Flattening of CR singular points and analyticity of the local hull of holomorphy I

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Abstract: This is the first article of the two papers in which we investigate the holomorphic and formal flattening problem for a codimension two real submanifold in $\mathbb{C}^n$ with $n \geq 3$ near a non-degenerate CR singular point. The problem is motivated from the study of the complex Plateau problem that seeks for the Levi-flat hypersurface bounded by a given real submanifold and is motivated by the classical complex analysis problem of finding the local hull of holomorphy of a real submanifold in a complex space. The present article is focused on the case of CR singular points with at least one elliptic direction. We solve the holomorphic flattening problem and thus provide a complete description of the local hull of holomorphy in this setting. The results in this paper and those in [HY3] are taken from our arxiv post [HY4]. We split [HY4] into two independent articles to avoid it being too long.

1 Introduction

A primary goal of this investigation is to study the question that asks when a real analytic submanifold M of codimension two in $\mathbb{C}^{n+1}$ bounds a real analytic (up to M) Levi-flat hypersurface $\widehat{M}$ near $p \in M$ such that $\widehat{M}$ is foliated by a family of complex hypersurfaces moving along the normal direction of M at p, and gives the invariant local hull of holomorphy of M near p. This question can be reduced to the study of the holomorphic flattening property for M near p.

To start with, we first discuss some basic holomorphic property for a real submanifold in a complex space. For a point q in a real submanifold $M \subset \mathbb{C}^{n+1}$, there is an immediate holomorphic invariant, namely, the complex dimension $CR_M(q)$ of the tangent space of type $(1,0)$ at q. $CR_M(q)$ is an upper semi-continuous function over M. q is called a CR point of M if $CR_M(q') \equiv CR_M(q)$ for any $q'(\approx q) \in M$. Otherwise, q is called a CR singular point of M. When M near p bounds a Levi-flat hypersurface foliated by a family of complex hypersurfaces moving along the normal direction of M at p, then the tangent space of M at p

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is a complex hyperplane. In this case p must be a CR singular point unless we are in the trivial and uninteresting situation that M is a complex hypersurface itself.

Investigations for CR manifolds and CR singular manifolds have very different nature. There is a vast amount of work related to the study of various problems for CR manifolds, which goes back to the work of Poincaré [Po], Cartan [Cat] and Chern-Moser [CM]. The study of submanifolds with CR singular points at least dates back to the fundamental paper of Bishop [Bis] in 1965. Since then, many efforts have been paid to understand both the geometric and analytic structures of such manifolds. Here, we mention the papers by Kenig-Webster [KW1-2], Moser-Webster [MW], Bedford-Gaveau [BG], Huang-Krantz [HK], Huang [Hu1], Gong [Gon1-3], Huang-Yin [HY1-2], Stolovitch [Sto], Dobeault-Tomassini-Zaitsev [DTZ1-2], Ahern-Gong [AG], Coffman [Cof1-2], Lebl [Le1-2], Burcea [Va1], etc, and many references therein.

Let $M \subset \mathbb{C}^{n+1}$ be a codimension two real submanifold with *CR singular points*. Then a simple linear algebra computation shows that $CR_M(q) = n-1$ when q is a CR point, and $CR_M(q) = n$ when q is a CR singular point. The general holomorphic (or, formal) flattening problem is then to ask when M can be transformed, by a biholomorphic mapping (formal equivalence, respectively), into the standard Levi-flat hyperplane $(\mathbb{C}^n \times \mathbb{R}^1) \times \{0\} \subset \mathbb{C}^{n+1}$. A good understanding to this problem is crucial for understanding many geometric, analytic and dynamic properties of the manifolds. For instance, by a classical theorem of Cartan, solving the problem when M bounds a real analytic (up to M) Levi-flat hypersurface is equivalent to solving the holomorphic flattening problem of the manifold. Here, we refer the reader to the papers by Kenig-Webster [KW1], Moser-Webster [MW], Huang-Krantz [HK], Gong [Gon1-3], Huang [Hu1], Stolovitch [Sto], Huang-Yin [HY1], Dobeault-Tomassini-Zaitsev[DTZ1], and many references therein, for investigations along these lines.

The major difficulty for getting the flattening property for M lies in the complicated nature of the CR singular points. And, in general, only non-degenerate CR singular points with a rich geometric structure could be flattened. To be more precise, we use $(z, w) := (z_1, \cdots, z_n, w)$ for the complex coordinates of $\mathbb{C}^{n+1}$. We first make the following definition. For related concepts and many intrinsic discussions on this matter, see the work in Stolovitch [Sto], Dobeault-Tomassini-Zaitsev [DTZ1], and Huang-Yin [HY2]:

Definition 1.1. Let M be a codimension two real submanifold in $\mathbb{C}^{n+1}$. We say that $q \in M$ is a non-degenerate CR singular point, or a non-degenerate complex tangent point, if there is a biholomorphic change of coordinates which maps p to 0 and in the new coordinates (z, w), M is defined near 0 by an equation of the following form:

$$w = \sum_{j=1}^{n} \left(|z_j|^2 + \lambda_j (z_j^2 + \overline{z_j^2}) \right) + o(|z|^2) \tag{1.1}$$

Here, $0 \leq \lambda_1 \leq \cdots \leq \lambda_n < \infty$. $\{\lambda_1, \cdots, \lambda_n\}$ (counting multiplicity) are called the Bishop invariants of M at 0. We call λ_j an elliptic, parabolic or hyperbolic Bishop invariant of M at 0 in terms of $0 \leq \lambda_j < 1/2$, $\lambda_j = 1/2$, or $\lambda_j > 1/2$, respectively.

Notice that the set of Bishop invariants at a non-degenerate CR singular point $p \in M$ consists of the only second order biholomorphic invariants of M at $p \in M$. By the results in Moser-Webster [MW] and Huang-Krantz [HK], in the case of complex dimension two ($n+1 = 2$), any real analytic real surface near an elliptic CR singular point can be holomorphically flattened. However, in higher dimension, the situation is quite different.

Example 1.1. *Let*

$$M := \{\ (z, w) \approx 0, w = \sum_{j=1}^{2} |z_j|^2 + 2\Re\left(\sum_{j_1+j_2\geq 3} a_{j_1 j_2} z_1^{j_1} z_2^{j_2}\right) + \sqrt{-1} \sum_{j_1\geq 2, j_2\geq 2} b_{j_1\overline{j_2}} z_1^{j_1} \overline{z_2}^{j_2},\ \ b_{j\bar{l}} = \overline{b_{l\bar{j}}}.\} \tag{1.2}$$

Here, we assume that the two power series in (1.2) are convergent. M then has a non-degenerate CR singular point at 0 and all Bishop invariants of M at 0 are 0 and thus are elliptic. It was shown in Huang-Yin [HY2] ([Remark 2.7, HY2]) that $(M, 0)$ can not even be flattened to the order m if $b_{j_1\overline{j_2}} \neq 0$ for some $j_1 + j_2 \leq m$. Namely, if $b_{j_1\overline{j_2}} \neq 0$ for some $j_1 + j_2 \leq m$, then there is no holomorphic change of variables (preserving the origin) such that in the new coordinates, M is defined near 0 by an equation of the form $w = \rho$ with the property that $\Im(\rho)$ vanishes at the origin to the order at least m. through the study of a formal normal form near a CR singular point with all vanishing (thus elliptic) Bishop invariants.

