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A Codimension Two CR Singular Submanifold That Is Formally Equivalent to a Symmetric Quadric

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Let $M \subset \mathbb{C}^{n+1}$ $(n \geq 2)$ be a real analytic submanifold defined by an equation of the form: $w = |z|^2 + O(|z|^3)$, where we use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. We first derive a pseudonormal form for M near 0. We then use it to prove that $(M, 0)$ is holomorphically equivalent to the quadric $(M_\infty : w = |z|^2,\ 0)$ if and only if it can be formally transformed to $(M_\infty, 0)$. We also use it to give a necessary and sufficient condition when $(M, 0)$ can be formally flattened. Our main theorem generalizes a classical result of Moser for the case of $n = 1$.

1 Introduction

Let $M \subset \mathbb{C}^{n+1}$ $(n \geq 1)$ be a submanifold. For a point $p \in M$, we define $\mathrm{CR}(p)$ to be the CR dimension of M at p, namely, the complex dimension of the space $T_p^{(0,1)}M$. A point $p \in M$ is called a CR point if $\mathrm{CR}(q) = CR(p)$ for $q (\in M) \approx p$. Otherwise, p is called a CR singular point of M. When M is a real hypersurface, points on M are always CR points.

The local equivalence problem in *several complex variables* is to find a complete set of holomorphic invariants of M near a fixed point $p \in M$. The investigation normally has quite different nature in terms of whether p is a CR point or a CR singular point.

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The CR case was first considered by Poincaré and Cartan. A complete set of invariants in the strongly pseudoconvex hypersurface case was give by Chern–Moser in [5]. (See the survey articles [1, 12, 25] for many references along these lines). The first place but also the most important place where one encounters (stable) CR singular points is when the real dimension of M is the same as the complex dimension of the ambient space where M is embedded. This, in the literature, is called the critical-dimensional case. The study for the CR singular points first appeared in the article of Bishop [4]. The systematic investigations on the precise holomorphic structure of M near a nondegenerate CR singular point, in the critical-dimensional case, can be found in the work of Moser–Webster [22], Moser [21], Gong [9, 10], Huang–Yin [16]. (The reader can find many references in [12] on this matter.)

Recently, there appeared several articles, in which CR singular points in the non-critical-dimensional case were considered (see [6–8, 24], to name a few). In [24], among other things, Stolovitch introduced a set of generalized Bishop invariants for a nondegenerate general CR singular point, and established some of the results of Moser–Webster [22] to the case of $\dim_{\mathbb{R}} M > \dim_{\mathbb{C}} \mathbb{C}^{n+1}$. Coffman in [7] studied a class of nonstable CR singular points in the noncritical-dimensional case. In [8], Dolbeault–Tomassini–Zaitsev introduced the concept of the elliptic flat CR singular points and studied global filling property by complex analytic varieties for a class of compact submanifold of real codimension two in $\mathbb{C}^{n+1}$ with exactly two elliptic flat CR singular points.

In this article, we study the local holomorphic structure of a manifold M near a CR singular point p, for which we can find a local holomorphic change of coordinates such that in the new coordinates system, $p = 0$ and M near p is defined by an equation of the form: $w = |z|^2 + O(|z|^3)$. Here we use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. Such a nondegenerate CR singular point has an intriguing nature that its quadric model has the largest possible symmetry. We will first derive a pseudonormal form for M near p (see Theorem 2.3). As expected, the holomorphic structure of M near p is influenced not only by the nature of the CR singularity, but also by the fact that (M, p) partially inherits the property of strongly pseudoconvex CR structures for $n > 1$. Unfortunately, as in the case of $n = 1$ first considered by Moser [21], our pseudonormal form is still subject to the simplification of the complicated infinite-dimensional formal automorphism group of the quadric $\mathrm{aut}_0(M_\infty)$, where M_∞ is defined by $w = |z|^2$. Thus, our pseudonormal form cannot be used to solve the local equivalence problem. However, with the rapid iteration procedure, we will show in Section 4 that if all higher-order terms in our pseudonormal form vanish, then M is biholomorphically equivalent to the model M_∞. Namely, we have the following:

Theorem 1.1. Let $M \subset \mathbb{C}^{n+1}$ ($n \geq 1$) be a real analytic submanifold defined by an equation of the form: $w = |z|^2 + O(|z|^3)$. Then $(M, 0)$ is holomorphically equivalent to the quadric $(M_\infty, 0)$ if and only if it can be formally transformed to $(M_\infty, 0)$. □

One of the differences of our consideration here from the case of $n = 1$ is that a generic $(M, 0)$ can not be formally mapped into the Levi-flat hypersurface $\mathrm{Im}(w) = 0$. As another application of the pseudonormal form to be obtained in Section 2, we will give a necessary and sufficient condition when $(M, 0)$ can be formally flattened (see Theorem 3.5).

Theorem 1.1, in the case of $n = 1$, is due to Moser [21]. Indeed, our proof of Theorem 1.1 uses the approach of Moser in [21] and Gong in [10], which is based on the rapidly convergent power series method. Convergence results along the lines of Theorem 1.1 near other type of CR singular points can be found in the earlier articles of Gong [9] and Stolovitch [24]. The articles of Coffman [6, 7] also contain the rapid convergence arguments in the setting of other CR singular cases.

2 A Formal Pseudonormal Form

We use $(z, w) = (z_1, \ldots, z_n, w)$ for the coordinates in $\mathbb{C}^{n+1}$ with $n \geq 2$ in all that follows. We first present some notation and definitions, which were already encountered in the previous articles of Stolovitch [24] and Dolbeault–Tomassini–Zaitsev [8].

Let $(M, 0)$ be a formal submanifold of codimenion two in $\mathbb{C}^{n+1}$ with $0 \in M$ as a CR singular point and $T_0^{(1,0)} M = \{w = 0\}$. Then, M can be defined by a formal equation of the form

$$w = q(z, \overline{z}) + o(|z|^2), \tag{2.1}$$

where $q(z, \overline{z})$ is a quadratic polynomial in $(z, \overline{z})$. We say that $0 \in M$ is a not-completely degenerate CR singular point if there is no change of coordinates in which we can make $q \equiv 0$. We further say that 0 is a not-completely degenerate flat CR singular point if we can make q real-valued after a linear change of variables.

Assume that 0 is a not-completely degenerate flat CR singular point with $q(z, \overline{z}) = A(z, \overline{z}) + B(z, \overline{z}) \in \mathbb{R}$ for each z. Here $A(z, \overline{z}) = \sum_{\alpha,\beta=1}^{n} a_{\alpha\overline{\beta}} z_\alpha \overline{z_\beta}$, $B(z, \overline{z}) = 2\mathrm{Re}(\sum_{\alpha,\beta=1}^{n} b_{\alpha\beta} z_a z_\beta)$.

Here, $\overline{a_{\alpha\overline{\beta}}} = a_{\beta\overline{\alpha}}$ and $b_{\alpha\beta} = b_{\beta\alpha}$. Then the assumption that $A(z,\overline{z})$ is definite is independent of the choice of the coordinates system. Suppose that A is definite. Then after a linear change of coordinates, we can assume that $A(z,\overline{z}) = |z|^2$. Write B_0 for the complex symmetric matrix $(b_{jk})_{1\le j,k\le n}$. Making use of the classical Takagi theorem [Corollary 4.4.4, p. 204, 17], there is an $n\times n$ unitary matrix U such that $U\cdot B_0\cdot U^t = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ with $0\le\lambda_1\le\lambda_2\le\cdots\lambda_n$. Next, applying the transformation $(z,w)\to(z\cdot U,w)$ and noticing that the quadratic form $|z|^2$ is invariant under this transformation, we see that, in the new coordinates, in the defining equation for $(M,0)$ as in (2.1), q takes the following special form:

$$q(z,\overline{z}) = \sum_{\alpha=1}^{n}\left\{|z_\alpha|^2 + \lambda_\alpha\left(z_\alpha^2 + \overline{z_\alpha}^2\right)\right\}, \tag{2.2}$$

where $0\le\lambda_\alpha<\infty$ with $0\le\lambda_1\le\cdots\le\lambda_n<\infty$. The set of non-negative numbers $0\le\lambda_1\le\cdots\le\lambda_n$ is the quadratic invariant for $(M,0)$. In terms of Stolovitch, we call $\{\lambda_1,\ldots,\lambda_n\}$ the set of generalized Bishop invariants. When $0\le\lambda_\alpha<1/2$ for all α, we say that 0 is an elliptic CR singular point of M. Note that $0\in M$ is an elliptic CR singular point if and only if in a certain defining equation of M of the form as in (2.1), we can make $q(z,\overline{z})>0$ for $z\ne 0$. (Hence the definition coincides with the notion of elliptic flat complex points in [8].) When $\lambda_\alpha>1/2$ for all α, we say $0\in M$ is a hyperbolic CR singular point. Note that, in the case other than elliptic and hyperbolic situations (but still with a normalization as in (2.2)), one can always find a two-dimensional linear subspace of $\mathbb{C}^{n+1}$ whose intersection with M has a parabolic complex tangent at 0. For a more general related notion on ellipticity and hyperbolicity, we refer the reader to the article of Stolovitch [24].

In terms of the terminology above, the manifold in Theorem 1.1 has vanishing generalized Bishop invariants at the CR singular point. In [9, 24] , one finds the study on the related convergence problem in the different situations, where, among other nondegeneracy conditions, all the generalized Bishop invariants are assumed to be nonzero. However, the method of studying CR singular points with vanishing Bishop invariants is different from that used in the nonvanishing Bishop invariants case (see [10, 16, 21, 22, 24]).

We now return to the manifolds with only vanishing generalized Bishop invariants.

Let $E(z,\bar{z})$ (respectively, $f(z,w)$) be a formal power series in $(z,\bar{z})$ (respectively, in (z,w)) without constant term. We say $\mathrm{Ord}\left(E(z,\bar{z})\right)\ge k$ if $E(tz,t\bar{z}) = O(t^k)$. Similarly, we say $\mathrm{Ord}_{wt}\left(f(z,w)\right)\ge k$ if $f(tz,t^2w) = O(t^k)$. Set the weight of $z,\bar{z}$ to be

1 and that of w to be 2. For a polynomial $h(z,w)$, we define its weighted degree, denoted by $\deg_{wt} h$, to be the degree counted in terms the weighted system just given. Write $E^{(t)}(z,\bar z)$ and $f^{(t)}(z,w)$ for the sum of monomials with weighted degree t in the Taylor expansion of E and f at 0, respectively.

Write $u_k=\sum_{i=1}^k |z_i|^2$ for $1\le k\le n$ and $v_k=\sum_{i=1}^{k-1}|z_i|^2-|z_k|^2$ for $2\le k\le n$. We also write $u=u_n=|z|^2$. In what follows, we make a convention that the sum $\sum_{p=j}^l a_p$ is defined to be 0 if $j>l$.

