(z,\overline{z}) + p(z,\overline{z})$. For any $c \in \mathbb{R}\setminus\{0\}$, define the real hypersurface K_c by the equation $q(z,\overline{z}) = c$. Then K_c intersects transversally M along a submanifold L_c of real dimension 3. Then L_c is a CR submanifold of CR dimension 1 if and only if $L(q) \equiv 0$ along L_c. Here

$$L = (G_2 - iE_2)\frac{\partial}{\partial z_1} - (G_1 - iE_1)\frac{\partial}{\partial z_2} + 2i(G_2E_1 - G_1E_1)\frac{\partial}{\partial w}, \tag{7.9}$$

that is non-zero and tangent to $M \setminus \{0\}$.

Write $\Psi = p(z,\overline{z}) - iE(z,\overline{z})$. Then the above is equivalent to the equation $\Psi_2 \cdot (\overline{z_1} + 2\lambda_1 z_1) = \Psi_1 \cdot (\overline{z_2} + 2\lambda_2 z_2)$. Namely, M is non-minimal at its CR points near 0 if and only if the just mentioned equation holds.

One solution is given by $\Psi = p(z,\overline{z}) - iE(z,\overline{z}) = \mu_1(|z_1|^2\overline{z_1} + \lambda_1|z_1|^2 z_1) + \mu_2(|z_2|^2\overline{z_2} + \lambda_2|z_2|^2 z_2) + \mu_1\overline{z_1}(|z_2|^2 + \lambda_2 z_2^2) + \mu_2\overline{z_2}(|z_1|^2 + \lambda_1 z_1^2)$, with $\mu_1, \mu_2 \in \mathbb{C}$. Then $\Psi_1 = (\mu_1\overline{z_1} + \mu_2\overline{z_2})(\overline{z_1} + 2\lambda_1 z_1)$ and $\Psi_2 = (\mu_2\overline{z_2} + \mu_1\overline{z_1})(\overline{z_2} + 2\lambda_1 z_2)$. Thus $\Psi_2 \cdot (\overline{z_1} + 2\lambda_1 z_1) = \Psi_1 \cdot (\overline{z_2} + 2\lambda_2 z_2)$ holds trivially.

Finally, We also mention a recent preprint [Bur2] for some generalization of the work in Kenig-Webster [KW] and Huang-Krantz [HK].

8 Appendix

In this appendix, for convenience of the reader, we give a detailed proof of Theorem 1.2 for the case $n = 2$, $m = 3$ to demonstrate the basic ideas of the complicated calculations for the proof of Theorem 1.2 performed in Sections 4-6 of this paper.

In Section 3, we have showed, by making use of the non-minimality, the following:

$$\begin{aligned}&\big(|w_n|^2 + |w_1|^2\big)\cdot\big(\overline{w_n}\Psi_1 - \overline{w_1}\Psi_n\big) + \big(2\lambda_n w_n\overline{w}_1 - 2\lambda_1 w_1\overline{w}_n\big)\cdot\Psi = 0, \quad \text{where}\\ &\Psi = w_n\overline{w_n}\Phi_1 - w_n\overline{w_1}\Phi_n + \overline{w_1}\cdot\Phi,\ \Phi = w_n H_{\overline{1}} - w_1 H_{\overline{n}},\\ &H = E^{(m)},\ \ w_j = z_j + 2\lambda_j\overline{z}_j \text{ for } 1 \le j \le n = 2.\end{aligned} \tag{8.1}$$

We use the following notations

$$\begin{aligned}&\xi = 2\lambda_n,\ \eta = 2\lambda_1,\ \theta = 1-\xi^2,\\ &H_{[tsrh]} = H_{(te_n+se_1, re_n+he_1)} \text{ for } t+s+r+h = m,\\ &\Phi_{[tsrh]} = \Phi_{(te_n+se_1, re_n+he_1)} \text{ for } t+s+r+h = m,\\ &\Psi_{[tsrh]} = \Psi_{(te_n+se_1, re_n+he_1)} \text{ for } t+s+r+h = m+1\end{aligned} \tag{8.2}$$

Here, for any homogeneous polynomial $\chi(z,\overline{z})$ of degree $k \ge 1$, we write

$$\chi = \sum_{\alpha\ge 0,\beta\ge 0,|\alpha|+|\beta|=k} H_{(\alpha,\beta)} z^\alpha \overline{z^\beta}.$$

We first set up more notations and establish formulas which are crucial in the general case discussed in §4 − §6.