Example 1.1 shows that in higher dimensions, the geometry from the nearby CR points also play a role in the flattening problem, while in the two variables case, the nearby points are totally real and can be locally holomorphically flattened. Thus the nearby points in the two dimension case has no influence for the holomorphic property at a non-degenerate CR singular point. Indeed, suppose M is already flattened and is defined by an equation of the form $u = q(z, \overline{z}), v = 0$, where $w = u + iv$. Then the complex hypersurface $S_{u_0} =: \{w = u_0 + i0\}$ with $u_0 \in \mathbb{R}$ intersects M along a CR submanifold E of CR dimension $(n-1)$ near p_0 if S_{u_0} intersects M (CR) transversally at p_0. The points where S_{u_0} is (CR) tangent to M are apparently CR singular points of M. Recall a well-known terminology (see [T] and [Tu]): A point p in a CR submanifold N is called a non-minimal point if N contains a proper CR submanifold S containing p such that $T_p^{(1,0)}S = T_p^{(1,0)}N$. Hence, in such a terminology, we have the following simple fact:

If M can be flattened, then all CR points in M are non-minimal CR points.

We mention that the necessary condition for the non-minimality of CR points already appeared in the earlier work of Dobeault-Tomassini-Zaitsev [DTZ1-2] and Lebl [Le1-2] on the study of the general complex Plateau problem, which looks for the Levi-flat varieties (even maybe in the sense of current) bounded by the given manifolds.

Our main results, which we state below, demonstrate that, with the non-minimality assumption at CR points, the existence of one elliptic Bishop invariant is good enough for the holomorphic flattening:

Theorem 1.2. *Let $M \subset \mathbb{C}^{n+1}$ with $n > 1$ be a codimension two real analytic CR manifold with $p \in M$ a non-degenerate complex tangent point. Suppose one of the Bishop invariants λ of M at p is elliptic. Also assume that all CR points of M near p are non-minimal. Then the local hull of holomorphy $\widehat{M}$ of M near p is a real analytic Levi-flat hypersurface which has M near p as part of its real analytic boundary. Moreover $\widehat{M}$ is foliated by a family of smooth complex hypersurfaces in $\mathbb{C}^{n+1}$, that moves along the transversal direction of the tangent space of M at p.*

Here, by saying that $\widehat{M}$ is the local hull of holomorphy of M near p, we mean that for any small $0 < \epsilon << 1$, the germ at p of the holomorphic hull of the set $M \cap \{|(z,w)| < \epsilon\}$ coincides with the germ of $\widehat{M}$ at p.

As we mentioned above, by the classical Cartan theorem ([Cat]), Theorem 1.2 gives the following flattening theorem:

Theorem 1.3. *Let $M \subset \mathbb{C}^{n+1}$ with $n > 1$ be a codimension two real analytic CR manifold with $p \in M$ a non-degenerate complex tangent point (namely, a non-degenerate CR singular point). Suppose that one of the Bishop invariants λ of M at p is elliptic. Then M near p can be holomorphically flattened if and only if all CR points of M near p are non-minimal.*

When M is merely smooth, our proof of Theorem 1.2 also produces a formal flattening result for M near p. (See Theorem 2.2). In fact, through a more lengthy (but quite different) argument which we will present in the second article to these series [HY3], as far as the formal flattening is concerned, we need only the existence of one non-parabolic Bishop invariant. (See Theorem 1.1 of [HY3])

Example 1.4. Define $M \subset \mathbb{C}^3$ with coordinates (z_1, z_2, w) by the following equation near 0:

$$w = q(z,\overline{z}) + p(z,\overline{z}) + iE(z,\overline{z}).$$

Here $q = |z_1|^2 + \lambda_1(z_1^2 + \overline{z_1^2}) + |z_2|^2 + \lambda_2(z_2^2 + \overline{z_2^2})$ with $0 \leq \lambda_1, \lambda_2 < \infty$, and

$$p(z,\overline{z}) + iE(z,\overline{z}) = \mu_1|z_1|^2(z_1 + \lambda_1\overline{z_1}) + \mu_2|z_2|^2(z_2 + \lambda_2\overline{z_2}) + \mu_1 z_1(|z_2|^2 + \lambda_2\overline{z_2^2}) + \mu_2 z_2(|z_1|^2 + \lambda_1\overline{z_1^2}).$$

Here μ_1, μ_2 are two complex numbers. Then, M is non-minimal at its CR points near its non-degenerate CR singular point 0. (See Example 3.1.) Hence, Theorem 1.1 of [HY3] says that when one of the λ_1, λ_2 is not $\frac{1}{2}$, then M can be formally flattened at 0; and when one of the λ_1, λ_2 is less than $\frac{1}{2}$, then Theorem 1.3 says that M can be holomorphically flattened near 0.

In this example, $M \setminus \{0\}$ near 0 is foliated by a family of three dimensional strongly pseudoconvex CR manifolds— the intersections of M with real hypersurfaces $K_c : q(z,\overline{z}) = c$ with $c \in \mathbb{R}$. (When both λ_1, λ_2 are elliptic, $c > 0$). Assume that one of the Bishop invariants $\{\lambda_1, \lambda_2\}$ is not elliptic. Then there is an orbit corresponding to $c = 0$, that extends to the CR singular point with it as its non-smooth point. Also none of the orbits closes up near 0.

We next say a few words about the proof of the above mentioned theorems: To prove Theorem 1.2, we first slice M near the complex tangent point p by a family of two dimensional complex planes transversal to the elliptic direction. We then get a family of elliptic Bishop surfaces. Now each one bounds a three dimensional Levi flat CR manifold and their union forms a codimension one subset $\widetilde{M}$ in $\mathbb{C}^{n+1}$ with M as part of its boundary. An analysis, based on Bishop disks, similar to that in Huang-Krantz [HK], shows that $\widetilde{M}$ is a real analytic hypersurface with M as part of its real analytic boundary. However, all we know from this construction is that $\widetilde{M}$ has only one Levi-flat direction (along the elliptic direction). And it is not clear at all if $\widetilde{M}$ is flat along the parameter directions. In fact, $\widetilde{M}$ can not be Levi flat without the non-minimality property from the nearby CR points. Now, the issue is that, with the assumption of the non-minimality at the nearby CR points, we can show that $\widetilde{M}$ has a nearby open piece which is Levi-flat. Since the Levi-form of a real analytic hypersurface can be made to be real analytic, by the uniqueness of real analytic functions, we conclude that $\widetilde{M}$ has its Levi-form vanishing everywhere. This proves $\widetilde{M}$ is Levi-flat. Next by the Cartan theorem, $\widetilde{M}$ can be holomorphically transformed to the standard Levi-flat hypersurface defined by $\Im w = 0$. Then one can easily see that $\widetilde{M}$ serves as the local hull of holomorphy of M near the CR singular point p.

Theorem 1.3 is equivalent to Theorem 1.2 by a classical result of Cartan which states that a real analytic hypersurface is Levi-flat if and only if it can be transformed locally to an open piece of the standard Levi-flat hyperplane defined by $\Im w = 0$. When all Bishop invariants at p are elliptic, we mention that Theorem 1.2 can also be derived by combining the results obtained in Dobeault-Tomassini-Zaitsev [DTZ1-2] (see a recent preprint by Burcea [Bur2]). (The work in Dobeault-Tomassini-Zaitsev [DTZ1-2] contains other very nice global results.) The arguments based on Dobeault-Tomassini-Zaitsev [DTZ1-2] depend strongly on all the ellipticity of Bishop invariants and requires that the CR orbits in M near the CR singular point form a family of compact strongly pseudoconvex manifolds shrinking down to the complex tangent point such that the Harvey-Lawson theorem applies. This is not the case even when one non-elliptic Bishop invariant at the CR singular point appears.

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2 The geometry for manifolds with at least one elliptic Bishop invariant

Let M be a (connected) real submanifold of real codimension two in $\mathbb{C}^{n+1}$ with $n \geq 3$. In what follows, we use $Z = (z,z', w) = (z_1, z_2, \cdots, z_n, w)$ for the coordinates of $\mathbb{C}^{n+1}$ with $z' = (z_2, \cdots, z_n)$. We assume that M is not a complex submanifold, neither a CR manifold. Then we have points in M where the complex dimension of holomorphic tangent space is n and we have points where the holomorphic tangent space is of complex dimension $(n-1)$. We write $\mathcal{S}_M$ for the points of M where the holomorphic tangent space has complex dimension n. Then $M \setminus \mathcal{S}_M$ is a CR submanifold of CR dimension $(n-1)$. We assume that $\mathcal{S}_M$ is nowhere dense in M. Notice that $\mathcal{S}_M (\neq \emptyset)$ is the set of the CR singular points of M.