We start with the following elementary algebraic lemma:

Lemma 2.1. $\mathrm{Span}_{\mathbb{C}}\{|z_1|^2,\dots,|z_n|^2\}=\mathrm{Span}\{u,v_2,\dots,v_n\}$. Moreover, for each index i with $1\le i\le n$, $|z_i|^2$ can be uniquely expressed as the following linear combination of $u,\ v_2,\dots,\ v_n$:

$$\begin{cases} |z_1|^2=2^{1-n}\left(u+\sum_{h=2}^n 2^{n-h}v_h\right),\\ |z_i|^2=2^{-(n+1-i)}\left(u+\sum_{h=i+1}^n 2^{n-h}v_h-2^{n-i}v_i\right) \text{ for } 2\le i\le n. \end{cases} \tag{2.3}$$

□

Proof of Lemma 2.1. By a direct computation, we have

$$2^{1-n}\left(u+\sum_{h=2}^n 2^{n-h}v_h\right)=2^{1-n}\left(\sum_{i=1}^n|z_i|^2+\sum_{h=2}^n 2^{n-h}\left(\sum_{i=1}^{h-1}|z_i|^2-|z_h|^2\right)\right)$$

$$=2^{1-n}\left(\left(1+\sum_{h=2}^n 2^{n-h}\right)|z_1|^2+\sum_{j=2}^{n-1}\left(1+\sum_{h=j+1}^n 2^{n-h}-2^{n-j}\right)|z_j|^2\right)$$

$$=2^{1-n}(2^{n-1}|z_1|^2)=|z_1|^2;$$

$$2^{-(n+1-i)}\left(u+\sum_{h=i+1}^n 2^{n-h}v_h-2^{n-i}v_i\right)$$

$$=2^{-(n+1-i)}\left(\sum_{i=1}^n|z_i|^2+\sum_{h=i+1}^n 2^{n-h}\left(\sum_{j=1}^{h-1}|z_j|^2-|z_h|^2\right)-2^{n-i}\left(\sum_{j=1}^{i-1}|z_j|^2-|z_i|^2\right)\right)$$

$$=2^{-(n+1-i)}\left(\sum_{j=1}^{i-1}\left(1+\sum_{h=i+1}^n 2^{n-h}-2^{n-i}\right)|z_j|^2+\left(1+\sum_{h=i+1}^n 2^{n-h}+2^{n-i}\right)|z_i|^2\right.$$

$$\left.+\sum_{j=i+1}^n\left(1+\sum_{h=j+1}^n 2^{n-h}-2^{n-j}\right)|z_j|^2\right)=|z_i|^2, \text{ for } 2\le i\le n.$$

Hence, we see that $\mathrm{span}_{\mathbb{C}}\{|z_1|^2, \ldots, |z_n|^2\} = \mathrm{span}_{\mathbb{C}}\{u, v_2, \ldots, v_n\}$. The uniqueness assertion in the lemma now is obvious. ■

For a formal (or holomorphic) transformation $f(z, w)$ of $(\mathbb{C}^n, 0)$ to itself, we write

$$\begin{cases} f(z, w) = \left(f_1(z, w), \ldots, f_n(z, w)\right), \\ f_k(z, w) = \sum_{(i_1,\ldots,i_n)} f_{k,(I)}(w) z^I, \ I = (i_1, \ldots, i_n) \text{ and } z^I = z_1^{i_1} \ldots z_n^{i_n}. \end{cases} \tag{2.4}$$

Let $E(z, \bar{z})$ be a formal power series with $E(0) = 0$. We next prove the following:

Lemma 2.2. $E(z, \bar{z})$ has the following expansion:

$$E(z, \bar{z}) = \sum_{\{i_k \cdot j_k = 0,\ k=1\ldots,n\}} E_{(I,J)}(u, v_2, \ldots, v_n) z^I \overline{z}^J = \sum_{\{i_k \cdot j_k = 0,\ k=1,\ldots,n\}} E^{(K)}_{(I,J)} z^I \overline{z}^J u^{k_1} {v_2}^{k_2} \ldots {v_n}^{k_n}. \tag{2.5}$$

Here and in what follows, we write $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_n)$, $K = (k_1, \ldots, k_n)$, $z^I = z_1^{i_1} \ldots z_n^{i_n}$ and $\overline{z}^J = \overline{z}_1^{j_1} \ldots \overline{z}_n^{j_n}$. Moreover, the coefficients $E^{(K)}_{(I,J)}$ are uniquely determined by E. □

Proof of Lemma 2.2. Since $\{|z_i|^2\}_{i=1}^n$ and $\{u, v_2, \ldots, v_n\}$ are the unique linear combinations of each other by Lemma 2.1, one sees the existence of the expansion in (2.5). Also, to complete the proof of Lemma 2.2, it suffices for us to prove the following statement:

$$\sum_{(I,J,K)\in A(N,N^*)} E^{(K)}_{(I,J)} z^I \overline{z}^J |z_1|^{2k_1} \ldots |z_n|^{2k_n} = 0 \quad \text{if and only if } E^{(K)}_{(I,J)} \equiv 0.$$

Here, we define $A(N, N^*) = \{(I, J, K) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^n,\ \ i_l \cdot j_l = 0,\ \ i_l, j_l, k_l \ge 0$ for $1 \le l \le n,\ \sum_{l=1}^n (i_l + k_l) = N,\ \sum_{l=1}^n (j_l + k_l) = N^*\}$. Let $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ with $p_1, \ldots, p_n, q_1, \ldots, q_n$ non-negative integers be such that $|P| = N, |Q| = N^*$. We define $A(N, N^*; P, Q) = \{(I, J, K) \in A(N, N^*) :\ \ i_l \cdot j_l = 0,\ \ i_l, j_l, k_l \ge 0, i_l + k_l = p_l,\ \ j_l + k_l = q_l$, for $1 \le l \le n\}$. Now, suppose that $\sum_{(I,J,K)\in A(N,N^*)} E^{(K)}_{(I,J)} z^I \overline{z}^J |z_1|^{2k_1} \ldots |z_n|^{2k_n} = 0$. We then get

$$\sum_{(I,J,K)\in A(N,N^*;P,Q)} E^{(K)}_{(I,J)} \equiv 0 \quad \text{for each } P,\ Q \text{ with } |P| = N,\ |Q| = N^*.$$

We next claim that there is at most one element in $A(N, N^*; P, Q)$. Indeed, $(I, J, K) \in A(N, N^*; P, Q)$ if and only if $i_l + k_l = p_l,\ j_l + k_l = q_l,\ i_l \cdot j_l = 0$, for $1 \le l \le n$. Now, if $i_l = 0$, then $k_l = p_l$. Since $j_l = q_l - p_l \geq 0$, thus this happens only when $q_l \geq p_l$. If $j_l = 0$, then $k_l = q_l$. Since $i_l = p_l - q_l \geq 0$, we see that this can only happen when $p_l \geq q_l$. Hence, we see that i_l, j_l are uniquely determined by p_l and q_l when $p_l \neq q_l$. When $p_l = q_l$, it is easy to see that $i_l = j_l = 0,\ k_l = q_l = p_l$. We thus conclude the argument for the claim. This completes the proof of Lemma 2.2. ■

We now let $M \subset \mathbb{C}^{n+1}$ be a formal submanifold defined by

$$w = |z|^2 + E(z, \bar{z}), \tag{2.6}$$

where E is a formal power series in $(z, \bar{z})$ with $\mathrm{Ord}(E) \geq 3$. We will subject (2.6) to the following formal power series transformation in (z, w):

$$\begin{cases} z' = F = z + f(z, w), & \mathrm{Ord}_{wt}(f) \geq 2, \\ w' = G = w + g(z, w), & \mathrm{Ord}_{wt}(g) \geq 3. \end{cases} \tag{2.7}$$

Write $e_j \in \mathbb{Z}^n$ for the vector whose component is 1 at the jth position and is 0 elsewhere. We next give a formal pseudonormal form for $(M, 0)$ in the following theorem.

Theorem 2.3. Given real valued formally power series $h_j(x) = O(x)$ in $x \in \mathbb{R}$ with $j = 2, \ldots, n$, there exits a unique formal transformation of the form in (2.7) with the normalization

$$\begin{cases} f_{i,(0)}(w) = 0,\ 1 \leq i \leq n, \\ f_{i,(e_j)}(w) = 0 \text{ for } 1 \leq j < i \leq n, \\ f_{1,(e_1)}(w) = 0,\ \mathrm{Im}\left(f_{i,(e_i)}(w)\right) = h_i(w) \text{ for } 2 \leq i \leq n \text{ and } w \text{ real}, \end{cases} \tag{2.8}$$

which transforms M to a formal submanifold defined in the following pseudonormal form:

$$w' = |z'|^2 + \varphi(z', \overline{z'}). \tag{2.9}$$

Here $\varphi(z', \overline{z'}) = O(|z'|^3)$ and in the following unique expansion:

$$\varphi(z', \overline{z'}) = \sum_{i_k \cdot j_k = 0, 1 \leq k \leq n} \varphi_{(I,J)}(u, v_2, \ldots, v_n) z'^I \overline{z}'^J = \sum_{i_k \cdot j_k = 0, 1 \leq k \leq n} \varphi^{(K)}_{(I,J)} z'^I \overline{z}'^J u^{k_1} {v_2}^{k_2} \ldots {v_n}^{k_n}, \tag{2.10}$$

we have, for any $k \geq 0$, $l \geq 1$ and $\tau \geq 2$, the following normalization condition:

$$\begin{cases} \varphi^{(\tau e_1)}_{(0,0)} = 0, \\ \mathrm{Re}\big(\varphi^{(le_1+e_i)}_{(0,0)}\big) = 0, \text{ for } 2 \leq i \leq n, \\ \varphi^{(le_1)}_{(e_i,e_j)} = 0, \text{ for } i > j, \\ \varphi^{(le_1)}_{(I,0)} = \varphi^{(le_1)}_{(0,I)} = \varphi^{(ke_1+e_j)}_{(0,I)} = 0, \text{ for } |I| \geq 1, \\ \varphi^{(ke_1)}_{(I,e_h)} = 0, \text{ for } h \geq 1, |I| \geq 2, i_h = 0, \\ \varphi^{(0)}_{(0,I)} = \overline{\varphi^{(0)}_{(I,0)}}, |I| > 2. \end{cases} \tag{2.11}$$

□

Proof of Theorem 2.3. We need to prove that the following equation, with unknowns in (f, g, φ), can be uniquely solved under the normalization conditions in (2.8) and (2.11):

$$w + g(z,w) = \sum_{i=1}^{n} (z_i + f_i(z,w))(\bar{z}_i + \overline{f_i(z,w)}) + \varphi(z + f(z,w), \bar{z} + \overline{f(z,w)}), \tag{2.12}$$

where $w = |z|^2 + E(z,\bar{z})$. Collecting terms of degree t in the above equation, we obtain for each $t \geq 3$ the following:

$$E^{(t)}(z,\bar{z}) + g^{(t)}(z,u) = 2\mathrm{Re}\left(\sum_{i=1}^{n} (\overline{z_i} f_i^{(t-1)}(z,u))\right) + \varphi^{(t)}(z,\bar{z}) + I^{(t)}(z,\bar{z}), \tag{2.13}$$

where $I^{(t)}(z,\bar{z})$ is a homogeneous polynomial of degree t depending only on $g^{(\sigma)}$, $f^{(\sigma-1)}$, $\varphi^{(\sigma)}$ (and also E, h_j) for $\sigma < t$. Then, by an induction argument, we need only to prove the following proposition:

Proposition 2.4. Consider the following linear equation in (f, g, φ):

$$\Gamma(z,\bar{z}) + g(z,u) = 2\mathrm{Re}\left(\sum_{i=1}^{n} \left(\overline{z_i} f_i(z,u)\right)\right) + \varphi(z,\bar{z}), \tag{2.14}$$

where φ satisfies the normalization in (2.11), f satisfies the normalization in (2.8), $g(z,w) = O_{wt}(3)$, and $\Gamma = O(|z|^3)$. Here $h_j(x) = O(x)$ are given real-valued formal power series in $x \in \mathbb{R}$ with $j = 2, \ldots, n$. Then (2.14) has a unique solution (f, g, φ) with $\mathrm{Im}(f_{j,(e_j)}(x)) = h_j(x)$ $(j = 2, \ldots, n)$. Moreover, g and φ are independent of the given formal real-valued functions $h_j(x)$. □

Assuming Proposition 2.4 for the moment, we start with (2.13) with $t = 3, \Gamma = E^{(3)}$. We then get the unique solution $(f^{(2)}, g^{(3)}, \varphi^{(3)})$. Suppose that we have uniquely determined $(f^{(\sigma-1)}, g^{(\sigma)}, \varphi^{(\sigma)})$ for $\sigma < m$. Now, consider (2.13) with $\sigma = m$. $I^{(\sigma)}$ is now also given. Hence, by Proposition 2.4, we can uniquely solve for $f^{(m-1)}$, $g^{(m)}, \varphi^{(m)}$. By an induction argument, we obtain uniquely $f^{(m-1)}$, $g^{(m)}, \varphi^{(m)}$ for any $m \geq 3$. Now, $f(z, w) = \sum_{m=3}^{\infty} f^{(m-1)}(z, w)$, $g(z, w) = \sum_{m=3}^{\infty} g^{(m)}(z, w), \varphi(z, \overline{z}) = \sum_{m=3}^{\infty} \varphi^{(m)}(z, \overline{z})$ are the unique solution to (2.11). Thus, to complete the proof of Theorem 2.3, it suffices for us to prove Proposition 2.4. ■