By (8.1), we have

$$\begin{aligned}\Phi_{[tsrh]} =& \xi(h+1)H_{[ts(r-1)(h+1)]} + (h+1)H_{[(t-1)sr(h+1)]}\\ &- (r+1)H_{[t(s-1)(r+1)h]} - \eta(r+1)H_{[ts(r+1)(h-1)]},\end{aligned} \tag{8.3}$$

and

$$\begin{aligned}\Psi_{[tsrh]} =& (s+1)\big\{\xi\Phi_{[t(s+1)(r-2)h]} + (1+\xi^2)\Phi_{[(t-1)(s+1)(r-1)h]} + \xi\Phi_{[(t-2)(s+1)rh]}\big\}\\ &- \xi(t+1)\Phi_{[(t+1)s(r-1)(h-1)]} - t\Phi_{[tsr(h-1)]} - \xi\eta(t+1)\Phi_{[(t+1)(s-1)(r-1)h]}\\ &- \eta t\Phi_{[t(s-1)rh]} + \Phi_{[tsr(h-1)]} + \eta\Phi_{[t(s-1)rh]}.\end{aligned} \tag{8.4}$$

Collecting the coefficients of $z_n^t z_1^{s-1}\overline{z_n}^{r+3}\overline{z_1}^h$ for $t \ge 0$, $s \ge 1$, $r \ge -3$ and $h = m+1-t-s-r \ge 0$ in (8.1), we get

$$\begin{aligned}&s\big\{\xi\Psi_{[tsrh]} + (2\xi^2+1)\Psi_{[(t-1)s(r+1)h]} + (\xi^3+2\xi)\Psi_{[(t-2)s(r+2)h]}\\ &\quad + \xi^2\Psi_{[(t-3)s(r+3)h]}\big\} + \mathcal{F}\big\{(\Psi_{[t's'r'h']})_{s'+h'\le s+h-2, s'\le s, h'\le h}\big\} = 0.\end{aligned}$$

Here, for a set of complex numbers (or polynomials) $\{a_j, b\}_{j=1}^k$, we say $b \in \mathcal{F}\{a_1, \cdots, a_k\}$ if $b = \sum_{j=1}^k (c_j a_j + d_j \overline{c_j})$ with $c_j, d_j \in \mathbb{C}$. Also, we set up the convention that $\chi_{[tsrh]} = 0$ if one of the indices is negative.

Thus for $s \geq 1$, we can inductively get

$$\Psi_{[tsrh]} = \mathcal{F}\big\{(\Psi_{[t's'r'h']})_{s'+h' \leq s+h-2, s' \leq s, h' \leq h}\big\}. \tag{8.5}$$

Substituting (8.4) into (8.5), we get, for $s \geq 1$, the following

$$\begin{aligned}
&(s+1)\big\{\xi\Phi_{[t(s+1)(r-2)h]} + (1+\xi^2)\Phi_{[(t-1)(s+1)(r-1)h]} + \xi\Phi_{[(t-2)(s+1)rh]}\big\} \\
&= \mathcal{F}\big\{(\Phi_{[t's'r'h']})_{s'+h' \leq s+h-1, s' \leq s+1, h' \leq h}\big\}.
\end{aligned}$$

Hence for $s \geq 2$, we can inductively obtain

$$\Phi_{[tsrh]} = \mathcal{F}\big\{(\Phi_{[t's'r'h']})_{s'+h' \leq s+h-2, s' \leq s, h' \leq h}\big\}. \tag{8.6}$$

Substituting (8.3) into (8.6), we get, for $s \geq 2$ and $h \geq 0$, the following

$$\begin{aligned}
&\xi(h+1)H_{[ts(r-1)(h+1)]} + (h+1)H_{[(t-1)sr(h+1)]} = \mathcal{F}\big\{(\Phi_{[t's'r'h']})_{s'+h' \leq s+h-2, s' \leq s, h' \leq h}\big\} \\
&+ (r+1)H_{[t(s-1)(r+1)h]} + \eta(r+1)H_{[ts(r+1)(h-1)]}. \\
&= \mathcal{F}\Big\{(H_{[t's'(m-t'-s'-h')h']})_{s'+h' \leq s+h-1, s' \leq s, h' \leq h+1}\Big\}.
\end{aligned}$$

Hence for $s \geq 2$ and $h \geq 1$, we can inductively get that

$$H_{[ts(m-t-s-h)h]} = \mathcal{F}\Big\{(H_{[t's'(m-t'-s'-h')h']})_{s'+h' \leq s+h-2, s' \leq s, h' \leq h}\Big\}.$$

Notice that $H_{[tsrh]} = \overline{H_{[rhts]}}$. Keeping applying the above until the assumption that $s \geq 2$ and $h \geq 1$ do not hold anymore, we can inductively get the following crucial formula:

$$H_{[ts(m-t-s-h)h]} = \mathcal{F}\Big\{(H_{[t'1(m-t'-2)1]})_{1 \leq t' \leq m-2}, (H_{[t'0(m-t'-i)i]})_{i \leq \max(s,h), 0 \leq t' \leq m-i}\Big\}. \tag{8.7}$$

Now, we assume $m = 3$. We first normalize $H := E^{(3)}$ without using the non-minimality condition.