Assume that $p \in M$ is a CR singular point. After a (holomorphic) linear change of coordinates, we assume that $p = 0$ and $\dim T^{(1,0)}_0 M = \{w = 0\}$. Then M near 0 is defined by an equation of the following form:

$$w = F(z, \overline{z}) = q(z, \overline{z}) + O(|z|^3), \tag{2.1}$$

where $q(z, \overline{z})$ is a quadratic polynomial in $(z, \overline{z})$. We assume that 0 is a non-degenerate CR singular point as defined in the introduction. Then q can be further simplified as follows, after a holomorphic change of coordinates:

$$q = \sum_{j=1}^{n} (|z_j|^2 + \lambda_j (z_j^2 + \overline{z_j^2})), \quad 0 \leq \lambda_j < \infty.$$

For such a manifold, the set of CR singular points is defined by the following system of equations: $\frac{\partial F}{\partial \overline{z}_j} = z_j + 2\lambda_j \overline{z}_j + o(|z|) = 0,\ j = 1, \cdots, n$; or $(1+2\lambda_j)x_j = o(|(x, y)|),\ (1-2\lambda_j)y_j = o(|(x, y)|)$ with $x = \Re(z), y = \Im(z)$. Hence, if not all $\lambda_j = 1/2$, we can easily see that $\mathcal{S}_M$ is contained in a submanifold of M of real dimension at most $(n-1)$, which is smaller than $2(n-1)$ for $n \geq 2$. Moreover $\rho = -u + \Re F(z, \overline{z}) = -u + \sum_{j=1}^{n} (|z_j|^2 + \lambda_j (z_j^2 + \overline{z_j^2})) + o(|z|^2)$ is plurisubharmonic in a neighborhood of 0 and $\rho = 0$ defines a strongly pseudoconvex hypersurface near 0 which contains M. Hence, if M is non-minimal at its CR points, then each CR foliation must be strongly pseudoconvex.

More generally, we consider the class of real codimension two submanifolds (M, p) with $p \in \mathcal{S}_M$, satisfying the following hypotheses:

Cond (1): *The Hausdorff dimension of $\mathcal{S}_M$ is less than $2(n-1)$ and the CR points of M are non-minimal. Also, for any $q(\approx p) \notin \mathcal{S}_M$, there exist a neighborhood D_q of q in $\mathbb{C}^{n+1}$ and a continuous plurisubharmonic function ρ_q over D_q such that the zero set of ρ_q contains $M \cap D_q$ but does not contain any germ of non-trivial holomorphic curve.*

Cond (2): *There exits a complex surface E, that is transversal to the holomorphic tangent space $T^{(1,0)}_p M$ of M at p, such that $E \cap M$ is a Bishop surface of E with an elliptic complex tangency at p.*

Here, we recall that a real surface M contained in a complex surface E is said to be a Bishop surface with an elliptic complex tangency at $p \in M$ if there is a holomorphic chart with coordinates $(z, w) \in \mathbb{C}^2$ of E near p, that maps p to the origin of $\mathbb{C}^2$ and maps M to a real surface near 0 defined by an equation of the form:

$$w = |z|^2 + \lambda(z^2 + \overline{z^2}) + O(|z|^3), \quad 0 \le \lambda < \frac{1}{2}.$$

We also give the following condition for the description of the holomorphic hull.

Cond (3): *M is a codimension two real submanifold in $\mathbb{C}^{n+1}$ with p being a CR singular point. There are a neighborhood D_0 of p in $\mathbb{C}^{n+1}$ and a continuous plurisubharmonic function ρ_0 over D_0 such that the zero set of ρ_0 contains $M \cap D_0$ but does not contain any non-trivial holomorphic curves.*

Our main theorem of this section states that under the hypotheses in Cond (1) and Cond (2), assuming that M is real analytic, then M bounds a real analytic Levi-flat hypersurface $\widetilde{M}$ which has M as part of its real analytic boundary. Moreover, $\widetilde{M}$ serves as the local hull of holomorphy of M near 0 in case (M, p) also satisfies Cond (3). In the smooth category, we will show that M can be formally flattened. Notice that the assumption that M is non-minimal at its CR points gives a CR foliation by CR manifolds of CR dimension $(n-1)$ near each of its CR points. Indeed, what is crucial for our argument is the hypotheses in Cond (2).

For the M as in Theorem 1.2, after a holomorphic change of coordinates which maps $p = 0$ and makes M near $p = 0$ into its second order normal form as in Definition 1.1, we easily see that Cond (1), Cond (2) and Cond (3) hold for such an $(M, 0)$ as argued before. We next state the main result of this section as follows, which includes Theorem 1.2 as a special case:

Theorem 2.1. *Let $M \subset \mathbb{C}^{n+1}$ with $n > 1$ be a real codimension two real analytic submanifold near the origin. Assume that $p \in M$ is a CR singular point and M is non-minimal at all of its CR points. Suppose that the hypotheses in Cond (1) and Cond (2) hold for (M, p). Then M near p bounds a Levi-flat hypersurface $\widetilde{M}$ which has M as part of its real analytic boundary. Moreover, $\widetilde{M}$ near p is the local hull of holomorphy of M near p if the hypothesis in Cond (3) also holds for (M, p).*

Here, by saying that $\widetilde{M}$ is the local hull of holomorphy of M near p, we mean that for any small $0 < \epsilon << 1$, the germ at p of the holomorphic hull of the set $M \cap \{|Z - p| < \epsilon\}$ coincides with the germ of $\widetilde{M}$ at p.

When M is merely smooth, we provide the following weaker result:

Theorem 2.2. *Let $M \subset \mathbb{C}^{n+1}$ with $n > 1$ be a real codimension two smooth submanifold with $p \in M$. Assume that p is a CR singular point. Suppose that the hypotheses in Cond (1) and Cond (2) hold for (M, p). Then M near p can be holomorphically flattened to any order. Namely, for any positive integer m, there is a holomorphic change of coordinates which maps p to 0 and maps M to a manifold defined by an equation of the form $w = \rho(z, \overline{z})$ with $\Im \rho$ vanishing at least to the order m at the origin.*

Our proof of Theorem 2.1 is based on the following parametrized version of a result of Huang and Krantz[HK].

Theorem 2.3. *Let M be a real analytic hypersurface with $0 \in M$ being its CR singular point. Suppose that there is a holomorphic change of coordinates preserving the origin such that M in the new coordinates (which for simplicity we still write as (z, w)) is defined by an equation of the form:*

$$w = G(z, \overline{z}) + iE(z, \overline{z}) = O(|z|^2), \ (G + iE)(z_1, 0, \overline{z_1}, 0) = |z_1|^2 + \lambda_1(z_1^2 + \overline{z_1}^2) + o(|z_1|^2). \quad (2.2)$$

Here the constant λ_1 is such that $0 \leq \lambda_1 < \frac{1}{2}$ and the real analytic functions G, E are real-valued. Then, there is a positive constant $\epsilon_0 > 0$ such that for any $\vec{t} \in \mathbb{C}^{n-1}$ with $|\vec{t}| < \epsilon_0$, $M_{\vec{t}} := M \cap \{z' = \vec{t}\}$ bounds a Levi-flat real-analytic three dimensional submanifold $\widetilde{M_{\vec{t}}}$ that has $M_{\vec{t}}$ as part of its real analytic boundary and $\widetilde{M_{\vec{t}}}$ is contained in a real analytic submanifold foliated by a family of holomorphic disks attached to $M_{\vec{t}} \cap \{|(z, w)| < \epsilon_0'\}$ with $0 < \epsilon_0 << \epsilon_0' << 1$, shrinking down to a certain point $P(\vec{t}) \in M$. Moreover $\widetilde{M^{\#}} := \cup_{|\vec{t}| < \epsilon_0} \widetilde{M_{\vec{t}}}$ is a real analytic hypersurface with M near 0 as part of its real analytic boundary and $P(t)$ is a real analytic embedding from $\{|\vec{t}|\} < \epsilon_0$ into M.