Proof of Proposition 2.4. Expand Γ, φ as in (2.5) and (2.10), respectively, and expand f, g as in (2.4). Then (2.14) takes the following form:

$$\begin{aligned}
&\sum_{i_l \cdot j_l = 0} \Gamma_{(I,J)}(u, v_2, \ldots, v_n) z^I \overline{z}^J + \sum_{|I| \geq 0} g_{(I)}(u) z^I - \sum_{i_l \cdot j_l = 0} \varphi_{(I,J)}(u, v_2, \ldots, v_n) z^I \overline{z}^J \\
&\quad = \sum_{i=1}^{n} |z_i|^2 f_{i,(e_i)}(u) + \sum_{1 \leq i \leq n, |J| > 0} |z_i|^2 z^J f_{i,(e_i+J)}(u) + \sum_{1 \leq i \leq n, k_i = 0} \overline{z}_i z^K f_{i,(K)}(u) \\
&\qquad + \sum_{i=1}^{n} |z_i|^2 \overline{f_{i,(e_i)}(u)} + \sum_{1 \leq i \leq n, |J| > 0} |z_i|^2 \overline{z}^J \overline{f_{i,(e_i+J)}(u)} + \sum_{1 \leq i \leq n, k_i = 0} z_i \overline{z}^K \overline{f_{i,(K)}(u)}.
\end{aligned}$$

Comparing the coefficients of $z^I \overline{z}^J$ with $i_l \cdot j_l = 0$, $l = 1, \ldots, n$ in the above equation, we get the following system:

$$z^0 \overline{z}^0 : \; -g_{(0)}(u) + \sum_{i=1}^{n} 2\mathrm{Re}\left(|z_i|^2 f_{i,(e_i)}(u)\right) + \varphi_{(0,0)} = \Gamma_{(0,0)}, \tag{2.15}$$

$$z_j,\ \overline{z_j} : \begin{cases} -g_{(e_j)}(u) + \overline{f_{j,(0)}(u)} + \sum_{i=1}^{n} |z_i|^2 f_{i,(e_i+e_j)}(u) + \varphi_{(e_j,0)} = \Gamma_{(e_j,0)} \\ f_{j,(0)}(u) + \sum_{i=1}^{n} |z_i|^2 \overline{f_{i,(e_i+e_j)}(u)} + \varphi_{(0,e_j)} = \Gamma_{(0,e_j)} \end{cases} \quad \text{for } 1 \leq j \leq n, \tag{2.16}$$

$$z_i \overline{z_j} : \begin{cases} f_{j,(e_i)}(u) + \overline{f_{i,(e_j)}(u)} + \varphi_{(e_i,e_j)} = \Gamma_{(e_i,e_j)} \\ \overline{f_{j,(e_i)}(u)} + f_{i,(e_j)(u)} + \varphi_{(e_j,e_i)} = \Gamma_{(e_j,e_i)} \end{cases} \quad \text{for } i \neq j, \tag{2.17}$$

$$z_i \overline{z}^J,\ z^J \overline{z_i} : \begin{cases} \overline{f_{i,(J)}(u)} + \varphi_{(e_i,J)} = \Gamma_{(e_i,J)} \\ f_{i,(J)}(u) + \varphi_{(J,e_i)} = \Gamma_{(J,e_i)} \end{cases} \quad \text{for } |J| \geq 2,\ j_i = 0, \tag{2.18}$$

$$z^I,\ \overline{z}^I : \begin{cases} -g_{(I)}(u) + \sum_{i=1}^{n} \left(|z_i|^2 f_{i,(I+e_i)}(u)\right) + \varphi_{(I,0)} = \Gamma_{(I,0)} \\ \sum_{i=1}^{n} \left(|z_i|^2 \overline{f_{i,(I+e_i)}}(u)\right) + \varphi_{(0,I)} = \Gamma_{(0,I)} \end{cases} \quad \text{for } |I| \geq 2, \tag{2.19}$$

$$z^I \overline{z}^J : \; \varphi_{(I,J)} = \Gamma_{(I,J)} \text{ for } |I|, |J| \geq 2,\ i_l \cdot j_l = 0,\ l = 1, \ldots, n. \tag{2.20}$$

We first show how the system (2.19) is uniquely solved. Substituting (2.3) to (2.19) and then collecting coefficients of the zeroth-order term, linear terms, and higher-order terms in $v_2, \ldots, v_n$, respectively, while taking u as a parameter, we obtain, by Lemma 2.2, the following:

$$\sum_{k\geq 0} \Gamma^{(ke_1)}_{(I,0)} u^k = -g_{(I)}(u) + 2^{1-n} u f_{1,(I+e_1)}(u) + \sum_{i=2}^{n} 2^{i-1-n} u f_{i,(I+e_i)}(u) + \sum_{k\geq 0} \varphi^{(ke_1)}_{(I,0)} u^k, \tag{2.21}$$

$$\sum_{k\geq 0} \Gamma^{(ke_1+e_j)}_{(I,0)} u^k = 2^{1-j} f_{1,(I+e_1)}(u) + \sum_{i=2}^{j-1} 2^{i-1-j} f_{i,(I+e_i)}(u) - 2^{-1} f_{j,(I+e_j)}(u) + \sum_{k\geq 0} \varphi^{(ke_1+e_j)}_{(I,0)} u^k, \ j \geq 2, \tag{2.22}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(I,0)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(I,0)}, \quad k_2+\cdots+k_n \geq 2, \tag{2.23}$$

$$\sum_{k\geq 0} \Gamma^{(ke_1)}_{(0,I)} u^k = 2^{1-n} u \overline{f_{1,(I+e_1)}(u)} + \sum_{i=2}^{n} 2^{i-1-n} u \overline{f_{i,(I+e_i)}(u)} + \sum_{k\geq 0} \varphi^{(ke_1)}_{(0,I)} u^k, \tag{2.24}$$

$$\sum_{k\geq 0} \Gamma^{(ke_1+e_j)}_{(0,I)} u^k = 2^{1-j} \overline{f_{1,(I+e_1)}(u)} + \sum_{i=2}^{j-1} 2^{i-1-j} \overline{f_{i,(I+e_i)}(u)} - 2^{-1} \overline{f_{j,(I+e_j)}(u)} + \sum_{k\geq 0} \varphi^{(ke_1+e_j)}_{(0,I)} u^k, \ j \geq 2, \tag{2.25}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,I)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,I)}, \quad k_2+\cdots+k_n \geq 2. \tag{2.26}$$

Using the normalization in φ and letting $u=0$ in (2.21) and (2.24), we get $\Gamma^{(0)}_{(I,0)} = -g_{(I)}(0) + \varphi^{(0)}_{(I,0)}$ and $\Gamma^{(0)}_{(0,I)} = \varphi^{(0)}_{(0,I)}$. By the normalization $\varphi^{(0)}_{(I,0)} = \overline{\varphi^{(0)}_{(0,I)}}$, we get $\varphi^{(0)}_{(I,0)} = \overline{\Gamma^{(0)}_{(0,I)}}$ and

$$g_{(I)}(0) = \overline{\Gamma^{(0)}_{(0,I)}} - \Gamma^{(0)}_{(I,0)}. \tag{2.27}$$

Summing up (2.25) with $j = 2, \ldots, n$ and then adding it to (2.24)$/u$, we get:

$$\sum_{k\geq 1}\Gamma^{(ke_1)}_{(0,I)}u^{k-1} + \sum_{j=2}^{n}\sum_{k\geq 0}\Gamma^{(ke_1+e_j)}_{(0,I)}u^k = \left(2^{1-n} + \sum_{j=2}^{n}2^{1-j}\right)\overline{f_{1,(I+e_1)}(u)}$$
$$+\sum_{i=2}^{n}\left(2^{i-1-n} + \sum_{j=i+1}^{n}2^{i-1-j} - 2^{-1}\right)\overline{f_{i,(I+e_i)}(u)} + \sum_{k\geq 1}\varphi^{(ke_1)}_{(0,I)}u^{k-1} + \sum_{j=2}^{n}\sum_{k\geq 0}\varphi^{(ke_1+e_j)}_{(0,I)}u^k. \tag{2.28}$$

After an immediate simplification, (2.28) takes the form

$$\sum_{k\geq 1}\Gamma^{(ke_1)}_{(0,I)}u^{k-1} + \sum_{j=2}^{n}\sum_{k\geq 0}\Gamma^{(ke_1+e_j)}_{(0,I)}u^k = \overline{f_{1,(I+e_1)}(u)} + \sum_{k\geq 1}\varphi^{(ke_1)}_{(0,I)}u^{k-1} + \sum_{j=2}^{n}\sum_{k\geq 0}\varphi^{(ke_1+e_j)}_{(0,I)}u^k. \tag{2.29}$$

By the the normalization condition $\varphi^{(le_1)}_{(0,I)} = \varphi^{(ke_1+e_j)}_{(0,I)} = 0$ for $k \geq 0, l \geq 1$, we obtain the following:

$$f_{1,(I+e_1)}(u) = \sum_{k\geq 1}\overline{\Gamma^{(ke_1)}_{(0,I)}}u^{k-1} + \sum_{k\geq 0}\sum_{j=2}^{n}\overline{\Gamma^{(ke_1+e_j)}_{(0,I)}}u^k. \tag{2.30}$$

Back to Equation (2.25) with $j = 2$, we get

$$\sum_{k\geq 0}\Gamma^{(ke_1+e_2)}_{(0,I)}u^k = 2^{-1}\overline{f_{1,(I+e_1)}(u)} - 2^{-1}\overline{f_{2,(I+e_2)}(u)} + \sum_{k\geq 0}\varphi^{(ke_1+e_2)}_{(0,I)}u^k.$$

By the normalization condition that $\varphi^{(ke_1+e_2)}_{(0,I)} = 0$, we obtain

$$\begin{aligned} f_{2,(I+e_2)}(u) &= f_{1,(I+e_1)}(u) - 2\sum_{k\geq 0}\Gamma^{(ke_1+e_2)}_{(0,I)}u^k \\ &= \sum_{k\geq 1}\overline{\Gamma^{(ke_1)}_{(0,I)}}u^{k-1} + \sum_{k\geq 0}\left(\sum_{i=3}^{n}\overline{\Gamma^{(ke_1+e_i)}_{(0,I)}} - \overline{\Gamma^{(ke_1+e_2)}_{(0,I)}}\right)u^k. \end{aligned}$$

Next, we inductively prove

$$f_{j,(I+e_j)}(u) = \sum_{k\geq 1}\overline{\Gamma^{(ke_1)}_{(0,I)}}u^{k-1} + \sum_{k\geq 0}\left(\sum_{i=j+1}^{n}\overline{\Gamma^{(ke_1+e_i)}_{(0,I)}} - \overline{\Gamma^{(ke_1+e_j)}_{(0,I)}}\right)u^k \quad \text{for } 2 \leq j \leq n. \tag{2.31}$$

Suppose that (2.31) holds for $i = 2, \ldots, j-1 (< n)$. By (2.25) and the normalization condition $\varphi_{(0,I)}^{(ke_1+e_j)} = 0$ for $2 \le j \le n$ and $k \ge 0$, we get

$$
\begin{aligned}
f_{j,(I+e_j)}(u) &= 2^{2-j} f_{1,(I+e_1)}(u) + \sum_{i=2}^{j-1} 2^{i-j} f_{i,(I+e_i)}(u) - 2\sum_{k\ge 0} \overline{\Gamma_{(0,I)}^{(ke_1+e_j)}} u^k \\
&= 2^{2-j}\left(\sum_{k\ge 1} \overline{\Gamma_{(0,I)}^{(ke_1)}} u^{k-1} + \sum_{k\ge 0}\sum_{j=2}^{n} \overline{\Gamma_{(0,I)}^{(ke_1+e_j)}} u^k\right) \\
&\quad + \sum_{i=2}^{j-1} 2^{i-j}\left(\sum_{k\ge 1} \overline{\Gamma_{(0,I)}^{(ke_1)}} u^{k-1} + \sum_{k\ge 0}\left(\sum_{h=i+1}^{n} \overline{\Gamma_{(0,I)}^{(ke_1+e_h)}} - \overline{\Gamma_{(0,I)}^{(ke_1+e_i)}}\right) u^k\right) - 2\sum_{k\ge 0} \overline{\Gamma_{(0,I)}^{(ke_1+e_j)}} u^k \\
&= \left(2^{2-j} + \sum_{i=2}^{j-1} 2^{i-j}\right) \sum_{k\ge 1} \overline{\Gamma_{(0,I)}^{(ke_1)}} u^{k-1} + \sum_{i=2}^{j-1}\sum_{k\ge 0}\left(2^{2-j} + \sum_{h=2}^{i-1} 2^{h-j} - 2^{i-j}\right) \overline{\Gamma_{(0,I)}^{(ke_1+e_i)}} u^k \\
&\quad + \sum_{k\ge 0}\left(2^{2-j} + \sum_{h=2}^{j-1} 2^{h-j} - 2\right) \overline{\Gamma_{(0,I)}^{(ke_1+e_j)}} u^k + \sum_{i=j+1}^{n}\sum_{k\ge 0}\left(2^{2-j} + \sum_{h=2}^{j-1} 2^{h-j}\right) \overline{\Gamma_{(0,I)}^{(ke_1+e_i)}} u^k \\
&= \sum_{k\ge 1} \overline{\Gamma_{(0,I)}^{(ke_1)}} u^{k-1} + \sum_{k\ge 0}\left(\sum_{i=j+1}^{n} \overline{\Gamma_{(0,I)}^{(ke_1+e_i)}} - \overline{\Gamma_{(0,I)}^{(ke_1+e_j)}}\right) u^k.
\end{aligned}
$$