Let $z' = z, w' = w + B(z,w)$ be a holomorphic transformation that transforms $w = G(z,\overline{z}) + iE(z,\overline{z})$ to $w' = G'(z',\overline{z}') + iE'(z',\overline{z}')$. Then we get

$$\Im B(z,w) = E'(z,\overline{z}) - E(z,\overline{z}).$$

Here $B(z,w)$ is a weighted holomorphic homogeneous polynomial in (z,w) of degree 3, with $wt(z) = 1$ and $wt(w) = 2$.

Sub-appendix I: In this part, we first prove the following:

Lemma 8.1. *After a holomorphic transformation, we can have $E(z, \overline{z})$ defined in (2.1) to satisfy the following normalization:*
(1) *When $\lambda_n = 0$, then*

$$E_{(3e_n,0)} = E_{(2e_n+e_1,0)} = E_{(e_n+2e_1,0)} = E_{(3e_1,0)} = E_{(2e_n,e_n)} = E_{(e_n+e_1,e_n)} = 0. \tag{8.8}$$

(2) *When $\lambda_n \neq 0$, then*

$$E_{(3e_n,0)} = E_{(2e_n+e_1,0)} = E_{(e_n+2e_1,0)} = E_{(3e_1,0)} = E_{(e_n+e_1,e_1)} = E_{(e_n+e_1,e_n)} = 0. \tag{8.9}$$

Proof. First, notice that the real dimension of the space of all such $B^{(3)}$ is

$$\begin{aligned} &2 \cdot \sharp\big\{(i_1, i_n, j) \in \mathbb{R}^3 : \ i_1, i_n, j \geq 0, \ i_1 + i_n + 2j = 3\big\} \\ =&2 \cdot \sharp\big\{(i_1, i_n, j) \in \mathbb{R}^3 : \ i_1 \neq 0, \ i_1 + i_n + 2j = 3\big\} + 4. \end{aligned} \tag{8.10}$$

(1) Assume that $\lambda_n = 0$. Set

$$\hat{P}^{(3)} = \big\{\text{polynomials of the form } 2\Re \sum_{i+j+2k=3} a_{(ijk)} z_1^i z_n^j |z_n|^{2k}\big\}.$$

To get the normalization condition (8.8), we only need to prove that

$$\Im\big(B^{(3)}(z, q(z, \overline{z}))\big)\big|_{\hat{P}^{(3)}} = Q^{(3)}(z, \overline{z}) \tag{8.11}$$

is solvable for any $Q^{(3)}(z, \overline{z}) \in \hat{P}^{(3)}$. Here, we choose an orthonormal basis $\{z^\alpha \overline{z^\beta}\}$ for the space of (not necessarily holomorphic) polynomials. Then for any subspace P and a polynomial A, we write $A|_P$ for the orthogonal projection of A to P.

Notice that $\hat{P}^{(3)}$ and the space $\{B^{(3)}(z, q(z, \overline{z}))\}$ have the same dimension. Hence to prove (8.11), we need to show that

$$\Im\big(B^{(3)}(z, q(z, \overline{z}))\big)\big|_{\hat{P}^{(3)}} = 0 \iff B = 0.$$

By considering the terms involving only z_n and $\overline{z}_n$, we get

$$\Im\big(b_{(011)} z_n |z_n|^2 + b_{(030)} z_n^3\big) = 0.$$

Thus we get $b_{(011)} = b_{(030)} = 0$. Hence, if $\Im\big(B^{(3)}(z, q(z, \overline{z}))\big)\big|_{\hat{P}^{(3)}} = 0$, we have that $B^{(3)}(z, |z_n|^2 + \lambda_1 z_1^2) = 0$. Namely, we have

$$b_{(120)} z_1 z_n^2 + b_{(101)} z_1 |z_n|^2 + b_{(210)} z_1^2 z_n + (b_{(300)} + \lambda_1 b_{101}) z_1^3 = 0. \tag{8.12}$$

Hence we get $b_{(ijk)} = 0$. Furthermore, we obtain $B^{(3)}(z, \overline{z}) = 0$.

(2) In this case, we assume $\lambda_n \neq 0$. Write

$$\hat{P}_{-3}^{(3)} = \{\text{homogeneous polynomials of the form:}$$
$$2\Re\Big(\sum_{i\geq 1, i+j+2k=3} a_{ijk} z_1^i z_n^j |z_n|^{2k} + b z_n^3 + c z_n |z_1|^2 \Big) \}.$$

To get the normalization condition (8.9), we only need to prove that

$$\Im\big(B^{(3)}(z, q(z,\overline{z}))\big)\big|_{\hat{P}_{-3}^{(3)}} = Q^{(3)}(z,\overline{z}) \tag{8.13}$$

is solvable for any real valued polynomial $Q^{(3)}(z,\overline{z}) \in \hat{P}_{-3}^{(3)}$. Notice that the real dimension of $\hat{P}_{-3}^{(3)}$ is

$$2 \cdot \sharp\big\{(i_1, i_n, j) \in \mathbb{R}^3 : \ i_1 \neq 0, \ i_1 + i_n + 2j = 3\big\} + 4,$$

which is the same as the real dimension of all such $B(z,w)'s$. Hence to prove (8.13), we need to show that

$$\Im\big(B^{(3)}(z, q(z,\overline{z}))\big)\big|_{\hat{P}_{-3}^{(3)}} = 0 \Longleftrightarrow B^{(3)} = 0.$$