We will present a proof of Theorem 2.3 in the next section. The above result in the smooth category or the parametrized result of Kenig-Webster [KW], which will be used to prove Theorem 2.2 was done in [Bur2] more or less in the same time as our work in [HY4] where a version of Theorem 2.3 was first given, also by modifying the construction of holomorphic disks in Huang-Krantz [HK].

Proofs of Theorem 2.1 and Theorem 2.2: We next proceed to the proof of Theorem 2.1. We assume the hypotheses in Cond (1) and Cond (2) hold for (M, p). After a holomorphic change of coordinates, we also assume that the CR singular point is at the origin. We first pick a point $q(\approx 0) \in M \setminus \mathcal{S}_M$. Notice that the hypothesis in Cond (1) holds near q and the CR leaf through q must be a CR submanifold of hypersurface type of CR dimension $(n-1)$. After a linear (holomorphic) change of coordinates, we assume that $q = 0$ and $T_0 M = \{y_1 = \Im z_1 = 0, \ v = 0\}$ and $T_0^{(1,0)} M = \{z_1 = w = 0\}$. Performing a linear transformation, we also assume that $\frac{\partial}{\partial x_1}|_0$ is tangent to the CR foliation at 0 and $\frac{\partial}{\partial u}|_0$ is transversal to the CR foliation of M near $q = 0$. We assume that M is of C^a-smoothness, where when $a = \infty$, M is smooth and when $a = \omega$, M is assumed to be real analytic. Now, by a basic fact from the foliation theory, we can find a real valued function $t(Z)$ defined over a certain neighborhood U_0 of $q = 0$ in M such that for each $t_0 \in I_{\delta_0} = (-\delta_0, \delta_0)$ with a certain small $0 < \delta_0 << 1$, $M_{t_0} = \{Z \in U_0, t(Z) = t_0\}$ is a connected CR submanifold of hypersurface type of CR dimension $(n-1)$. Moreover $dt|_{U_0} \neq 0$. We assume that $0 \in M_{t_0=0}$. Define $\Psi : U_0 \to \mathbb{C}^n \times \mathbb{R}$ by sending $Z = (z_1, z', w) \in U_0$ to $\Psi(Z) = (z_1, z', t(Z))$. After shrinking U_0 and δ_0 if needed, we can assume that Ψ is a C^a-embedding. Write ${M^*}_t$ for $\Psi(M_t)$ for each $t \in I_{\delta_0}$. Since each component of Ψ is the restriction of a holomorphic function over M_t, Ψ is a CR diffeomorphism from M_t to M_t^*.

Now, by the hypothesis in Cond (1), shrinking U_0 and δ_0 if needed, we can find a neighborhood D of $q = 0$ in $\mathbb{C}^{n+1}$ and a continuous plurisubharmonic function ρ_D such that $U_0 \subset H = \{\rho_D = 0\}$ and $\rho_D < 0$ defines the pseudoconvex side D^- of H. Hence, $\{M_t^*\}_t$ is a family of real hypersurface, depending C^a-smoothly on the parameter. Since M_t^* cannot contain any non-trivial holomorphic curve, M_t^* is a minimal CR manifold. By the Trepreau theorem, there is a domain $\widetilde{M^*}_t$ in $\mathbb{C}^n \times \{t = t(M_t)\}$ having M_t^* as part of its C^a-smooth boundary, that can be filled in by small holomorphic disks attached to M_t^*, such that any CR function defined over $\overline{M_t^*}$ extends holomorphically to $\widetilde{M^*}_t$. Writing Ψ_t^{-1} for the holomorphic extension of $\Psi^{-1}|_{M_t^*}$ to $\widetilde{M^*}_t$ and considering the plurisubharmonic function $\rho_D \circ \Psi_t^{-1}$ over $\widetilde{M^*}_t$, since we assumed that the zero set of ρ_D does not contain any non-trivial holomorphic curve, we see that $\rho_D \circ \Psi_t^{-1} < 0$ in $\widetilde{M^*}_t$. Hence, each $\widetilde{M^*}_t$ is pseudoconvex.

In what follows, we write $\epsilon_0, \epsilon_1, \epsilon_2$, for sufficiently small positive constant. Choose a curve $\gamma(t)$ in U_0 through $p = 0$, over which $t(Z)$ is a diffeomorphism to I_{δ_0} with $t(\gamma(t)) = t$. We construct $\widetilde{M^*}_t$ such that for a certain sufficiently small (fixed) number ϵ', $\widetilde{M^*}_t$ contains all points in the pseudoconvex side of M_t^* that is at most ϵ'-distance away from $\Psi(\gamma(t))$ for any $t \in I_{\delta_0}$. Write $\widetilde{M^*} = \left(\cup_{t\in I_{\delta_0}} \widetilde{M^*}_t\right) \cap \{|Z^*| < \epsilon_1\}$, where Z^* is also the coordinate for $\mathbb{C}^{n+1}$ and $\epsilon_1 << \epsilon'$. Then $\widetilde{M^*}$ is a C^a-regular Levi-flat hypersurface with $M^* \cap \{|Z^*| < \epsilon_1\}$ as part of its C^a-regular boundary. Write $\Phi = \Psi^{-1} : M^* \to U_0$. Shrinking U_0 and δ_0 appropriately if needed, Φ is a diffeomorphism and is a CR diffeomorphism when restricted to each M_t^*. Moreover, Φ extends to a holomorphic embedding when restricted to each $\widetilde{M^*}_t$. Still denote the extended map by Φ and write its inverse as Ψ. By tracing the Baouendi-Treves approximation theorem and the disk filling property of each $\widetilde{M^*}_t$, we can see that Φ is a smooth embedding over $\widetilde{M^*}$ and holomorphic over each leaf $\widetilde{M^*}_t \cap \{|Z^*| < \epsilon_1\}$. Now, as in [Hu1], we can apply the Whitney extension theorem to extend Φ almost holomorphically ($\overline{\partial}-$ flatly) from $\widetilde{M^*}$ to a neighborhood of $\widetilde{M^*}$ in $\mathbb{C}^n \times \mathbb{C}$. Namely $\overline{\partial}\Phi$ vanishes to infinitely order along $\widetilde{M^*}$. Still write this $\overline{\partial}$-flat extension as Φ and write Ψ as its inverse. Write $Z^* = (z, t + \sqrt{-1}\tau)$ as mentioned above. Fix ϵ_0 and ϵ_0^* such that $0 < \epsilon_0 << \epsilon_0^* << 1$. For any $0 < \epsilon < 1$, we define $\rho_\epsilon^\pm = \pm\tau + \epsilon(|Z^*|^2 - 1)$ where $|Z^*| < \epsilon_0^*$. Define $\Omega_\epsilon = \{Z \in \{|Z| < \epsilon_0^*\} : \rho_\epsilon^\pm \circ \Psi(Z) < 0, \rho_D < \epsilon\}$. Write $\widetilde{M} = \Phi(\widetilde{M^*})$. Then it is easy to see that $(\cap_{0<\epsilon<<1}\Omega_\epsilon) \cap \{|Z| \le \epsilon_0\} = \widetilde{M} \cap \{|Z| \le \epsilon_0\}$.