Subtracting the complex conjugate of (2.25) from (2.22) and making use of the normalization condition $\varphi_{(0,I)}^{(ke_1+e_j)} = 0$ for $2 \le j \le n$ and $k \ge 0$, we obtain

$$
\varphi_{(I,0)}^{(ke_1+e_j)} = \Gamma_{(I,0)}^{(ke_1+e_j)} - \overline{\Gamma_{(0,I)}^{(ke_1+e_j)}}, \quad j \ge 2,\ k \ge 0. \tag{2.32}
$$

From (2.21), (2.24), and $\varphi_{(0,I)}^{(ke_1)} = \varphi_{(I,0)}^{(ke_1)} = 0$ for $k \ge 1$, we can similarly get

$$
g_{(I)}(u) = \sum_{k=0}^{\infty}\left(\overline{\Gamma_{(0,I)}^{(ke_1)}} - \Gamma_{(I,0)}^{(ke_1)}\right) u^k, \quad |I| \ge 2. \tag{2.33}
$$

Under the normalization condition $f_{i,(0)}(u) = 0$, (2.16) can be solved exactly in the same way as for (2.19). (The only difference is that the role of I is now played by e_j.) We obtain the following:

$$
f_{1,(e_1+e_j)}(u) = \sum_{k\ge 1} \overline{\Gamma_{(0,e_j)}^{(ke_1)}} u^{k-1} + \sum_{k\ge 0}\sum_{i=2}^{n} \overline{\Gamma_{(0,e_j)}^{(ke_1+e_i)}} u^k, \tag{2.34}
$$

$$f_{i,(e_j+e_i)}(u) = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,e_j)}} u^{k-1} + \sum_{k\geq 0} \left(\sum_{l=i+1}^{n} \overline{\Gamma^{(ke_1+e_l)}_{(0,e_j)}} - \overline{\Gamma^{(ke_1+e_i)}_{(0,e_j)}} \right) u^k \quad \text{for } 2 \leq i \leq n, \tag{2.35}$$

$$g_{(e_j)}(u) = \sum_{k=1}^{\infty} \left(\overline{\Gamma^{(ke_1)}_{(0,e_j)}} - \Gamma^{(ke_1)}_{(e_j,0)} \right) u^k, \tag{2.36}$$

$$\varphi^{(ke_1+e_l)}_{(e_j,0)} = \Gamma^{(ke_1+e_l)}_{(e_j,0)} - \overline{\Gamma^{(ke_1+e_l)}_{(0,e_j)}}, \quad l \geq 2,\ k \geq 0, \tag{2.37}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_j,0)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_j,0)}, \quad k_2 + \cdots + k_n \geq 2, \tag{2.38}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,e_j)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,e_j)}, \quad k_2 + \cdots + k_n \geq 2. \tag{2.39}$$

Now we proceed to solve (2.15). The procedure presented to solve (2.19) can now be applied to get the following system, which is similar to (2.21) and (2.22):

$$\sum_{k\geq 2} \Gamma^{(ke_1)}_{(0,0)} u^k = -g_{(0)}(u) + 2^{2-n} u \mathrm{Re}(f_{1,(e_1)}(u)) + \sum_{i=2}^{n} 2^{i-n} u \mathrm{Re}(f_{i,(e_i)}(u)) + \sum_{k\geq 2} \varphi^{(ke_1)}_{(0,0)} u^k, \tag{2.40}$$

$$\sum_{k\geq 1} \Gamma^{(ke_1+e_j)}_{(0,0)} u^k = 2^{2-j} \mathrm{Re}(f_{1,(e_1)}(u)) + \sum_{i=2}^{j-1} 2^{i-j} \mathrm{Re}(f_{i,(e_i)}(u)) - \mathrm{Re}(f_{j,(e_j)}(u)) + \sum_{k\geq 1} \varphi^{(ke_1+e_j)}_{(0,0)} u^k, \quad j \geq 2. \tag{2.41}$$

By the normalization

$$f_{1,(e_1)} = 0, \qquad \varphi^{(\tau e_1)}_{(0,0)} = \mathrm{Re}(\varphi^{(le_1+e_i)}_{(0,0)}) = 0 \quad \text{for } \tau \geq 2, 2 \leq i \leq n, l \geq 1, \tag{2.42}$$

we can obtain the following solution:

$$g_{(0)}(u) = \sum_{k\geq 2} \left(-\Gamma^{(ke_1)}_{(0,0)} u^k \right) - \mathrm{Re} \left(\sum_{k\geq 1;\ j=2,\ldots,n} \Gamma^{(ke_1+e_j)}_{(0,0)} u^{k+1} \right), \tag{2.43}$$

$$\mathrm{Re}(f_{h,(e_h)})(u) = \frac{1}{2} \sum_{k\geq 1} \left(-\sum_{j=2}^{h-1} \mathrm{Re}(\Gamma^{(ke_1+e_j)}_{(0,0)} u^k) - 2\mathrm{Re}(\Gamma^{(ke_1+e_h)}_{(0,0)} u^k) \right), \ h \geq 2, \tag{2.44}$$

$$\varphi_{(0,0)} = \Gamma_{(0,0)} - \sum_{k\geq 2} \Gamma^{(ke_1)}_{(0,0)} u^k - \mathrm{Re} \left(\sum_{k\geq 1, j=2,\ldots,n} \Gamma^{(ke_1+e_j)}_{(0,0)} u^k v_j \right). \tag{2.45}$$

In fact, making use of (2.41) with $j = 2$, we get

$$\mathrm{Re}\big(f_{1,(e_1)}\big)(u) - \mathrm{Re}\big(f_{2,(e_2)}\big)(u) + \sum_{k\geq 1} \varphi^{(ke_1+e_2)}_{(0,0)} u^k = \sum_{k\geq 1} \Gamma^{(ke_1+e_2)}_{(0,0)} u^k.$$

Under the normalization in (2.42), we have

$$\mathrm{Re}(f_{2,(e_2)})(u) = -\mathrm{Re}\big(\Gamma^{(ke_1+e_2)}_{(0,0)}\big)u^k, \text{ which satisfies (2.44) with } h = 2.$$

Suppose that (2.44) holds for $h = 2, \ldots, j-1$. We can inductively solve $\mathrm{Re}(f_{j,(e_j)})$ as follows:

$$\begin{aligned}
\mathrm{Re}(f_{j,(e_j)})(u) &= \sum_{i=2}^{j-1} 2^{i-j}\mathrm{Re}(f_{i,(e_i)})(u) - \sum_{k\geq 1}\mathrm{Re}\big(\Gamma^{(ke_1+e_j)}_{(0,0)}u^k\big)\\
&= \sum_{i=2}^{j-1} 2^{i-j}\cdot\frac{1}{2}\sum_{k\geq 1}\left(-\sum_{h=2}^{i-1}\mathrm{Re}\big(\Gamma^{(ke_1+e_h)}_{(0,0)}u^k\big) - 2\mathrm{Re}\big(\Gamma^{(ke_1+e_i)}_{(0,0)}u^k\big)\right) - \sum_{k\geq 1}\mathrm{Re}\big(\Gamma^{(ke_1+e_j)}_{(0,0)}u^k\big)\\
&= -\sum_{k\geq 1}\sum_{i=2}^{j-1}\left(\sum_{h=i+1}^{j-1} 2^{h-1-j} + 2\cdot 2^{i-1-j}\right)\cdot\mathrm{Re}\big(\Gamma^{(ke_1+e_i)}_{(0,0)}u^k\big) - \sum_{k\geq 1}\mathrm{Re}\big(\Gamma^{(ke_1+e_j)}_{(0,0)}u^k\big)\\
&= \frac{1}{2}\sum_{k\geq 1}\left(-\sum_{h=2}^{j-1}\mathrm{Re}\big(\Gamma^{(ke_1+e_h)}_{(0,0)}u^k\big) - 2\mathrm{Re}\big(\Gamma^{(ke_1+e_j)}_{(0,0)}u^k\big)\right).
\end{aligned}$$

This shows that (2.44) holds for $2 \leq h \leq n$.

Back to (2.41), we get $\varphi^{(ke_1+e_j)}_{(0,0)} = \sqrt{-1}\mathrm{Im}(\Gamma^{(ke_1+e_j)}_{(0,0)})$. Summing up (2.41) with $j = 2, \ldots, n$ and adding it to $(2.40)/u$, we get

$$-g_{(0)}(u) + 2\mathrm{Re}\big(f_{1,(e_1)}\big) + \sum_{k\geq 2}\varphi^{(ke_1)}_{(0,0)}u^{k-1} + \sum_{k\geq 1}\sum_{j=2}^{n}\varphi^{(ke_1+e_j)}_{(0,0)}u^k = \sum_{k\geq 2}\Gamma^{(ke_1)}_{(0,0)}u^{k-1} + \sum_{k\geq 1}\sum_{j=2}^{n}\Gamma^{(ke_1+e_j)}_{(0,0)}u^k.$$

This immediately gives (2.43).

From (2.17), we get

$$f_{i,(e_j)}(u) = \textstyle\sum_{k=1}^{\infty}\Gamma^{(ke_1)}_{(e_j,e_i)}u^k, \quad i < j, \tag{2.46}$$

$$\varphi^{ke_1}_{(e_i,e_j)} = \Gamma^{(ke_1)}_{(e_i,e_j)} - \overline{\Gamma^{(ke_1)}_{(e_j,e_i)}}, \quad i < j,\ k \geq 1, \tag{2.47}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,e_j)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,e_j)}, \quad \text{for } \mathrm{k}_2 + \cdots + \mathrm{k}_\mathrm{n} \geq 1. \tag{2.48}$$

From (2.18), we obtain

$$f_{i,(J)}(u) = \sum_{k\geq 0} \Gamma^{(ke_1)}_{(J,e_i)} u^k, \quad 1 \leq i \leq n, \tag{2.49}$$

$$\varphi^{(ke_1)}_{(e_i,J)} = \Gamma^{(ke_1)}_{(e_i,J)} - \overline{\Gamma^{(ke_1)}_{(J,e_i)}}, \quad 1 \leq i \leq n, k \geq 0, \tag{2.50}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(J,e_i)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(J,e_i)} \quad \text{for } \mathrm{k}_2 + \cdots + \mathrm{k}_\mathrm{n} \geq 1, \tag{2.51}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,J)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,J)} \quad \text{for } \mathrm{k}_2 + \cdots + \mathrm{k}_\mathrm{n} \geq 1, \tag{2.52}$$

where $|J| \geq 2$ and $j_i = 0$.