Notice that

$$\begin{aligned} 0 &= \Im B^{(3)}(z,w)\big|_{\hat{P}_{-3}^{(3)}} \\ &= \Im\Big(\sum_{i\geq 1, i_j+2k=3} b_{(ijk)} z_1^i z_n^j w^k + b_{(030)} z_n^3 + b_{(011)} z_n (|z_n|^2 + \lambda_n z_n^2 + \lambda_n \overline{z_n}^2 + \lambda_1 z_1^2 + |z_1|^2) \Big)\Big|_{\hat{P}_{-3}^{(3)}}. \end{aligned} \tag{8.14}$$

Collecting the coefficients of $z_n|z_1|^2$ and z_n^3, respectively, in (8.14), we get $b_{(011)} = 0$ and $b_{(030)} + \lambda_n b_{(011)} = 0$. Thus we get $b_{(030)} = b_{(011)} = 0$. Hence $\Im B^{(3)}(z,w)\big|_{\hat{P}_{-3}^{(3)}} = 0$ implies that

$$b_{(300)} z_1^3 + b_{(210)} z_1^2 z_n + b_{(120)} z_1 z_n^2 + b_{(101)} z_1 (|z_n|^2 + \lambda_n z_n^2 + z_1^2) = 0.$$

Now it is obvious that $b_{(101)} = b_{(120)} = b_{(300)} = b_{(210)} = 0$. Hence we have get $B^{(3)} = 0$. This completes the proof of (8.9). □

Sub-appendix II: Now we proceed to prove Theorem 1.2 for $n = 2$ and $m = 3$.

Case I: In this case, we assume that $\lambda_n = \lambda_1 = 0$. Then (8.1) has the following form:

$$\overline{z}_n \Psi_1 = \overline{z_1} \Psi_n. \tag{8.15}$$

By considering the coefficients of $z_n^t z_1^{s-1} \overline{z_n}^{r+1} \overline{z_1}^h$ for $t \geq 0$, $s \geq 1$, $r \geq 0$ and $h = (m+1) - t - s - r \geq 0$ in (8.15), we get

$$s\Psi_{[tsrh]} = (t+1)\Psi_{[(t+1)(s-1)(r+1)(h-1)]}. \tag{8.16}$$

Setting $h=0$ in (8.16), we get $\Psi_{[tsr0]}=0$ for $s\geq 1$. Combining this with (8.16), we inductively get $\Psi_{[tsrh]}=0$ for $s\geq h+1$. Now, we will apply (8.4). Notice that we now have $\xi=\eta=0$. We thus obtain:

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)h]}=(t-1)\Phi_{[tsr(h-1)]} \text{ for } s\geq h+1. \tag{8.17}$$

Setting $h=0$ in (8.17), we get $\Phi_{[tsr0]}=0$ for $s\geq 2$. Combining this with (8.17), we inductively get $\Phi_{[tsrh]}=0$ for $s\geq h+2$. Together with (8.3), we get

$$(h+1)H_{[(t-1)sr(h+1)]}=(r+1)H_{[t(s-1)(r+1)h]} \text{ for } s\geq h+2. \tag{8.18}$$

Setting $t=0$, we get $H_{[0srh]}=0$ for $s\geq h+1,\ r\geq 1$. Then we inductively get $H_{[tsrh]}=0$ for $s\geq h+1,\ r\geq t+1$. When $s\geq h+1, r\leq t$, from (8.18), we inductively get $H_{[tsrh]}=\mathcal{F}\{(H_{[t's'r'0]})_{t'\geq r'}\}$, which is 0 by our normalization in (8.8). Thus we have proved

$$H_{[tsrh]}=0 \text{ for } s\geq h+1. \tag{8.19}$$

Next we will prove that $H_{[tsrs]}=0$. Setting $s=h\geq 1$, $t\geq 0$ and $r=-1$ in (8.16), we get $\Psi_{[ts0s]}=0$ for $t\geq 1$. Substituting it back to (8.16), we inductively get

$$\Psi_{[tsrs]}=0 \text{ for } t\geq r+1.$$

Substituting (8.4) into this equation, we get

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)s]}=(t-1)\Phi_{[tsr(s-1)]} \text{ for } t\geq r+1. \tag{8.20}$$

Setting $s=0$, we get $\Phi_{[t1r0]}=0$ for $t\geq r+1$. Substituting this back to (8.20), we get $\Phi_{[t(s+1)rs]}=0$ for $t\geq r+1$. Together with (8.3), we get

$$(s+1)H_{[(t-1)(s+1)r(s+1)]}=(r+1)H_{[ts(r+1)s]} \text{ for } t\geq r+1.$$

Notice that $H_{[t0r0]}=0$ by our normalization. Hence we inductively get

$$H_{[tsrs]}=0 \text{ for } t\geq r. \tag{8.21}$$

Since $H_{[tsrh]}=\overline{H_{[rhts]}}$, (8.19) and (8.21) imply $H\equiv 0$ for the case $\lambda_n=\lambda_1=0$.