We claim that $\rho_\epsilon^\pm \circ \Psi(Z)$ are strongly plurisubharmonic in a neighborhood of Ω_ϵ when $\epsilon > 0$ is sufficiently small. To see that, since Φ is $\overline{\partial}$-flat along $\widetilde{M^*}$, we see that its inverse Ψ is also $\overline{\partial}$-flat along $\widetilde{M}$. For $Z \in \overline{\Omega_\epsilon}$ with $dist(Z, \widetilde{M} \cap \{|Z| \le \epsilon_0\}) \le \epsilon_1$, we have the the following computation:

$$\begin{aligned}\partial\overline{\partial}\rho_\epsilon^\pm \circ \Psi(Z) &= \partial\overline{\partial}\{\pm\psi_{n+1} + \epsilon(\sum_{j=1}^{n+1} |\psi_j|^2 - 1)\} \\ &= \sum_{j=1}^{n+1} \mathrm{C}(\epsilon + O(\epsilon^2)) dz_j \wedge \overline{dz_j} + \sum_{j,k=1}^{n+1} O(\epsilon^2) dz_j \wedge \overline{dz_k}.\end{aligned} \tag{2.3}$$

In the above, we have written $\Psi = (\psi_1, \cdots, \psi_{n+1})$ and $w = z_{n+1}$. Also C is a certain positive constant. Hence, the complex Hessian of $\rho^{\pm}_{\epsilon}$ are positive definite over Ω_ϵ for $0 < \epsilon << 1$. Now from the way Ω_ϵ is defined, it is apparently pseudoconvex and we thus conclude that $\widetilde{M} \cap \{|Z| \leq \epsilon_0\}$ is holomrphically convex. By the continuity principle, we see that its germ at 0 is precisely the germ of holomorphic hull of $M \cap \{|Z| \leq \epsilon_1\}$ with $0 < \epsilon_1 << \epsilon_0 << 1$. As an immediate consequence of the above construction, we have the following:

Proposition 2.4. *There exits a sufficiently small positive number ϵ such that for any continuous family of holomorphic disks $\{\phi_t\}_{0 \leq t \leq 1}$ attached to $M \cap \{|Z|\} < \epsilon$ with $\phi_{t=0}(\Delta)$ being a point in M, it holds that $\phi_t(\Delta) \subset \widetilde{M}$ for any $0 \leq t \leq 1$.*

Here, we recall that a holomorphic disk ϕ is said to be attached to M if ϕ is a holomorphic map over the unit disk Δ, continuous up to the boundary (the unit circle S^1) and it holds that $\phi(S^1) \subset M$. We say $\{\phi_t\}$ is a continuous family with parameter t if ϕ is a continuous function in (ξ, t) for $\xi \in \overline{\Delta}$.

Proof of Proposition 2.4: Fix a sufficiently small positive number ϵ. By the hypothesis, for any $0 < \epsilon' << 1$, we have $\rho^{\pm}_{\epsilon'} \circ \phi_t(\xi) < 0$, $\rho_D \circ \phi_t(\xi) < \epsilon'$ for $|\xi| = 1$ and $t \in [0, 1]$. For $0 \leq t << 1$, we also have $\rho^{\pm}_{\epsilon'} \circ \phi_t(\xi) < 0$, $\rho^{\pm}_D \circ \phi_t(\xi) < \epsilon'$ for $\xi \in \Delta$. Now suppose t_0 is smallest t such that the above does not hold. Then we have a $\xi_0 \in \Delta$ such that either $\rho^{+}_{\epsilon'} \circ \phi_t(\xi_0) = 0$, or $\rho^{-}_{\epsilon'} \circ \phi_t(\xi_0) = 0$, or $\rho_D \circ \phi_t(\xi_0) = \epsilon'$. By the maximum principle for subharmonic functions, we get either $\rho^{+}_{\epsilon'} \circ \phi_t \equiv 0$, or $\rho^{-}_{\epsilon'} \circ \phi_t \equiv 0$, or $\rho_D \circ \phi_t \equiv \epsilon'$. This is a contradiction. Since $\epsilon' << 1$ is arbitrary, we see the proof of the proposition. ■

Now let $\widetilde{M^{\#}}$ and $\widetilde{M_{\vec{t}}}$ be as constructed in Theorem 2.3. Let $p(t)$ be the embedding from $\{t \in \mathbb{C}^{n-1} : |t| < \epsilon_2\}$ with $0 < \epsilon_2 << 1$ into M with $p(0) = 0$, where $p(t)$ is the complex tangent point of $M_{\vec{t}}$. Since $\mathcal{S}_M$ is assumed to have Hausdorff dimension less than $2(n-1)$, we can find a sequence of points $\{p_j = p(t_j)\}$ with $t_j \to 0$ such that the assumption in Cond(1) holds. Let $\widetilde{M_j}$ be the local hull of holomorphy of M near p_j, as constructed above, which is a C^a-smooth Levi-flat hypersurface with M near p_j as part of its C^a-smooth boundary. Notice that $\widetilde{M_{\vec{t_j}}}$ is foliated by holomorphic disks shrinking down to p_j. Hence, by Proposition 2.4, we see that for a certain small ϵ, $\widetilde{M_{\vec{t}}} \cap \{|Z - p_j| < \epsilon\}$ is contained in $\widetilde{M_j}$ for each $\vec{t} \approx \vec{t_j}$. Hence the germ of $\widetilde{M^{\#}}$ at p_j is contained in the germ of $\widetilde{M_j}$ at p_j. Since near p_j, both $\widetilde{M_j}$ and $\widetilde{M^{\#}}$ are smooth, share the same boundary, and have the same dimension. We can easily conclude that they completely coincide near p_j.

Now, when M is real analytic, $\widetilde{M^{\#}}$ is also real analytic and thus its Levi-form can be made to be real analytic too. Since the Levi form vanishes in a small neighborhood of p_j in $\widetilde{M^{\#}}$, we conclude that $\widetilde{M^{\#}}$ is Levi flat everywhere. Hence, $\widetilde{M^{\#}}$ is a real analytic Levi-flat hypersurface with M near 0 as part of its smooth boundary. By the classical Cartan theorem, $\widetilde{M^{\#}}$ can be biholomorphically transformed into $v = 0$. Now, assuming that the hypothesis in Cond (3) holds, it is easy to argue, as above, that $\widetilde{M^{\#}}$ also serves as the local hull of holomorphy of

M near 0. This completes the proof of Theorem 2.1. Thus the proof of Theorem 1.2 is also complete. ■

Now, we assume that M is merely smooth. As we mentioned before, modifying the argument in Kenig-Webster [KW1] and Huang-Krantz [HK], Burcear in [Bur2] also constructed $\widetilde{M\#}$ which is now only a smooth real-hypersurface with M as part of its smooth boundary. We first make $p = 0$ and $T_0\widetilde{M\#} = \{v = 0\}$. We extend $\widetilde{M\#}$ smoothly across M, which we still denote by $\widetilde{M\#}$. We then have the following property for $\widetilde{M\#}$:

There is a sequence of points $\{p_j\} \subset \widetilde{M\#}$, converging to 0, and an open neighborhood U_j of p_j in $\widetilde{M\#}$ for each j, such that $\widetilde{M\#}$ is Levi-flat over U_j.

Now the proof of Theorem 2.2 follows from this property and the following lemma:

Lemma 2.5. *Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface containing the origin. Suppose that there is a sequence $\{p_j\} \subset M$ with $p_j \to 0$ and there is an open subset $p_j \in U_j$ of M such that U_j is Levi-flat for each j. Then for any positive integer N, there is a holomorphic change of coordinates preserving the origin such that in the new coordinates, M is defined by an equation of the form: $v = \rho_N(z, \overline{z}, u) = O(|(z, u)|^N)$.*

Proof of Lemma 2.5: Suppose that the statement in the lemma does not hold. Then we have a sufficiently large integer N such that for any holomorphic change of coordinates preserving the origin, M can can not be defined by an equation of the form: $v = O(|(z, u)|^{N+1})$.

We first choose a coordinate system such that M is defined at the point under study by an equation of the form:

$$v = \rho = \rho_N(z, \overline{z}, u) + O(|(z, u)|^{N+1}),$$

where $\rho_N(z, \overline{z}, u) = o(|(z, u)|)$ is a real-valued polynomial of degree N. By Lemma 3.2 of Chern-Moser [CM], we can find a holomorphic change of coordinates preserving the origin such that in the new system, we can make $\rho_N(z, 0, u) = 0$.