Summarizing the solutions just obtained, we have the following formula (one can also directly verify that they are indeed the solutions of (2.14) with the normalization conditions given in (2.8) and (2.11)):

$$F_1(z,w) = z_1 + f_1(z,w) = z_1 + \sum_{k\geq 0, j_1=0, |J|\geq 1} z^J \Gamma^{(ke_1)}_{(J,e_1)} w^k + \sum_{|I|\geq 1} z^{I+e_1} S^{(1)}_I,$$

$$F_j(z,w) = z_j + f_j(z,w) = z_j + \frac{1}{2} z_j \sum_{k\geq 1} \left(-\sum_{l=2}^{j-1} \mathrm{Re}\big(\Gamma^{(ke_1+e_l)}_{(0,0)}\big) - 2\mathrm{Re}\big(\Gamma^{(ke_1+e_j)}_{(0,0)}\big) \right) w^k$$
$$+ \sqrt{-1} z_j h_j(w) + \sum_{k\geq 1, i>j} z_i \Gamma^{(ke_1)}_{(e_i,e_j)} w^k + \sum_{k\geq 0, i_j=0, |I|\geq 2} z^I \Gamma^{(ke_1)}_{(I,e_j)} w^k + \sum_{|I|\geq 1} z^{I+e_j} S^{(j)}_I,$$

$$G(z,w) = w + g(z,w) = w + \left(-\sum_{k\geq 2} \Gamma^{(ke_1)}_{(0,0)} w^k - \mathrm{Re}\left(\sum_{k\geq 1, j=2,\ldots,n} \Gamma^{(ke_1+e_j)}_{(0,0)} \right) w^{k+1} \right)$$
$$+ \sum_{k\geq 0, |I|\geq 1} z^I w^k \big(\overline{\Gamma^{(ke_1)}_{(0,I)}} - \Gamma^{(ke_1)}_{(I,0)} \big),$$

$$\varphi(z,\overline{z}) = \Gamma(z,\bar{z}) + g(z,u) - 2\mathrm{Re}\left(\sum_{i=1}^n \big(\overline{z_i} f_i(z,u) \big) \right), \tag{2.53}$$

where $2 \leq j \leq n$ and

$$\begin{cases} S^{(1)}_I = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,I)}} w^{k-1} + \sum_{k\geq 0} \sum_{i=2}^n \overline{\Gamma^{(ke_1+e_i)}_{(0,I)}} w^k, \\ S^{(j)}_I = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,I)}} w^{k-1} + \left(\sum_{k\geq 0} \sum_{i=j+1}^n \overline{\Gamma^{(ke_1+e_i)}_{(0,I)}} w^k \right) - \sum_{k\geq 0} \overline{\Gamma^{(ke_1+e_j)}_{(0,I)}} w^k, \text{ for } 2 \leq j \leq n. \end{cases} \tag{2.54}$$

This completes the proof of Proposition 2.4 and thus the proof of Theorem 2.3. ■

Before proceeding further, we give the following definition:

Definition 2.5.

(1) We say a formal transformation $H = (F, G)$ in (2.7) is well normalized or satisfies the good normalization condition if $H = (F, G)$ is of the form as in (2.8) with $h_j = 0$ for $2 \le j \le n$.

(2) Let $(M, 0)$ be as in (2.6). We say that $(M^*, 0)$ is a formal pseudonormal form for $(M, 0)$ if $(M^*, 0)$ is formally equivalent to $(M, 0)$ and M^* is defined by $w = |z|^2 + \varphi$ with φ satisfying the normalization as in (2.11).

(3) We say that a formal submanifold $(M, 0) \subset (\mathbb{C}^{n+1}, 0)$ of real dimension $2n$ defined by (2.6) can be formally flattened near 0 if there is a formal change of coordinates $(z', w') = H(z, w)$ with $H(0) = 0$ such that in the new coordinates, $(M, 0)$ is defined by a formal function of the form $w' = E^*(z', \overline{z}')$ with $E^*(z', \overline{z}') = \overline{E^*(z', \overline{z}')}$.

(4) We say a pseudonormal form of $(M, 0)$ given by $w = |z|^2 + \varphi(z, \overline{z})$ with φ satisfying the normalizations in (2.11) is a flat pseudonormal form if φ is formally real-valued. □

We now state an immediate corollary of Theorem 2.3, which will be used later:

Corollary 2.6.

(1) Let $(M, 0)$ be as in (2.6). Then there exists a unique formal transformation satisfying good normalization condition, that transforms M into a formal pseudonormal form.

(2) A well-normalized formal transformation, which transforms a formal pseudonormal form to another formal pseudonormal form, must be the identity.

(3) Let $(M, 0)$ be as in (2.6) with $\overline{E(z, \overline{z})} = E(z, \overline{z})$ and let $G(z, w), \varphi$ be as in Theorem 2.3. Then $g(z, w) = g(w)$ with $\overline{G(w)} = G(\overline{w})$ and φ is also real valued. □

Proof of Corollary 2.6. (1) is an immediate consequence of Theorem 2.3. To prove (2), note that in (2.13), we have $I^{(3)} = 0$, then by (2.53), we get $(f^{(2)}, g^{(3)}) = 0$ under the good normalization condition (in (2.8) with $h_j = 0$). By an inductive process, we can show that $I^{(t)} = 0$ and $(f^{(t-1)}, g^{(t)}) = 0$ for all $t \ge 3$. Hence the transformation must be the identity.

Now we turn to the proof of (3). In the normalized map $H(z, w) = (F(z, w), G(z, w))$ transforming M into its pseudonormal form in Theorem 2.3, the w-component $G(z, w)$ can be inductively proved to be only a function in w and to be formally real-valued for real w, by the formula in (2.53). This is due to the fact that the Γ in (2.53) obtained from each induction stage in the process of the proof of Theorem 2.3 is formally real-valued. Hence, we can conclude, by the last equation in (2.53), that the φ in the pseudonormalization of M obtained in Theorem 2.3 is also formally real-valued. ■

Remark 2.7. (A) The pseudonormal form obtained in Theorem 2.3 contains information reflecting both the singular CR structure and partial strongly pseudoconvex CR structure at the point under study. For instance, the following submanifold in $\mathbb{C}^3$ is given in a pseudonormal form:

$$M: \ w = |z|^2 + 2\mathrm{Re} \sum_{j_1+j_2\geq 3} \left(a_{j_1 j_2} z_1^{j_1} z_2^{j_2}\right) + \sum_{j_1\geq 2, j_2\geq 2} b_{j_1 \overline{j_2}} z_1^{j_1} \overline{z_2}^{j_2}. \tag{2.55}$$

Here the harmonic terms $\mathrm{Re}\sum_{j_1+j_2\geq 3}(a_{j_1 j_2} z_1^{j_1} z_2^{j_2})$ are presented due to the nature of CR singularity of M at 0, which may be compared with the Moser pseudonormal form in [21] in the pure CR singularity setting. Typical mixed terms like $\sum_{j_1\geq 2, j_2\geq 2} b_{j_1 \overline{j_2}} z_1^{j_1} \overline{z_2}^{j_2}$ are associated with the partial CR structure near 0, which can be compared with the Chern–Moser normal form in the pure CR setting [5].

(B) The phenomenon in Corollary 2.6 (3) is different from the two-dimensional case. In the two-dimensional case, $(M, 0)$ can always be flattened. This is no longer true for a general M when $n+1 > 2$. Indeed, we will see in Theorem 3.5 that M can be formally flattened if and only if its pseudonormal form is given by a formal real-valued function. Also, note that pseudonormal forms of $(M, 0)$ are far from being unique for a given $(M, 0)$. □

3 Normalization of Holomorphic Maps by Automorphisms of the Quadric and a Formal Flattening Theorem

In this section, we first compute the isotropic automorphism group of the model space $M_\infty \subset \mathbb{C}^{n+1}$ defined by the equation $w = \sum_{i=1}^{n} |z_i|^2$. Write $\mathrm{Aut}_0(M_\infty)$ for the set of biholomorphic self-maps of $(M_\infty, 0)$. We have the following:

Proposition 3.1. $\mathrm{Aut}_0(M_\infty)$ consists of the transformations given in the following (3.1) or (3.2):

$$\begin{cases} z' = b(w)\dfrac{wa(w) - \frac{\langle z, \bar{a}(w)\rangle}{\langle a(w), \bar{a}(w)\rangle} a(w) + \sqrt{1 - wa(w)\bar{a}(w)}\left(z - \frac{\langle z, \bar{a}(w)\rangle}{\langle a(w), \bar{a}(w)\rangle} a(w)\right)}{1 - \langle z, \bar{a}(w)\rangle} U(w) \\ w' = b(w)\bar{b}(w) w \end{cases} \tag{3.1}$$

$$(z', w') = (b(w)zU(w), b(w)\bar{b}(w)w), \tag{3.2}$$

where $a = (a_1, \cdots, a_n)$, $\sum_{j=1}^{n} a_j(0)\bar{a}_j(0) < 1$, $\langle z, \bar{a}\rangle = \sum_{i=1}^{n} \bar{a}_i z_i$, $b(0) \neq 0$, $a(0) \neq 0$, $U(\mathrm{Re}(w))$ is a unitary matrix and $a(w), b(w), U(w)$ are holomorphic in w near 0. □

Proof of Proposition 3.1. Write $w = x + \sqrt{-1}y$. Let $(F, G) \in \mathrm{Aut}_0(M_\infty)$. Then $\mathrm{Im}(G(z, |z|^2)) \equiv 0$ for $z \approx 0$. Since M_∞ bounds a family of balls near 0 defined by

$$B_r = \{(z, w) \in \mathbb{C}^{n+1} : w = x + \sqrt{-1}y, y = 0, x = r^2 \geq |z^2|\}.$$

We see that $\mathrm{Im}(G(z, x)) \equiv 0$ for $z \approx 0$ and $x(\in \mathbb{R}) \approx 0$. Therefore, $G(z, w) = G(w) = cw + o(w)$ $(c > 0)$ is independent of z and takes real value when $w = x$ is real. Now $F(z, r^2)$ must be a biholomorphic map from $|z|^2 < r^2$ to $|z|^2 < G(r^2)$ for any sufficiently small positive r. Using the explicit expression for automorphisms of the unit ball (see [23]), we obtain either:

$$F(z, r^2) = \sqrt{G(r^2)}\frac{a(r) - \frac{\langle \frac{z}{r}, \bar{a}(r)\rangle}{\langle a(r), \bar{a}(r)\rangle}a(r) + v\left(\frac{z}{r} - \frac{\langle \frac{z}{r}, \bar{a}(r)\rangle}{\langle a(r), \bar{a}(r)\rangle}a(r)\right)}{1 - \langle \frac{z}{r}, \bar{a}(r)\rangle}U(r), \tag{3.3}$$

where $U(r)$ is a unitary matrix and $v = \sqrt{1 - a(r)\bar{a}(r)}$, $a(r) \neq 0$; or we have

$$F(z, r^2) = \sqrt{G(r^2)}\left(\frac{z}{r}\right)U(r). \tag{3.4}$$

Write $G(x) = xb(x)\bar{b}(x)$ with $b(0) \neq 0$ and $b(w)$ holomorphic in w. In the case of (3.4), $F(z, x) = b(x)zU(r)e^{\sqrt{-1}\theta(x)}$ is real analytic, where $\theta(x)$ is real-valued real analytic function in x. Hence, $b(x)U(r)e^{\sqrt{-1}\theta(x)}$ is the Jacobian matrix of F in z. Since both $e^{\sqrt{-1}\theta(x)}$ and $b(x)(\neq 0)$ are real analytic for $x \approx 0$, we conclude that $U(r)$ is real analytic in x. Still, write $U(x)$ for $U(r)e^{i\theta(x)}$. Then, $U(w)$ is also holomorphic in w. We see the proof of Proposition 3.1 in the case of (3.2).