Step II: In this step, we assume that $\lambda_n=0$ and $\lambda_1\neq 0$. Theorem 1.2 with $m=3$ in this setting is an immediate consequence of the following lemma:

Lemma 8.2. *Suppose that $\lambda_n=0$ and $\lambda_1\neq 0$. Assume that there exists an $h_0\geq -1$ such that*

$$\Psi_{[tsrh]}=\Phi_{[tsrh]}=0 \textit{ for } h\leq h_0,\ H_{[tsrh]}=0 \textit{ for } \max(s,h)\leq h_0+1. \tag{8.22}$$

Then we have

$$\Psi_{[tsrh]}=\Phi_{[tsrh]}=0 \textit{ for } h\leq h_0+1,\ H_{[tsrh]}=0 \textit{ for } \max(s,h)\leq h_0+2. \tag{8.23}$$

Once we have Lemma 8.2 at our disposal, since (8.22) holds for $h_0 = -1$ by our normalization, hence (8.23) holds for $h_0 = -1$. Then by an induction, we see that (8.23) holds for all $h_0 \leq m-2$. This completes the proof of Theorem 1.2 in this setting.

Proof of Lemma 8.2. First, notice that (8.1) has the following form:

$$(|z_n|^2 + \eta z_1^2)(\overline{z_n}\Psi_1 - \eta z_1 \Psi_n) - \eta z_1 \overline{z_n}\Psi + \overline{z_1}\mathcal{F}\{\Psi_1, \Psi_n, \Psi\} = 0.$$

Collecting the coefficients of $z_n^t z_1^{s-1} \overline{z_n}^{r+3} \overline{z_1}^{h_0+1}$ in the above equation and making use of the assumptions in Lemma 8.2, we have:

$$\begin{aligned}
&s\Psi_{[(t-1)s(r+1)(h_0+1)]} + (s-2)\eta\Psi_{[t(s-2)(r+2)(h_0+1)]} - t\eta\Psi_{[t(s-2)(r+2)(h_0+1)]} \\
&- (t+1)\eta^2\Psi_{[(t+1)(s-4)(r+3)(h_0+1)]} - \eta\Psi_{[t(s-2)(r+2)(h_0+1)]} = 0.
\end{aligned}$$

Namely, we have

$$s\Psi_{[(t-1)s(r+1)(h_0+1)]} = (t+3-s)\eta\Psi_{[t(s-2)(r+2)(h_0+1)]} + (t+1)\eta^2\Psi_{[(t+1)(s-4)(r+3)(h_0+1)]}. \tag{8.24}$$

By setting $r = -3$ in (8.24), we get $\Psi_{[ts0(h_0+1)]} = 0$ for $t \geq 1$. Substituting this back to (8.24), we inductively get that $\Psi_{[tsr(h_0+1)]} = 0$ for $t \geq r+1$. Combining this with (8.4) and the hypothesis, we obtain

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)(h_0+1)]} = (t-1)\eta\Phi_{[t(s-1)r(h_0+1)]} \text{ for } t \geq r+1. \tag{8.25}$$

Setting $r = 0$ in (8.25), we get $\Phi_{[ts0(h_0+1)]} = 0$ for $t \geq 2$. Hence we inductively get $\Phi_{[tsr(h_0+1)]} = 0$ for $t \geq r+2$. In particular, we have $\Phi_{[t0r(h_0+1)]} = 0$ for $t \geq r+2$. Combining this with (8.3),the hypothesis and $\lambda_n = 0$, we get $(h_0+2)H_{[(t-1)0r(h_0+2)]} = 0$ for $t \geq r+2$. Namely, we obtain $H_{[t0r(h_0+2)]} = 0$ for $t \geq r+1$. Together with our normalization (8.8) and the reality of H, we obtain:

$$H_{[t0r(h_0+2)]} = 0. \tag{8.26}$$

(1) When $h_0 = -1$, setting $s = 0$ in (8.25), we get $\Phi_{[(t-1)1(r-1)0]} = 0$ for $t \geq r+1$. Together with (8.3) and (8.8), we get that $H_{[(t-1)1r1]} = (r+1)H_{[t0(r+1)0]} = 0$ for $t \geq r+1$. Namely, we obtain $H_{[t1r1]} = 0$ for $t \geq r$. By the reality of H, we get for all t, r the following:

$$H_{[t1r1]} = 0. \tag{8.27}$$

From (8.3), (8.26) and (8.27), we obtain $\Phi_{[t0r0]} = \Phi_{[t1r0]} = 0$. Together with (8.4), we see that $\Psi_{[t0r0]} = 0$.