After a linear change of coordinates, we can assume that $\rho_N((z_1, 0), \overline{(z_1, 0)}, u) \not\equiv 0$.

Now, for each $t \in \mathbb{C}^{n-1}$ near 0, define $r_t = -v + \rho((z_1, t), \overline{(z_1, t)}, u)$ and

$$\begin{aligned} L_t &= \frac{\partial}{\partial z_1} + 2i \frac{\rho_{z_1}((z_1, t), \overline{(z_1, t)}, u))}{1 - i\rho_u((z_1, t), \overline{(z_1, t)}, u))} \frac{\partial}{\partial w}; \\ \lambda_t &=< \partial\overline{\partial} r_t, L_t \wedge \overline{L_t} >; \\ T_t &= \frac{\partial}{\partial u} + \rho_u((z_1, t), \overline{(z_1, t)}, u) \frac{\partial}{\partial v}. \end{aligned} \tag{2.4}$$

L_t is a non-vanishing CR vector field along the three dimensional CR manifold M_t, T_t is a tangential, but normal to the CR-direction, vector field along M_t and λ_t is the Levi-function over M_t with respect to the frames such chosen. We will treat t as a parameter. Notice all these quantities depend smoothly on t. We consider the following two cases.

Case I: Assume that $\rho((z_1,0),\overline{(z_1,0)},0) = P_k(z_1,\overline{z_1}) + o(|z_1|^k)$ with $2 \le k \le N$, where $P_k(z_1,\overline{z_1})$ is a non-zero homogeneous real-valued polynomial of degree k. Assume that $\frac{\partial^k P_k}{\partial z_1^{\alpha+1}\partial \overline{z_1}^{\beta+1}} \neq 0$ for a certain pair $\alpha,\beta \ge 0$ with $\alpha+\beta = k-2$. Next, we notice that (see [(3.19), Ko], for instance)

$$\lambda_t = \frac{4}{|1-i\rho_u|^2}\{r_{z_1\overline{z_1}}|r_w|^2 + r_{w\overline{w}}|r_z|^2 - r_{z_1\overline{w}}r_w r_{\overline{z}_1} - r_{\overline{z_1}w}r_{\overline{w}}r_{z_1}\}.$$

We compute that

$$L_{t=0} = \frac{\partial}{\partial z_1} + O(|z_1|^{k-1} + u|z_1|)\frac{\partial}{\partial w},\ \ \lambda_{t=0} = (P_k)_{z_1\overline{z_1}} + O(|z_1|^{k-1} + u|z_1|^2).$$

Hence, one sees that

$$L_{t=0}\lambda_{t=0} = (P_k)_{z_1^2\overline{z_1}} + O(|z_1|^{k-2} + u|z_1|), \overline{L_t}\lambda_{t=0} = (P_k)_{z_1\overline{z_1^2}} + O(|z_1|^{k-2} + u|z_1|).$$

By an induction argument, we have

$$L^{\alpha}_{t=0}\overline{L^{\beta}_{t=0}}\lambda_{t=0}|_0 = \frac{\partial^k P_k}{\partial z_1^{\alpha+1}\partial \overline{z_1}^{\beta+1}} \neq 0.$$

However, by the assumption, there is a sequence of t_j such that M_{t_j} is Levi flat over a certain open piece $U'_j = \{z' = t_j\} \cap U_j$ for each j, for which 0 is an accumulation point of $\{U'_j\}$. Since the Levi function vanishes identically over a Levi-flat piece, by passing to a limit, we reach a contradiction.

Case II: Assume that $\rho((z_1,0),\overline{(z_1,0)},u) = u^j\left(Q_{k-j}(z_1,\overline{z_1}) + O(|z_1|^{k-j+1}) + O(|uz_1^2|)\right) + O(|z_1|^{N+1})$ for some $j \ge 1$ with $2 \le k \le N$, where $Q_{k-j}(z_1,\overline{z_1})$ is a non vanishing homogeneous real-valued polynomial of degree $k-j \ge 2$. Then

$$L_{t=0} = \frac{\partial}{\partial z_1} + O(|u^j z_1|)\frac{\partial}{\partial w},\ \lambda_{t=0} = u^j\left((Q_{k-j})_{z_1\overline{z_1}} + O(|z_1|^{k-j-1}) + O(|u|)\right) + O(|(z_1,u)|^{N-1}).$$

Hence $T^j_{t=0}\lambda_{t=0} = j!(Q_{k-j})_{z_1\overline{z_1}} + O(|z_1|^{k-j-1}) + O(|u|)$, and

$$L_{t=0}T^j_{t=0}\lambda_{t=0} = j!(Q_{k-j})_{z_1^2\overline{z_1}} + O(|z_1|^{k-j-2} + u), \overline{L_{t=0}}T^j_{t=0}\lambda_{t=0} = j!(Q_{k-j})_{z_1\overline{z_1^2}} + O(|z_1|^{k-j-2} + u).$$

By an induction argument, we have

$$L^{\alpha}_{t=0}\overline{L^{\beta}_{t=0}}T^j_{t=0}\lambda_{t=0}|_0 \neq 0$$

for a certain pair (α,β) with $\alpha+\beta = k-j$. Again $\lambda_{t_j} \equiv 0$ over $U_j \cap \{z' = t\}$ and thus $L^{\alpha}_{t=t_j}\overline{L^{\beta}_{t=t_j}}T^j_{t=t_j}\lambda_{t=t_j}|_0 \equiv 0$ over $U_j \cap M_{t_j}$ as above. Passing to the limit, this is again a contradiction.

The proof of Lemma 2.5 and thus the proof of Theorem 2.2 are complete. ∎

Hence, the proof of Theorem 1.3 is complete. ∎.

3 A parametrized version of a result of Huang-Krantz

In this section, we give the proof of Theorem 2.3. Our proof is fundamentally based on the construction of holomorphic disks attached to an elliptic real surface in a complex surface. Since the proof is more or less a generalization of that in Huang-Krantz [HK], we only give the necessary argument needed for dealing with the parameter.

Proof of Theorem 2.3: We now proceed to the proof of Theorem 2.3. We use the same notation set up in the statement of Theorem 2.3. The special form for the change of coordinates in the theorem suggests us to slice M along the $t := (z_2, \cdots, z_n) = const$–direction and apply the two dimensional result in [HK]. By the stability of the elliptic tangency (see [For] for instance), we get a family of elliptic Bishop surfaces parametrized by t. By the work in Huang-Krantz [HK], each surface bounds a three dimensional real-analytic Levi-flat manifold. Putting these manifolds together and tracing the construction of these manifolds through the Bishop disks, we will obtain a real-analytic hypersurface $\widehat{M}$. We next give the details on these.