Suppose that $a \neq 0$. Still, write $G(w) = wb(w)\bar{b}(w)$ with $b(0) \neq 0$. We have

$$F(z, r^2) = b(r^2)\frac{ra(r) - \frac{\langle z, \bar{a}(r)\rangle}{\langle a(r), \bar{a}(r)\rangle}a(r) + v\left(z - \frac{\langle z, \bar{a}(w)\rangle}{\langle a(r), \bar{a}(r)\rangle}a(r)\right)}{1 - \left\langle z, \frac{\bar{a}(r)}{r}\right\rangle}e^{i\theta}U(r).$$

Since $f(z, w)$ is holomorphic in (z, w) and $f(0, w) = b(w)\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ with $U^* = e^{i\theta}U$, we see that $\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ is holomorphic in w. In particular, $|a(\sqrt{w})|^2$ is real analytic in w. Moreover,

$$\frac{\partial F}{\partial z_i}(0, w) = b(w)\left(\frac{|a|^2 - v - 1}{|a|^2}\bar{a}_i a + v e_i\right) U^*(\sqrt{w})$$

is holomorphic in w. Since $\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ is real analytic in w, we see that

$$\left(\frac{|a|^2 - v - 1}{|a|^2}\bar{a}_i a + v e_i\right) U^*(\sqrt{w})\overline{U^*(\sqrt{w})}^t\, \overline{a(\sqrt{w})}^t r = ((|a|^2 - v - 1) + v) r \bar{a}_i$$

is real analytic in w, too. Here $(\cdot)^t$ denotes the matrix transpose. Since $(|a|^2 - v - 1) + v = |a|^2 - 1$ is real analytic, we conclude that both ra_i and a_i/r are real analytic in w. Since both $\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ and ra_i are real analytic, we see that $U^*(\sqrt{w})$ is real analytic in w. Still, write a for a/r. We further obtain the following with the given properties stated in the proposition:

$$\begin{cases} F(z, w) = b(w)\dfrac{wa(w) - \frac{\langle z, \bar{a}(w)\rangle}{\langle a(w), \bar{a}(w)\rangle} a(w) + \sqrt{1 - wa(w)\bar{a}(w)}\left(z - \frac{\langle z, \bar{a}(w)\rangle}{\langle a(w), \bar{a}(w)\rangle} a(w)\right)}{1 - \langle z, \bar{a}(w)\rangle} U^*(w) \\ G(w) = b(w)\bar{b}(w)w. \end{cases}.$$

This completes the proof of Proposition 3.1. ∎

Remark 3.2. In Proposition 3.1, if we let $a(w), b(w), U(w)$ be formal power series in w with $a(0)$, $b(0) \neq 0$ and $\langle a(0), \overline{a}(0)\rangle < 1$, $U(x) \cdot \overline{U(x)}^t = I$, then (3.1) and (3.2) give formal automorphisms of M_∞, which are not convergent. Write the set of automorphisms obtained in this way as $\mathrm{aut}_0(M_\infty)$. One may prove that $\mathrm{aut}_0(M_\infty)$ consists of all the formal automorphisms of $(M_\infty, 0)$. □

We now suppose that $H = (F, G)$ is a formal equivalence self-map of $(\mathbb{C}^{n+1}, 0)$, mapping a formal submanifold of the form $w = |z|^2 + O(|z|^3)$ to a submanifold of the form $w = |z|^2 + O(|z|^3)$. The following lemma shows that we can always normalize H by composing it from the left with an element from $\mathrm{aut}_0(M_\infty)$ to get a well-normalized mapping. This fact will be used in the proof of Theorem 1.1. In what follows, we set $v(g, a) = \sqrt{1 - g \cdot a(g) \cdot \bar{a}(g)}$.

Lemma 3.3. There exists a unique automorphism $T \in \mathrm{aut}_0(M_\infty)$ such that $T \circ H$ satisfies the good normalization condition (as in (2.8) with $h_j = 0$). When H is biholomorphic, $T \in \mathrm{Aut}_0(M_\infty)$. □

Proof of Lemma 3.3. First, it is easy to see that by composing an automorphism of the form $w' = |c|^2 w,\ z' = czU$, we can assume that $F = z + O_{wt}(2)$ and $G = w + O_{wt}(3)$ (see [12]). Here c is a nonzero constant and U is a certain $n \times n$-unitary matrix.

Let $b(w) = 1, a_j = \alpha_j(w), a_1 = \cdots = a_{j-1} = a_{j+1} = \cdots = a_n = 0$, and $U = I$ in (3.1). We get the following automorphism of M_∞:

$$T_j = \left(\frac{v(w,\alpha_j) z_1}{1 - \bar{\alpha_j} z_j}, \cdots, \frac{v(w,\alpha_j) z_{j-1}}{1 - \bar{\alpha_j} z_j}, \frac{z_j - w\alpha_j}{1 - \bar{\alpha_j} z_j}, \frac{v(w,\alpha_j) z_{j+1}}{1 - \bar{\alpha_j} z_j}, \cdots, \frac{v(w,\alpha_j) z_n}{1 - \bar{\alpha_j} z_j}, w \right).$$

Write

$$\begin{cases} H_j = \left({}_{(j)}F, {}_{(j)}G \right) = T_j \circ T_{j-1} \circ \cdots \circ T_1 \circ H, \ H_0 = H; \\ \alpha_j = \frac{{}_{(j-1)}F_{j,(0)}(u)}{{}_{(j-1)}G_{(0)}(u)} \circ \left({}_{(j-1)}G_{(0)}(u) \right)^{-1}. \end{cases} \tag{3.5}$$

Then a direct computation shows that $({}_{(j)}F)_{i,(0)}(u) = 0$ for $1 \le i \le j$. In particular, we have $({}_{(n)}F)_{i,(0)}(u) = 0$ for all $1 \le i \le n$.

Still, write H for H_n. Next, for $i < j$, let $b(w) = 1, a = 0$, and let

$$U_j^i = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta_j^i) & 0 & -\sin(\theta_j^i) & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & \sin(\theta_j^i) & 0 & \cos(\theta_j^i) & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

in (3.1), where $\cos(\theta_j^i)$ is at the ith row and the jth column. Then we get an automorphism T_j^i. Set

$$H_j^i = \left({}_j^iF, {}_j^iG \right) = T_j^i \circ \cdots \circ T_{i+1}^i \circ T_n^{i-1} \circ \cdots T_i^{i-1} \circ \cdots \circ T_n^1 \circ \cdots \circ T_2^1 \circ H,$$
$$\theta_j^i = \begin{cases} \tan^{-1}\left(\frac{({}_n^{i-1}F)_{j,(e_i)}}{({}_n^{i-1}F)_{i,(e_i)}} \right) \circ \left({}_n^{i-1}G_{(0)}(w) \right)^{-1}, & i = j-1, \\ \tan^{-1}\left(\frac{({}_{j-1}^{i}F)_{j,(e_i)}}{({}_{j-1}^{i}F)_{i,(e_i)}} \right) \circ \left({}_{j-1}^{i}G_{(0)}(w) \right)^{-1}, & 1 \le i < j-1. \end{cases} \tag{3.6}$$

Then we can inductively prove that H_j^i satisfies

$$\left({}_j^iF \right)_{(0)} = 0, \left({}_j^iF \right)_{k,(e_l)} = 0 \quad \text{for } l = i, i+1 \le k \le j \quad \text{or } l < i, \quad l+1 \le k \le n.$$

In particular, we see that H_n^{n-1} satisfies $({}_n^{n-1}F)_{(0)} = 0$, $({}_n^{n-1}F)_{i,(e_j)} = 0$ for $1 \le j < i \le n$. Still, write H for H_n^{n-1} and set $H' = T \circ H = (F', G')$ with

$$T = (d(w)z, d(w)\bar{d}(w)w), \qquad d = \frac{1}{F_{1,(e_1)}(w)} \circ \left(G_{(0)}(w)\right)^{-1}.$$

Then H' satisfies

$$(F')_{(0)} = 0, \qquad (F')_{1,(e_1)} = 1, \qquad (F')_{i,(e_j)} = 0 \text{ for } 1 \le j < i \le n.$$

At last, a composition from the left with the rotation map as follows:

$$\hat{T} = (z_1, \beta_2 z_2, \cdots, \beta_n z_n, w), \qquad \beta_i = \frac{(\bar{F}')_{i,(e_i)}(w)}{\sqrt{(F')_{i,(e_i)}(w) \cdot (\bar{F}')_{i,(e_i)}(w)}} \circ \left(G'_{(0)}(w)\right)^{-1}$$

makes H' satisfy the good normalization condition. This proves the existence part of the lemma.

Next, suppose that both $H = (F, G) = (z + O_{wt}(2), w + O_{wt}(3))$ and $\hat{H} = (\hat{F}, \hat{G}) = T \circ H = (z + O_{wt}(2), w + O_{wt}(3))$ satisfy the good normalization condition. Here T is an automorphism of M_∞. Then T must be of the form in (3.2), for $T(0, w) = 0$. Hence,

$$T = (b(w)zU(w), b(w)\bar{b}(w)w).$$

By the good normalization condition (2.8) on H, $\hat{H}$, we have

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ \hat{F}_{1,(e_2)} & \hat{F}_{2,(e_2)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \hat{F}_{1,(e_n)} & \hat{F}_{2,(e_n)} & \dots & \hat{F}_{n,(e_n)} \end{pmatrix} = b\left(G_{(0)}(w)\right) \begin{pmatrix} 1 & 0 & \cdot & 0 \\ F_{1,(e_2)} & F_{2,(e_2)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ F_{1,(e_n)} & F_{2,(e_n)} & \dots & F_{n,(e_n)} \end{pmatrix} U(G_0(w)) \qquad (3.7)$$

with $U(x)$ unitary and $\mathrm{Im}(\hat{F}_{i,(e_i)}(0, u)) = \mathrm{Im}(F_{i,(e_i)}(0, u)) = 0$. Considering the norm of the first row of the right-hand side, we get $b(G_{(0)}(w)) \cdot \overline{b}(G_{(0)}(w)) = 1$ in case $G_{(0)}(w) = \overline{G_{(0)}(w)}$. Since

$G_0(w) = w + o(w)$, this implies that $b(w)\overline{b}(w) \equiv 1$ and thus $T = (b(w)zU(w), w)$. Write

$$b(w)U(w) = \widetilde{U}(w) = \begin{pmatrix} u_{11} & \ldots & u_{nn} \\ \vdots & \ddots & \vdots \\ u_{n1} & \ldots & u_{nn} \end{pmatrix}.$$

We note that $\widetilde{U}$ is a lower triangular matrix and is unitary when $w = x$. Thus we have $u_{ii}(w)\overline{u_{ii}}(w) = 1$ and $u_{ij} = 0$ for $i \neq j$. Note that

$$u_{11} \equiv 1,\ \hat{F}_{i,(e_i)}(w) = u_{ii}(w) \cdot F_{i,(e_i)}(w) \text{ for } 2 \leq \text{i} \leq n.$$

Since $\hat{F}_{i,(e_i)}(x), F_{i,(e_i)}(x) = 1 + O(x)$ are real, we get $u_{ii}(x) = 1$. This proves the uniqueness part of the lemma. ■

Lemma 3.4. Suppose that H with $H(0) = 0$ is an equivalence map from $w = |z|^2 + \varphi(z, \bar{z})$ to $w' = |z'|^2 + \varphi'(z', \bar{z}')$. Here φ and φ' are normalized as in (2.11). Let s, s' be the lowest order of vanishing in φ and φ', respectively. Then $s = s'$. □

Proof of Lemma 3.4. We seek for a contradiction if $s \neq s'$. Assume, for instance, that $s < s'$. Let T be an automorphism of M_∞ with $T \circ H$ being well normalized. Suppose that T transforms $w' = |z'|^2 + \varphi'(z', \bar{z}')$ to $w'' = |z''|^2 + \varphi''(z'', \overline{z''})$ with s'' the lowest vanishing order for φ''. (Note that φ'' does not necessarily satisfy the normalization (2.11).) We claim that $s' = s''$. To see this, we assume, without loss of generality, that $s', s'' \neq \infty$. Since T is an automorphism of M_∞, by Proposition 3.1, we can write

$$(z'', w'') = (p(z', w'), q(w')) = (z'B + O(|z'|^2 + |w'|), dw' + O(|w'|^2))$$

with $B \in GL(n, \mathbb{C})$, $d \neq 0$ and $q(u') = |p(z', u')|^2$. Here $u' = |z'|^2$. Note that

$$p(z', w') = p(z', u') + O(|z|^{s'}),\ q(w') = q(u') + d \cdot \varphi'^{(s')}(z', \overline{z}') + O(|z'|^{s'+1}).$$

This immediately gives the following:

$$\begin{aligned} q(w') &= |z''|^2 + \varphi''(z'', \overline{z''}) \\ &= |p(z', u') + O(|z'|^{s'})|^2 + \varphi''^{(s'')}(z'B, \overline{z'B}) + O(|z'|^{s''+1}) \\ &= |p(z', u')|^2 + O(|z'|^{s'+1}) + \varphi''^{(s'')}(z'B, \overline{z'B}) + O(|z'|^{s''+1}). \end{aligned}$$

Hence

$$d \cdot \varphi'^{(s')}(z', \overline{z}') + O(|z'|^{s'+1}) = q(w') - q(u') = O(|z'|^{s'+1}) + \varphi''^{(s'')}(z'B, \overline{z'B}) + O(|z'|^{s''+1}).$$

This shows that $s' = s''$.