Setting $h = 0$ in (8.5) and making use of $\Psi_{[t0r0]} = 0$, we first get $\Psi_{[t1r0]} = \Psi_{[t2r0]} = 0$, then inductively get $\Psi_{[tsr0]} = 0$. Combining this with (8.4), we get

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)0]} = (t-1)\eta\Phi_{[t(s-1)r0]}, \tag{8.28}$$

Setting $s=0$ in (8.28), we obtain $\Phi_{[t1r0]}=0$. By an induction argument, we get $\Phi_{[tsr0]}=0$. This proves (8.23) for the case $h_0=-1$.

(2) When $h_0\geq 0$, from (8.7),(8.26) and (8.27), we inductively get $H_{[tsr(h_0+2)]}=0$ for $s\leq h_0+2$. Combining this with (8.3) and (8.27), we get $\Phi_{[t0r(h_0+1)]}=\Phi_{[t1r(h_0+1)]}=0$. Substituting this back to (8.4), we obtain $\Psi_{[t0r(h_0+1)]}=0$. Together with (8.5), we inductively get $\Psi_{[tsr(h_0+1)]}=0$. Combining this with (8.4), we obtain

$$(s+1)\Phi_{[(t-1)(s+1)(r-1)(h_0+1)]}=(t-1)\eta\Phi_{[t(s-1)r(h_0+1)]}.$$

As in Case I, we inductively get $\Phi_{[tsr(h_0+1)]}=0$. This proves (8.23) for the case $h_0\geq 0$ and thus completes the proof of Theorem 1.2 for the case $\lambda_n=0$ and $\lambda_1\neq 0$. $\square$

Case III: In this case, we assume $\lambda_n\neq 0$ and $\lambda_1\neq 0$. Considering the coefficients of $z_1^3\overline{z_n}^3$, $z_1^3z_n\overline{z_n}^2$, $z_1^3z_n^2\overline{z_n}$ and $z_1^3z_n^3$, respectively, in (8.1), we get

$$\begin{aligned}
&4\xi\Psi_{[0400]}+2\eta\Psi_{[0220]}-\eta\xi\Psi_{[1210]}-\eta^2\Psi_{[1030]}-\eta\theta\Psi_{[0220]}=0,\\
&4(2\xi^2+1)\Psi_{[0400]}+2\eta(\Psi_{[1210]}+\xi\Psi_{[0220]})-\eta(2\xi\Psi_{[2200]}\\
&\qquad+(1+\xi^2)\Psi_{[1210]})-2\eta^2\Psi_{[2020]}-\eta\theta\Psi_{[1210]}=0,\\
&4(\xi^3+2\xi)\Psi_{[0400]}+2\eta(\Psi_{[2200]}+\xi\Psi_{[1210]})-\eta(2(1+\xi^2)\Psi_{[2200]}\\
&\qquad+\xi\Psi_{[1210]})-3\eta^2\Psi_{[3010]}-\eta\theta\Psi_{[2200]}=0,\\
&4\xi^2\Psi_{[0400]}+2\eta\xi\Psi_{[2200]}-\eta 2\xi\Psi_{[2200]}-4\eta^2\Psi_{[4000]}=0.
\end{aligned}$$

Simplifying the above from the last equation to the first one, we get

$$4\xi^2\Psi_{[0400]}=4\eta^2\Psi_{[4000]}, \tag{8.29}$$
$$4(\xi^3+2\xi)\Psi_{[0400]}+\eta\big\{(-2\xi^2-\theta)\Psi_{[2200]}+\xi\Psi_{[1210]}\big\}=3\eta^2\Psi_{[3010]}, \tag{8.30}$$
$$4(2\xi^2+1)\Psi_{[0400]}+\eta\big\{-2\xi\Psi_{[2200]}+2\xi\Psi_{[0220]}\big\}=2\eta^2\Psi_{[2020]}, \tag{8.31}$$
$$4\xi\Psi_{[0400]}+\eta\big\{-\xi\Psi_{[1210]}+(2-\theta)\Psi_{[0220]}\big\}=\eta^2\Psi_{[1030]}. \tag{8.32}$$

A direct computation shows that

$$\begin{aligned}
&(2-3\theta)\xi^2-(2-2\theta)\xi(\xi^3+2\xi)+(2-\theta)\xi^2(2\xi^2+1)-2\xi^3\xi\\
=&2\xi^2-2\xi(\xi^3+2\xi)+2\xi^2(2\xi^2+1)-2\xi^3\xi+\theta(-3\xi^2+2\xi(\xi^3+2\xi)-\xi^2(2\xi^2+1))=0.
\end{aligned}\tag{8.33}$$