In this section, we write $t = (z_2, \cdots, z_n)$. We also write $u = \Re w$, $v = \Im w$. For $|t|$ small, define $M_t = \{(z, w) \in M : (z_2, \cdots, z_n) = t\}$. Then M_t is a small deformation of the original M_0, which has a unique elliptic complex tangent point at $z_1 = 0$ for $|z_1| < \epsilon_0 << 1$. Since a small deformation of the surface will only move the complex tangent point to a nearby point and elliptic complex tangency is stable under small deformation, intuitively, M_t must have an elliptic complex tangent near $z_1 \approx 0$, which is completely determined by the equation:

$$\frac{\partial w}{\partial \overline{z_1}} = 2\lambda_1 \overline{z_1} + z_1 + \frac{\partial (p + iE)}{\partial \overline{z_1}}(z_1, t, \overline{z_1}, \overline{t}) = 0.$$

Here, we write $p(z, \overline{z}) = G(z, \overline{z}) - q(z, \overline{z})$. By the implicit function theorem, one solves uniquely $z_1 = a(t, \overline{t}) = O(|t|)$, which is C^ω in t. Then

$$P(t) = \left(a(t, \overline{t}), t, (G + \sqrt{-1}E)(a, t, \overline{a}, \overline{t})\right)$$

is the elliptic complex tangent point over M_t obtained by deforming the 0 on M_0 to M_t. Next, by making use of the Kenig-Webster formal normalization in the two dimensional setting, after a change of holomorphic coordinates of the form $Z' = (z_1, z', w + A(z_1, w) = w + O(|(z, w)|^2)$ if needed, we can assume without loss of generality that $E(z_1, 0, \overline{z_1}, 0) = O(|z_1|^7)$. We expand $w = G + iE$ at $(a(t, \overline{t}), t)$ to get:

$$\begin{aligned} w &= w_0(t, \overline{t}) + b(t, \overline{t})(z_1 - a(t, \overline{t})) + 2\Re\left(c(t, \overline{t})(z_1 - a(t, \overline{t}))^2\right) + d(t, \overline{t})|z_1 - a(t, \overline{t})|^2 + \\ &h^*(z_1 - a(t, \overline{t}), t, \overline{z_1 - a(t, \overline{t})}, \overline{t}) + \sqrt{-1}G^*\left(z_1 - a(t, \overline{t}), t, \overline{z_1 - a(t, \overline{t})}, \overline{t}\right) \end{aligned} \tag{3.1}$$

Here, all functions appeared above depend C^ω-smoothly on their variables with $w_0(0, 0) = 0$, $d(0, 0) = 1$, $b(0, 0) = 0$, $c(0, 0) = \lambda_1$. Moreover, $h^*(\eta, t, \overline{\eta}, \overline{t}) = O(|\eta|^3)$, $G^*(\eta, t, \overline{\eta}, \overline{t}) = O(|\eta|^7 + |t||\eta|^2)$ and $d(t, \overline{t})$ are all real-valued. By continuity and the ellipticity of $0 \leq \lambda_1 < 1/2$, for $|t|$ small, we have $A(\eta, \overline{\eta}, t, \overline{t}) := 2\Re\left(c(t, \overline{t})\eta^2\right) + d(t, \overline{t})|\eta|^2 \geq C|\eta|^2$ for a certain positive constant C independent of $|t|$. Hence, for $|t|$ small and for a real number r with $|r| << 1$, the following defines a simply connected (convex) domain D_t in $\mathbb{C}$ with a real analytic boundary:

$$D_t := \{\eta \in \mathbb{C} : \quad 2\Re\left(c(t,\bar{t})\eta^2\right) + d(t,\bar{t})|\eta|^2 + r^{-2}h^*(r\eta, t, r\bar{\eta}, \bar{t}) \leq 1\}.$$

Let $\sigma(\xi, t, \bar{t}, r)$ be the Riemann mapping from the unit disk to D_t preserving the origin and with positive derivative at the origin. By [Lemma 2.1, Hu1], $\sigma(\xi, t, \bar{t}, r)$ depends C^ω on its variables and is holomorphic in ξ in a fixed neighborhood of $\overline{\Delta}$. (See also [Lemma 4.1, Hu2] for a detailed proof on this.)

Now, we construct a family of holomorphic disks with parameter (t, r) for $|t|, |r| << 1$ attached to M, which takes the following form:

$$\begin{aligned}
&z_1(\xi, t, \bar{t}, r) = a(t, \bar{t}) + r\sigma(\xi, t, \bar{t}, r)(1 + \psi_1(\xi, t, \bar{t}, r)),\\
&(z_2, \cdots, z_n) = t,\\
&w(\xi, t, \bar{t}, r) = w_0(t, \bar{t}) + b(t, \bar{t}) \cdot r\sigma(\xi, t, \bar{t}, r)(1 + \psi_1(\xi, t, \bar{t}, r)) + r^2(1 + \psi_2(\xi, t, \bar{t}, r)),\\
&\Re\psi_1(0, t, \bar{t}, r) = 0, \quad \Im\psi_2(0, t, \bar{t}, r) = 0,\\
&\psi = (z_1(\xi, t, \bar{t}, r), t, w(\xi, t, \bar{t}, r))
\end{aligned} \tag{3.2}$$

Here ψ_1, ψ_2 are holomorphic functions in $\xi \in \Delta$, and are C^ω on (ξ, t, r) over $\overline{\Delta} \times \{t \in \mathbb{C}^{n-2} : |t| < \epsilon_0\} \times \{r \in \mathbb{R} : |r| < \epsilon_0\}$. Substituting (3.2) into (3.1) with $|\xi| = 1$, we get the following:

$$\psi_2(\xi, t, \bar{t}, r) = \Omega_1 + \Omega_2 + \sqrt{-1}\Omega_3. \tag{3.3}$$

Here $\Omega_1 = 2\Re\left(\{\frac{\partial A}{\partial \eta}(\sigma, t, \bar{\sigma}, \bar{t})\sigma + \sigma r^{-1}\frac{\partial h^*}{\partial \eta}(r\sigma, t, r\bar{\sigma}, \bar{t})\}\psi_1\right)$, $\Omega_2 = O(|\psi_1|^2)$, and $\Omega_3 = O(|t| + |r|^5)$ are all real-valued. Moreover, Ω_j $(j = 1, 2, 3)$ depend C^ω on there variables (ψ_1, t, r) in a certain suitable Banach space defined in [§5, Hu1]. Write $g(\xi, \bar{\xi}, t, \bar{t}, r) = 2\sigma\{\frac{\partial A}{\partial \eta}(\sigma, t, \bar{\sigma}, \bar{t}) + r^{-1}\frac{\partial h^*}{\partial \eta}(r\sigma, t, r\bar{\sigma}, \bar{t})\}$. Then we similarly have $\Re g > 0$, which makes results in [Lemma 5.1, Hu1] applicable in our setting. Write $\mathcal{H}$ for the standard Hilbert transform, we obtain the following singular Bishop equation:

$$\Re\{g(\xi, \bar{\xi}, t, \bar{t}, r)\psi_1\} + \Omega_2(\psi_1, \overline{\psi_1}, t, \bar{t}, r) = -\mathcal{H}(\Omega_3). \tag{3.4}$$

Now, write $\psi_1 = U(\xi, \bar{\xi}, t, \bar{t}, r) + \sqrt{-1}\mathcal{H}(U(\xi, \bar{\xi}, t, \bar{t}, r))$ for $|\xi| = 1$. By the argument in [§5, Hu1], from (3.4), one can uniquely solve $U(\xi, \bar{\xi}, t, \bar{t}, r)$ for $|t|, |r| << 1$. Moreover, $U(\xi, \bar{\xi}, t, \bar{t}, r)$ depends C^ω on $(\xi, \bar{\xi}, t, \bar{t}, r)$ and $U(\xi, \bar{\xi}, t, \bar{t}, r) = O(|t| + |r|^5)$. Hence

$$U(\xi, \bar{\xi}, t, \bar{t}, r) + \sqrt{-1}\mathcal{H}(U(\xi, \bar{\xi}, t, \bar{t}, r))$$

extends to a holomorphic function in ξ which also depends C^ω on its variables $(\xi, \bar{\xi}, t, \bar{t}, r)$ with $|\xi| \leq 1$. Moreover, we have the estimates