Now, $T \circ H$ transforms $w = |z|^2 + \varphi$ to $w'' = |z''|^2 + \varphi''$ with $T \circ H$ being well normalized and φ being normalized as in (2.11). Also, $s < s''$. We see that $T \circ H$ transforms $w = |z|^2 + \varphi^{(s)}$ to $w = |z|^2$, moduling $O(|(z_1, \ldots, z_n)|^{s+1})$. This will create a contradiction to the uniqueness part of Proposition 2.4 with $h_j = 0$, when we apply an induction argument. The proof of Lemma 3.4 is complete. ■

An immediate application of Lemma 3.3 and Corollary 2.6 is that if $(M, 0)$ has a flat pseudonormal form, then all of its other pseudonormal forms are flat. Indeed, we will show in the following that any pseudonormal form of $(M, 0)$ is flat if $(M, 0)$ can be flattened:

Suppose that $(M, 0)$ can be flattened. For a given pseudonormal form of $(M, 0)$, there is a formal equivalence map H mapping it into $\mathrm{Im} w = 0$. Now, by Lemma 3.3, we can compose H with an element T of $\mathrm{aut}_0(M_\infty)$ such that $T \circ H := (F_T, G_T)$ satisfies the good normalization condition in (2.8) with $h_j = 0$. Next, since T maps any flattened submanifold to a flattened submanifold, there is a formal transformation $H^* := (F_*, G_*)$ satisfying the normalizations in (2.8) with h_j to be determined later such that $H^* \circ T \circ H := (F_H, G_H)$ maps the pseudonormal form given at the beginning to a flat pseudonormal form. Since both $T \circ H$ and H^* satisfy the normalizations in (2.8), we have

$$\begin{aligned}
(F_H)_1 &= (F_T)_1 + O((F_T)_2, \ldots, (F_T)_n) + O(|(F_T)_1, \ldots, (F_T)_n|^2) \\
&= z_1 + O(z_2, \ldots, z_n) + O\big((z_1, \ldots, z_n)^2\big), \\
(F_H)_i &= (F_T)_i \cdot (F_*)_{i,(e_i)}(G_T) + O((F_T)_{i+1}, \ldots, (F_T)_n) + O(|(F_T)_1, \ldots, (F_T)_n|^2) \\
&= z_i \cdot (F_T)_{i,(e_i)}(w) \cdot (F_*)_{i,(e_i)}((G_T)_{(0)}(w)) + O(z_{i+1}, \ldots, z_n) + O(|(z_1, \ldots, z_n)|^2).
\end{aligned}$$

Here $2 \le i \le n$.

We claim that we can make $(F_*)_{i,(e_i)}\big((G_T)_{(0)}(x)\big)$ real for $x \in \mathbb{R}$ by suitably choosing $h_j's$. Once the claim is proved, we see that $(F_H)_{i,(e_i)}(x)$ is real for $x \in \mathbb{R}$. Hence $H^* \circ T \circ H$ also has the good normalization (as in (2.8) with $h_j = 0$). By Corollary 2.6 (2), we see that $H^* \circ T \circ H = \mathrm{id}$ and two pseudonormal forms are the same. Thus, the pseudonormal at the beginning is also a flattened one.

Now we prove the claim. Set

$$(F_*)_{i,(e_i)}(y) = \widetilde{h}_i(y) + \sqrt{-1}h_i(y) = \sum_{k=1}^{\infty} a_{ik} y^k + \sqrt{-1}\sum_{k=1}^{\infty} b_{ik} y^k,$$
$$(G_T)_{(0)}(x) = x + \alpha(x) + \sqrt{-1}\beta(x),\ \alpha(x) = O(x^2),\ \beta(x) = O(x^2).$$

Here $a_{ik}, b_{ik} \in \mathbb{R}$ for $1 \le k < \infty$ and $\widetilde{h}_i(x),\ h_i(x),\ \alpha(x),\ \beta(x)$ are all real-valued for $x \in \mathbb{R}$.

With these notations, the claim is equivalent to solving the following equation with $(b_{ik})_{k=1}^{\infty}$ as its unknowns:

$$\mathrm{Im}\left(\sum_{k=1}^{\infty}(a_{ik} + \sqrt{-1}b_{ik}) \cdot (x + \alpha(x) + \sqrt{-1}\beta(x))^k\right) = 0, \tag{3.8}$$

with $a_{ik}, \alpha(x), \beta(x)$ given and $x \in \mathbb{R}$.

Collecting terms of degree 1 in (3.8), we get $b_{i1} = 0$. Suppose that we can find b_{ij} with $j = 2, \ldots, k-1$ such that (3.8) holds up to degree $\le k-1$. Then collecting terms of degree k in (3.8), we get

$$\Omega^{(k)}(x) + b_{ik}x^k = 0.$$

Here $\Omega^{(k)}(x)$ is a monomial of degree k determined by b_{ij} with $j = 1, \ldots, k-1$ and other known data. Hence (3.8) can be (uniquely) solved. The claim follows.

Summarizing the above, we have proved the following.

Theorem 3.5. Let $(M, 0)$ be a formal submanifold defined by an equation of the form: $w = |z|^2 + E(z, \overline{z})$ with $E = O(|z|^3)$. Then the following statements are equivalent:

(1) $(M, 0)$ can be flattened.

(2) $(M, 0)$ has a flat pseudonormal form. Namely, M has a pseudonormal form given by an equation of the form: $w' = |z'|^2 + \varphi(z', \overline{z'})$ with φ satisfying the normalizations in (2.11) and the reality condition $\varphi(z', \overline{z'}) = \overline{\varphi(z', \overline{z'})}$.

(3) Any pseudonormal form of $(M, 0)$ is flat. □

Corollary 3.6. M defined in (2.55) can be formally flattened if and only if $b_{i\bar{j}} = \overline{b_{j\bar{i}}}$ for all i, j. □

For a real analytic manifold M in $\mathbb{C}^n$ of real dimension n, it was proved in Moser–Webster [22] and in a article of the first author [13] that M can always be flattened near a point with an elliptic complex tangency. (See also the articles [15, 18, 19].) However, this is no longer the case, as in Corollary 3.6, for a codimension two real analytic manifold $M \subset \mathbb{C}^{n+1}$ near an elliptic complex tangent when $n+1 \geq 3$. For a codimension two real analytic manifold $M \subset \mathbb{C}^{n+1}$ with an elliptic complex tangent point $p_0 \in M$, one can easily show that any other point $p(\neq p_0) \in M$ is a CR point. If M can be holomorphically flattened, or, if M can be holomorphically mapped into $\mathbb{C}^n \times \mathbb{R}$, for any $p(\neq p_0) \in M$, there is obviously a CR submanifold in M of CR dimension $n-1$ passing through p. Namely, p is not a CR minimal point in the sense of Tumanov (see [2], for instance, for the definition). We conjecture this is also the sufficient condition for M being able to be holomorphically flattened near an elliptic complex tangent point. At this point, the first author would like to use this opportunity to correct some typos in [13]:

The x' on [13, p. 679, lines 6–15] and the x_α on [13, p. 679, line 11] should be changed to $\widetilde{x'}$ and $\widetilde{x}_\alpha$, respectively. The first x' in σ and ψ_1^* on [13, p. 679, lines 31–37], and also the second x' in ϕ_n on [13, p. 680, lines 9, 16] should be changed to $\widetilde{x'}$. (The first author also uses this opportunity to remark that the following trivial fact was implicitly used in [14, (4.5)′]: Since $x_\alpha = \widetilde{x}_\alpha + Re(\psi_\alpha^*(\xi, \widetilde{x}', r))$, hence $\widetilde{x}_\alpha = \widetilde{x}_\alpha(x', \xi, r) = x_\alpha + o(r^2)$.)

4 Proof of Theorem 1.1

We now give a proof of Theorem 1.1 by using the rapidly convergent power series method. We let $M \subset \mathbb{C}^{n+1} (n \geq 2)$ be defined by

$$w = \Phi(z, \bar{z}) = |z|^2 + E(z, \bar{z}), \tag{4.1}$$

where $E(z, \xi)$ is holomorphic near $z = \xi = 0$ with vanishing order ≥ 3. Assume that $H = (F, G) = (z + f, w + g)$ is a formal map satisfying the normalization condition in (2.7). We define

$$R = (r_1, r_2, \ldots, r_n) = \left(2^{\frac{2-n}{2}} r,\ 2^{\frac{2-n}{2}} r,\ 2^{\frac{3-n}{2}} r \ldots, 2^{-\frac{1}{2}} r, r\right). \tag{4.2}$$

Then $|R|^2 = 2^{(2-n)} r^2 + \sum_{i=2}^n (2^{\frac{i-n}{2}} r)^2 = 2r^2$. Define the domains:

$$\begin{aligned} \Delta_r &= \{(z, w) : |z_i| < r_i, |w| < 2r^2\}, \\ D_r &= \{(z, \xi) : |z_i| < r_i, |\xi_i| < r_i \text{ for } 1 \leq i \leq n\}. \end{aligned} \tag{4.3}$$

When $E(z,\xi)$ is defined over $\overline{D_r}$, we set the norm of $E(z,\xi)$ on D_r by

$$\|E\|_r = \sup_{(z,\xi)\in D_r} |E(z,\xi)|. \tag{4.4}$$

Also, for a holomorphic map $h(z,w)$ defined on $\overline{\Delta_r}$, we define

$$|h|_r = \sup_{(z,w)\in \Delta_r} |h(z,w)|. \tag{4.5}$$

After a scaling transformation $(z,\xi,w) \longrightarrow (az, a\xi, a^2 w)$, we may assume that E is holomorphic on $\overline{D_1}$ with $|E|_1 \le \eta$ for a given small $\eta > 0$.

Let M be a manifold with the following defining function:

$$w = |z|^2 + E(z,\overline{z}) \text{ with } \mathrm{Ord}(E) \ge d. \tag{4.6}$$

We want to find a polynomial change of coordinates

$$z' = z + \hat{f}(z,w),\ w' = w + \hat{g}(z,w) \quad \text{with } \mathrm{Ord}_{wt}(\hat{f}) \ge d-1,\ \mathrm{Ord}_{wt}(\hat{g}) \ge d,\ \hat{f}^{(t-1)} = \hat{g}^{(t)} = 0 \text{ for } t \ge 2d-2 \tag{4.7}$$

such that in the new coordinates, M is defined by

$$w' = |z'|^2 + \hat{\varphi}(z',\overline{z}') + E'(z',\overline{z}') \text{ with } \quad \mathrm{Ord}(E') \ge 2d-2, \tag{4.8}$$

where $\hat{\varphi}(z',\overline{z}')$ with $\deg(\hat{\varphi}(z',\overline{z}')) \ge d$ is either identical 0 or has degree $\deg(\hat{\varphi}) \le 2d-3$, and $\hat{\varphi}$ also satisfies the normalization in (2.11).

Applying (4.8), this amounts to solving the following equation:

$$w + \hat{g}(z,w) = |z + \hat{f}(z,w)|^2 + \hat{\varphi}(z + \hat{f}(z,w), \overline{z} + \overline{\hat{f}(z,w)}) + E'(z + \hat{f}(z,w), \overline{z} + \overline{\hat{f}(z,w)}), \tag{4.9}$$

with $w = |z|^2 + E(z,\overline{z})$. Instead of solving the nonlinear equation (4.9), we consider the following linear equation with $(\hat{f}, \hat{g}, \hat{\varphi})$ as its unknowns:

$$J^{2d-3} E(z,\bar{z}) = -\hat{g}(z,u) + 2\mathrm{Re}\left(\sum_{i=1}^{n} \bar{z}_i \hat{f}_i(z,u)\right) + \hat{\varphi}(z,\bar{z}), \tag{4.10}$$

where $J^{2d-3}E(z,\bar{z}) = \sum_{i\le 2d-3} E^{(i)}(z,\bar{z})$. Suppose we are able to solve (4.10). Define

$$\hat{F} = z + \hat{f},\ \hat{G} = w + \hat{g},\ \hat{H} = (\hat{F}, \hat{G}).$$

We can verify that under such a transformation, E' indeed satisfies the vanishing condition in (4.8). (See the proof in Lemma (4.1) for a detailed proof on this).