We also have the following computation:

$$\begin{aligned}
&-(2-2\theta)\xi\big\{(-2\xi^2-\theta)\Psi_{[2200]}+\xi\Psi_{[1210]}\big\}+(2-\theta)\xi^2\big\{-2\xi\Psi_{[2200]}+2\xi\Psi_{[0220]}\big\}\\
&-2\xi^3\big\{-\xi\Psi_{[1210]}+(2-\theta)\Psi_{[0220]}\big\}\\
=&(-\xi)\big\{(2-2\theta)(\theta-2)+2\xi^2(2-\theta)\big\}\Psi_{[2200]}+\xi^2\big\{-(2-2\theta)+2\xi^2\big\}\Psi_{[1210]}\\
&-\xi^3\big\{-2(2-\theta)+2(2-\theta)\big\}\Psi_{[0220]}\\
=&0.
\end{aligned}\tag{8.34}$$

Computing $(2-3\theta)(8.29)-(2-2\theta)\xi(8.30)+(2-\theta)\xi^2(8.31)-2\xi^3(8.32)$ and making use of (8.33)-(8.34), we get

$$(2-3\theta)4\Psi_{[4000]}-(2-2\theta)\xi 3\Psi_{[3010]}+(2-\theta)\xi^2 2\Psi_{[2020]}-2\xi^3\Psi_{[1030]}=0. \tag{8.35}$$

Substituting (8.4) into (8.35), we get

$$\begin{aligned}
&(2-3\theta)4\Psi_{[4000]}-(2-2\theta)\xi 3\Psi_{[3010]}+(2-\theta)\xi^2 2\Psi_{[2020]}-2\xi^3\Psi_{[1030]}\\
=&(2-3\theta)4\xi\Phi_{[2100]}-(2-2\theta)\xi 3\big\{(1+\xi^2)\Phi_{[2100]}+\xi\Phi_{[1110]}\big\}\\
&+(2-\theta)\xi^2 2\big\{\xi\Phi_{[2100]}+(1+\xi^2)\Phi_{[1110]}+\xi\Phi_{[0120]}\big\}\\
&-2\xi^3\big\{\xi\Phi_{[1110]}+(1+\xi^2)\Phi_{[0120]}\big\}\\
=&\xi\big\{4(2-3\theta)-3(2-2\theta)(1+\xi^2)+2\xi^2(2-\theta)\big\}\Phi_{[2100]}\\
&+\xi^2\big\{-3(2-2\theta)+2(2-\theta)(1+\xi^2)-2\xi^2\big\}\Phi_{[1110]}\\
&+\xi^3\big\{2(2-\theta)-2-2\xi^2\big\}\Phi_{[0120]}\\
=&-4\xi\theta^2\Phi_{[2100]}+\xi^2 2\theta^2\Phi_{[1110]}=-2\xi\theta^2(2\Phi_{[2100]}+(-\xi)\Phi_{[1110]}).
\end{aligned} \tag{8.36}$$

Since $\theta\neq 0$ with the assumption that $\lambda_n\neq\frac{1}{2}$. Hence we get $2\Phi_{[2100]}+(-\xi)\Phi_{[1110]}=0$. From (8.3), we get

$$0=2\Phi_{[2100]}+(-\xi)\Phi_{[1110]}=2(H_{[1101]}-H_{[2010]})-\xi\big(\xi H_{[1101]}+H_{[0111]}-2H_{[1020]}\big). \tag{8.37}$$

By our normalization (8.9), we have $H_{[1101]}=H_{[0111]}=0$. Thus we get $2H_{[2010]}-2\xi H_{[1020]}=0$. By the reality of H, we obtain:

$$2(1-\xi)\Re H_{[2010]}+\sqrt{-1}\cdot 2(1+\xi)\Im H_{[2010]}=0.$$

When $\lambda_n\neq 1/2$, then $1-\xi\neq 0$. Hence we get $H_{[2010]}=0$.

Collecting the terms of the form $z_n^t\overline{z_n}^{6-t}$ $(0\leq t\leq 6)$ in (8.1), we get

$$|w_n|^2\overline{w_n}\sum_{t'+r'=3}\Psi_{[t'1r'0]}z_n^{t'}\overline{z_n}^{r'}=0.$$

Thus we get

$$\Psi_{[t1r0]}=0 \text{ for } t+r=3. \tag{8.38}$$

Combining this with (8.4), we get

$$\begin{aligned}
0=&\Psi_{[3100]}+(-\xi)\Psi_{[2110]}+(-\xi)^2\Psi_{[1120]}+(-\xi)^3\Psi_{[0130]}\\
=&2\xi\Phi_{[1200]}-2\eta\Phi_{[3000]}+(-\xi)\big\{2(1+\xi^2)\Phi_{[1200]}+2\xi\Phi_{[0210]}-3\xi\eta\Phi_{[3000]}-\eta\Phi_{[2010]}\big\}\\
&+(-\xi)^2\big\{2\xi\Phi_{[1200]}+2(1+\xi^2)\Phi_{[0210]}-2\xi\eta\Phi_{[2010]}\big\}\\
&+(-\xi)^3\big\{2\xi\Phi_{[0210]}-\xi\eta\Phi_{[1020]}+\eta\Phi_{[0030]}\big\}\\
=&(-2+3\xi^2)\eta\Phi_{[3000]}+(\xi-2\xi^3)\eta\Phi_{[2010]}+\xi^4\eta\Phi_{[1020]}-\xi^3\eta\Phi_{[0030]}\\
=&(1-3\theta)\eta\Phi_{[3000]}+(-\xi)(1-2\theta)\eta\Phi_{[2010]}+\xi^2(1-\theta)\eta\Phi_{[1020]}+(-\xi)^3\eta\Phi_{[0030]}.
\end{aligned} \tag{8.39}$$