$$\psi_1, \psi_2 = O(|t| + |r|^5). \tag{3.5}$$

Next, we let $\widehat{M}=\bigcup_{0\le r<<1,\ |t|<<1,\xi\in\overline{\Delta}}\psi(\xi,t,\overline{t},r)$. Let $\widetilde{M}=\pi(\widehat{M})$ where π is the projection from $\mathbb{C}^{n+1}$ into the (z,u)-space. By the result in Huang-Krantz [HK], for each fixed t, $\widehat{M_t}=\widehat{M}\cap\{z'=t\}\cap B_{P(t,\overline{t})}(r_0)$ must be the local hull of holomorphic of M_t, that is a manifold C^ω-regular up to the boundary M_t. Here $B_{P(t,\overline{t})}(r_0)$ is the ball centered at $P(t,\overline{t})$ with a certain fixed radius $r_0>0$. Also, since $u=G(z_1,t,\overline{z_1},\overline{t})$ defines a strongly pseudoconvex hypersurface in $\mathbb{C}^2$ for each fixed t with $|t|<<1$, we see that $\pi(\widehat{M_t})\subset\widetilde{M^*}_t$, where $\widetilde{M^*}:=\{(z,u):u\ge G(z,\overline{z})\}$ and $\widetilde{M_t^*}=\widetilde{M^*}\cap\{(z_2,\cdots,z_n)=t\}$. Indeed, π, when restricted to $\widehat{M_t}$ is a C^ω-diffeomorphism to a neighborhood of $\pi(P(t))$ in $\widetilde{M_t^*}$ from the intersection of $\widehat{M_t}$ with the ball centered at $P(t)$ with a certain fixed radius $1>>r_0>0$. To see this, by the normalization presented by Kenig-Webster [KW], we have a change of variables in (z_1,w):

$$z_1'=z_1-a(t,\overline{t}),w'=\sum_{\alpha+\beta=1}^{6}b_{\alpha\beta}(t,\overline{t})(w-w_0(t,\overline{t}))^\alpha(z_1-a(t,\overline{t}))^\beta,$$

where $w_0,a\ b_{\alpha\beta}$ depend smoothly on t and takes values 0 at 0 for $\alpha+\beta\ge 2$. Also, $b_{10}(0,0)=1,b_{01}(0,0)=0$. In thses coordinates, M_t is mapped to M_t' that is flattened to order 6 at 0. Hence, the holomorphic hull of M_t' near 0 now is tangent to (z_1',u')-space (See [KW] [HK]), in particular, must be transversal to the $v'-$axis. Since the holomorphic hull is a biholomorphic invariant, and the (z_1',u')-plane is pulled to a real hypersurface near $P(t)$ defined by $v=E(a(t,\overline{t}),t,\overline{a(t,\overline{t})},\overline{t})+O(t)$. We see that $\widehat{M_t}$ has to be transversal to the v-axis when $|t|$ is small. Hence, π is an embedding from $\widehat{M_t}\cap B_{P(t)}(\epsilon_1)$ to $\widetilde{M_t^*}$ with a certain fixed $\epsilon_1<<1$ and also the image of $\pi\left(\widehat{M_t}\cap B_{P(t)}(\epsilon_1)\right)$ contains $\widetilde{M_t^*}\cap B_{\pi(P(t))}(\epsilon_1')$ with ϵ_1' a sufficiently small positive number independent of t. Thus π can be seen to be a one to one and onto map from $\widehat{M}$ to $\widetilde{M^*}$ near 0. Write the inverse map of π as $v(z_1,\overline{z_1},t,\overline{t})$, which is defined over $\widetilde{M^*}$ near 0. Notice that it is the graph function of $\widehat{M}$ near 0 and has to be C^ω-regular for each fixed t.

Next, we solve $v(z_1,\overline{z_1},t,\overline{t})$ from (3.2). For this, we use the computation in [HK]. First, we let

$$\begin{aligned}&z_1'=z_1-a(t,\overline{t})\\&w'=w-\big(w_0(t,\overline{t})+b(t,\overline{t})(z_1-a(t,\overline{t}))\big)\end{aligned}\tag{3.6}$$

Then (3.2) can be rewritten as

$$\begin{aligned}&z_1'(\xi,t,\overline{t},r)=r\sigma(\xi,t,\overline{t},r)(1+\psi_1(\xi,t,\overline{t},r)),\\&(z_2,\cdots,z_n)=t,\\&w'(\xi,t,\overline{t},r)=r^2(1+\psi_2(\xi,t,\overline{t},r)).\end{aligned}\tag{3.7}$$

Write $w'=u'+\sqrt{-1}v'$. Now, by the proof in [HK, pp 225], we see that for each (z_1',u',t), there is a unique v' satisfying (3.7). Moreover v', as a function in (z_1',u',t), has the following generalized Puiseux expansion:

$$v'=\sum_{i,j,s,\alpha,\beta\ge 0}S_{ijs\alpha\beta}u'^{\frac{i-j-s}{2}}z_1^j\overline{z_1}^st^\alpha\overline{t^\beta},$$

where $|S_{ijk\alpha\beta}| \lesssim C^{i+j+k}$ for some positive constant C. By the regularity of $\widehat{M_t}$ as mentioned above, we know that $\frac{\partial^{j+s} v'}{\partial {z'}_1^j \partial \overline{z'_1}^s}|_{z_1=0}$ must be smooth for each $|t|$ small and $u' \geq 0$. This shows that $S_{ijs\alpha\beta} = 0$ when $\frac{i-j-s}{2}$ is not a positive integer. As in [HK, pp 227], we see that v' is a real analytic function in (z'_1, u', t) near 0. By (3.6), we see that v is an analytic function in (z_1, u, t). Hence, we proved that $\widehat{M}$ is a real analytic manifold, which can be represented as a graph over $\widetilde{M^*}$ in (z, u)-space.

This completes the proof of Theorem 2.3. ■

Example 3.1. Define $M \subset \mathbb{C}^3$ by the following equation near 0:

$$w = q(z, \overline{z}) + p(z, \overline{z}) + iE(z, \overline{z}).$$

Here as before $q = |z_1|^2 + \lambda_1(z_1^2 + \overline{z_1^2}) + |z_2|^2 + \lambda_2(z_2^2 + \overline{z_2^2})$ with $0 \leq \lambda_1, \lambda_2 < \infty$, and $p, E = O(|z|^3)$ are real-valued. Also $G(z, \overline{z}) := q(z, \overline{z}) + p(z, \overline{z})$. For any $c \in \mathbb{R} \setminus \{0\}$, define the real hypersurface K_c by the equation $q(z, \overline{z}) = c$. Then K_c intersects transversally M along a submanifold L_c of real dimension 3. Then L_c is a CR submanifold of CR dimension 1 if and only if $L(q) \equiv 0$ along L_c. Here

$$L = (G_2 - iE_2)\frac{\partial}{\partial z_1} - (G_1 - iE_1)\frac{\partial}{\partial z_2} + 2i(G_2E_1 - G_1E_1)\frac{\partial}{\partial w}, \tag{3.8}$$

that is non-zero and tangent to $M \setminus \{0\}$.

Write $\Psi = p(z, \overline{z}) - iE(z, \overline{z})$. Then the above is equivalent to the equation $\Psi_2 \cdot (\overline{z_1} + 2\lambda_1 z_1) = \Psi_1 \cdot (\overline{z_2} + 2\lambda_2 z_2)$. Namely, M is non-minimal at its CR points near 0 if and only if the just mentioned equation holds.

One solution is given by $\Psi = p(z, \overline{z}) - iE(z, \overline{z}) = \mu_1(|z_1|^2\overline{z_1} + \lambda_1|z_1|^2 z_1) + \mu_2(|z_2|^2\overline{z_2} + \lambda_2|z_2|^2 z_2) + \mu_1\overline{z_1}(|z_2|^2 + \lambda_2 z_2^2) + \mu_2\overline{z_2}(|z_1|^2 + \lambda_1 z_1^2)$, with $\mu_1, \mu_2 \in \mathbb{C}$. Then $\Psi_1 = (\mu_1\overline{z_1} + \mu_2\overline{z_2})(\overline{z_1} + 2\lambda_1 z_1)$ and $\Psi_2 = (\mu_2\overline{z_2} + \mu_1\overline{z_1})(\overline{z_2} + 2\lambda_1 z_2)$. Thus $\Psi_2 \cdot (\overline{z_1} + 2\lambda_1 z_1) = \Psi_1 \cdot (\overline{z_2} + 2\lambda_2 z_2)$ holds trivially.

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