Substituting (8.3) into the equation above, we get

$$
\begin{aligned}
&(1-3\theta)\eta H_{[2001]} + (-\xi)(1-2\theta)\eta\big(\xi H_{[2001]} + H_{[1011]}\big) \\
&+ (-\xi)^2(1-\theta)\eta\big(\xi H_{[1011]} + H_{[0021]}\big) + (-\xi)^3\eta\xi H_{[0021]} = 0.
\end{aligned} \tag{8.40}
$$

By (8.9), we have $H_{[1011]} = H_{[0021]} = 0$. Hence we get $-2\theta^2 H_{[2001]} = 0$. When $\theta \neq 0$ or $\xi \neq \frac{1}{2}$, we conclude that $H_{[2001]} = 0$. Together with $H_{[2010]}$, the normalization in (8.3) and the reality of H, we see that $H_{[tsrh]} = 0$ for $s, h \leq 1$. Substituting this back to (8.3) and (8.4), we get $\Phi_{[t'0r'0]} = \Phi_{[t''1r''0]} = 0$ and thus also $\Psi_{[t0r0]} = 0$ for $t + r = 4$ and $t' + r' = 3$, $t'' + r'' + 1 = 3$.

Collecting terms of the form $z_n^t \overline{z_1} \cdot \overline{z_n}^{5-t}$ $(0 \leq t \leq 5)$ in (8.1), and making use of $\Psi_{[t'0r'0]} = 0$, we get

$$
|w_n|^2 \overline{w_n} \sum_{t'+r'=3} \Psi_{[t'1r'1]} z_n^{t'} \overline{z_n}^{r'} = 0.
$$

Thus we get $\Psi_{[t1r1]} = 0$. Combining this with (8.4), we get

$$
\begin{aligned}
0 =& \Psi_{[2101]} + (-\xi)\Psi_{[1111]} + (-\xi)^2\Psi_{[0121]} \\
=& 2\xi\Phi_{[0201]} - \eta\Phi_{[2001]} + (-\xi)\big\{2(1+\xi^2)\Phi_{[0201]} - 2\xi\eta\Phi_{[2001]}\big\} \\
&+ (-\xi)^2\big\{2\Phi_{[0201]} - \xi\eta\Phi_{[1011]} + \eta\Phi_{[0021]}\big\} \\
=& (-1+2\xi^2)\eta\Phi_{[2001]} - \xi^3\eta\Phi_{[1011]} + \xi^2\eta\Phi_{[0021]} \\
=& (1-2\theta)\eta\Phi_{[2001]} + (-\xi)(1-\theta)\eta\Phi_{[1011]} + \xi^2\eta\Phi_{[0021]}.
\end{aligned} \tag{8.41}
$$

Substituting (8.3) into this equation, we get

$$
(1-2\theta)\eta 2H_{[1002]} + (-\xi)(1-\theta)\eta\big(2\xi H_{[1002]} + 2H_{[0012]}\big) + \xi^2\eta\xi 2H_{[0012]} = 0. \tag{8.42}
$$

By (8.9), we have $H_{[0012]} = 0$. Hence we get $-\theta^2 H_{[1002]} = 0$. Since $\theta \neq 0$, we see that $H_{[1002]} = 0$.

By (8.38), we have $\Psi_{[0130]} = 0$. Combining this with (8.4), we get

$$
\begin{aligned}
\Psi_{[0130]} =& 2\xi\Phi_{[0210]} - (\xi\eta\Phi_{[1020]} - \eta\Phi_{[0030]}) \\
=& 2\xi(\xi H_{[0201]} - 2H_{[0120]}) = 2\xi^2 H_{[0201]}.
\end{aligned} \tag{8.43}
$$

Here, we used the fact that $\Phi_{[1020]} = \Phi_{[0030]} = 0$ and $H_{[tsrh]} = 0$ for $s, h \leq 1$. Thus we get $H_{[0201]} = 0$. Now, combing the normalization in (8.4) with $H_{[0201]} = 0, H_{[2010]} = 0, H_{[tsrh]} = 0$ for $s, h \leq 1$, we conclude that $H \equiv 0$. This completes the proof of Theorem 1.2 for the case of $\lambda_n \neq 0$, $n = 2$ and $m = 3$.

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