By Proposition 2.4, we can indeed uniquely solve (4.10) under the normalizations in (2.8) with $h_j = 0$ and under the assumption that $\hat{\varphi}(z,\overline{z})$ satisfies the normalization in (2.11). By (2.53), its solution is given by the following:

$$\begin{cases} \hat{F}_1(z,w) = z_1 + A_1 + A_2 + A_3, \\ \hat{F}_h(z,w) = z_h + B_{h1} + B_{h2} + B_{h3} + B_{h4} + B_{h5}, \text{ for } 2 \le h \le n, \\ \hat{G}(z,w) = w + C_1 + C_2 + C_3, \\ \hat{\varphi}(z,\overline{z}) = J^{2d-3}E(z,\bar{z}) + \hat{g}(z,u) - 2\mathrm{Re}\left(\sum_{i=1}^{n} (\overline{z_i}\hat{f}_i(z,u))\right), \end{cases} \tag{4.11}$$

where

$$A_1 = \sum_{j_1=0,|J|\ge 1, d-1\le |J|+2k\le 2d-4} E^{(ke_1)}_{(J,e_1)} z^J w^k,\ A_2 = \sum_{|I|\ge 1,k\ge 1,d-1\le |I|+2k-1\le 2d-4} \overline{E^{(ke_1)}_{(0,I)}} z^{I+e_1} w^{k-1},$$

$$A_3 = \sum_{|I|\ge 1,d-1\le |I|+2k+1\le 2d-4} \sum_{i=2}^{n} \overline{E^{(ke_1+e_i)}_{(0,I)}} z^{I+e_1} w^k,$$

$$B_{h1} = \sum_{k\ge 1,d-1\le 1+2k\le 2d-4} \frac{1}{2}\left(-\sum_{j=2}^{h-1} \mathrm{Re}\big(E^{(ke_1+e_j)}_{(0,0)}\big) - 2\mathrm{Re}\big(E^{(ke_1+e_h)}_{(0,0)}\big)\right) z_h w^k,$$

$$B_{h2} = \sum_{i>h,d-1\le 1+2k\le 2d-4} E^{(ke_1)}_{(e_i,e_h)} z_i w^k,\ B_{h3} = \sum_{j_h=0,|J|\ge 2,d-1\le |J|+2k\le 2d-4} E^{(ke_1)}_{(J,e_h)} z^J w^k,$$

$$B_{h4} = \sum_{|I|\ge 1,k\ge 1,d-1\le |I|+2k-1\le 2d-4} \overline{E^{(ke_1)}_{(0,I)}} z^{I+e_h} w^{k-1},$$

$$B_{h5} = \sum_{|I|\ge 1,d-1\le |I|+2k+1\le 2d-4} \left(\sum_{i=h+1}^{n} \overline{E^{(ke_1+e_i)}_{(0,I)}} - \overline{E^{(ke_1+e_h)}_{(0,I)}}\right) z^{I+e_h} w^k,$$

$$C_1 = -\sum_{d\le 2k\le 2d-3} E^{(ke_1)}_{(0,0)} w^k,\ C_2 = -\sum_{d\le 2k+2\le 2d-3} \mathrm{Re}\left(\sum_{j=2}^{n} E^{(ke_1+e_j)}_{(0,0)}\right) w^{k+1},$$

$$C_3 = \sum_{d\le |I|+2k\le 2d-3,|I|\ge 1} \big(\overline{E^{(ke_1)}_{(0,I)}} - E^{(ke_1)}_{(I,0)}\big) z^I w^k.$$

As in the article of Moser [21], the following lemma will be fundamental for applying the rapid iteration procedure of Moser to prove Theorem 1.1.

Lemma 4.1. Let $M : w = |z|^2 + E(z, \bar{z})$ be defined as in (4.1). Suppose that $\mathrm{Ord}(E) \geq d$. Let $\hat{H}$ and E' be defined above. Then $\mathrm{Ord}(E') \geq 2d - 2$. Moreover, if $(M, 0)$ is formally equivalent to the quadric $(M_\infty, 0)$ as in Theorem 1.1, then $\hat{\varphi} \equiv 0$. □

Proof of Lemma 4.1. Making use of (4.9) and (4.10), we have

$$E'(z', \overline{z'}) = \big(\hat{g}(z, \Phi) - \hat{g}(z, u)\big) - 2\mathrm{Re}\left(\sum_{i=1}^{n} \overline{z_i}\big(\hat{f}_i(z, \Phi) - \hat{f}_i(z, u)\big)\right) - |\hat{f}(z, \Phi)|^2$$
$$-\big(\hat{\varphi}(\hat{F}, \overline{\hat{F}}) - \hat{\varphi}(z, \overline{z})\big) + \big(E(z, \bar{z}) - J^{2d-3} E(z, \bar{z})\big). \tag{4.12}$$

Since $\mathrm{Ord}_{wt}(\hat{f}) \geq d - 1$, $\mathrm{Ord}_{wt}(\hat{g}) \geq d$, $Ord(\hat{\varphi}) \geq d$, we have

$$\mathrm{Ord}(\hat{g}(z, \Phi) - \hat{g}(z, u)) \geq \min\{(d-1) + d, 2d - 2\} = 2d - 2,$$
$$\mathrm{Ord}(\hat{f}_i(z, \Phi) - \hat{f}_i(z, u)) \geq \min\{(d-2) + d, 2d - 3\} = 2d - 3,$$
$$\mathrm{Ord}(|\hat{f}(z, \Phi)|^2) \geq 2(d-1) = 2d - 2,$$
$$\mathrm{Ord}(\hat{\varphi}(\hat{F}, \overline{\hat{F}}) - \hat{\varphi}(z, \overline{z})) \geq \min\{(d-1) + d, 2d - 2\} = 2d - 2.$$

Thus $\mathrm{Ord}\,(E') \geq 2d - 2$.

Now, assume that $(M, 0)$ is formally equivalent to the quadric $w = |z|^2$. By Lemma 3.3, we have $\hat{\varphi} \equiv 0$. The lemma follows. ■

In all that follows, we assume that M is as in Theorem 1.1.

Choose r', σ, ϱ, r to be such that

$$\tfrac{1}{2} < r' < \sigma < \varrho < r \leq 1, \qquad \varrho = \tfrac{1}{3}(2r' + r), \qquad \sigma = \tfrac{1}{3}(2r' + \varrho). \tag{4.13}$$

Before proceeding to the estimates of the solution given in (4.11), we need the following lemma:

Lemma 4.2. If E is holomorphic in $\overline{D_r}$, then we have

$$\big|E^{(ke_1)}_{(I,T)}\big| \leq \frac{(k+2)^n \|E\|_r}{R^{I+T} \cdot (2r^2)^k}, \qquad \big|E^{(ke_1+e_j)}_{(I,T)}\big| \leq \frac{2^n (k+2)^n \|E\|_r}{R^{I+T} (2r^2)^{k+1}}.$$ □

Proof of Lemma 4.2. We here give the estimates for $|E^{(ke_1)}_{(0,I)}|$, $|E^{(ke_1+e_j)}_{(0,I)}|$. The others can be done in exactly the same way. Suppose that $E = \sum a_{i_1 \ldots i_n j_1 \ldots j_n} z_1^{i_1} \ldots z_n^{i_n} \overline{z}_1^{j_1} \ldots \overline{z}_n^{j_n}$. Then by

(2.3), we have

$$\begin{aligned}
E_{(0,I)} &= \sum_J a_{j_1\dots j_n(i_1+j_1)\dots(i_n+j_n)}|z_1|^{2j_1}\dots|z_n|^{2j_n}\\
&= \sum_J a_{J(I+J)}\left(2^{1-n}\left(u+\sum_{i=2}^n 2^{n-i}v_i\right)\right)^{j_1}\cdot\prod_{h=2}^n\left(2^{h-n-1}\left(u+\sum_{i=h+1}^n 2^{n-i}v_i-2^{n-h}v_h\right)\right)^{j_h}\\
&= \sum_J a_{J(I+J)}\cdot 2^{-\left((n-1)j_1+\sum_{h=2}^n(n-h+1)j_h\right)}\cdot\left(u^{|J|}+\sum_{k=2}^n 2^{n-k}\left(\sum_{h=1}^{k-1}j_h-j_k\right)u^{|J|-1}v_k\right)\\
&\quad +O(|(v_2,\dots,v_n)|^2).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
E_{(0,I)}^{(ke_1)} &= \sum_{|J|=k} a_{J(I+J)}\cdot 2^{-((n-1)j_1+\sum_{h=2}^n(n-h+1)j_h)},\\
E_{(0,I)}^{(ke_1+e_l)} &= \sum_{|J|=k+1} a_{J(I+J)}\cdot 2^{-((n-1)j_1+\sum_{h=2}^n(n-h+1)j_h)}\cdot 2^{n-l}\cdot\left(\sum_{h=1}^{l-1}j_h-j_l\right).
\end{aligned}\tag{4.14}$$

By the Cauchy estimates, we get

$$\begin{aligned}
\left|E_{(0,I)}^{(ke_1)}\right| &= \left|\sum_{|J|=k} a_{J(I+J)}\cdot 2^{-((n-1)j_1+\sum_{h=2}^n(n-h+1)j_h)}\right|\\
&\le \sum_{|J|=k}\frac{\|E\|_r}{R^{I+2J}}\cdot 2^{-\left((n-1)j_1+\sum_{h=2}^n(n-h+1)j_h\right)}\\
&= \sum_{|J|=k}\frac{\|E\|_r}{R^I}\frac{2^{-((n-1)j_1+(n-1)j_2+\dots+j_n)}}{(2^{2-n}r^2)^{j_1}\cdot(2^{2-n}r^2)^{j_2}\dots(r^2)^{j_n}}\\
&\le \frac{(k+1)^{n-1}\|E\|_r}{R^I\cdot(2r^2)^k}.\\
\left|E_{(0,I)}^{(ke_1+e_l)}\right| &= \left|\sum_{|J|=k+1} a_{J(I+J)}2^{-((n-1)j_1+\sum_{h=2}^n(n-h+1)j_h)}2^{n-l}\left(\sum_{h=1}^{l-1}j_h-j_l\right)\right|\\
&\le \sum_{|J|=k+1}\frac{\|E\|_r}{R^{I+2J}}\cdot 2^{-((n-1)j_1+\sum_{h=2}^n(n-h+1)j_h)}\cdot 2^{n-l}\left|\sum_{h=1}^{l-1}j_h-j_l\right|\\
&\le \sum_{|J|=k+1}\frac{\|E\|_r}{R^I\cdot(2r^2)^{k+1}}\cdot 2^n(k+1)\\
&= \frac{2^n(k+2)^n\|E\|_r}{R^I\cdot(2r^2)^{k+1}}.
\end{aligned}$$

Here we have used the fact that

$$\sharp\{(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n : j_h \geq 0 \text{ for } 1 \leq h \leq n,\ j_1 + j_2 + \cdots + j_n = k\} \leq (k+1)^{n-1}.$$

This completes the proof of Lemma 4.2. ∎

To carry out the rapid iteration procedure, we need the following estimates of the solution given by (4.11) for the Equation (4.10).

Proposition 4.3. Suppose that $w = |z|^2 + E(z, \bar{z})$ is formally equivalent to M_∞ with E holomorphic over $\overline{D_r}$ and $Ord(E) \geq d$. Then the solution given in (4.11) satisfies the following estimates:

$$\begin{aligned} |\hat{f}_h|_\varrho, |\hat{g}|_\varrho &\leq \frac{C(n)(2d)^{2n}\|E\|_r}{r-\varrho}\left(\frac{\varrho}{r}\right)^{d-1}, \\ |\nabla \hat{f}_h|_\varrho, |\nabla \hat{g}|_\varrho &\leq \frac{C(n)(2d)^{2n}\|E\|_r}{(r-\varrho)^2}\left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}}, \\ \|E(z,\xi) - J^{2d-3}E(z,\xi)\|_\varrho &\leq \frac{(2d)^{2n}\|E\|_r}{(r-\varrho)^{2n}}\left(\frac{\varrho}{r}\right)^{2d-2}, \end{aligned} \tag{4.15}$$

where $C(n)$ is a constant depending only on n and ϱ, r satisfy (4.13). (Indeed, we can take $C(n) = 3^2 n(n+1)2^{2n+3}$.) □

Proof of Proposition 4.3. For B_{h1}, we have

$$\begin{aligned} |B_{h1}|_\varrho &= \left| \sum_{k\geq 1, d-1\leq 1+2k\leq 2d-4} \frac{1}{2}\left(-\sum_{j=2}^{h-1}\mathrm{Re}\left(E^{(ke_1+e_j)}_{(0,0)}\right) - 2\mathrm{Re}\left(E^{(ke_1+e_h)}_{(0,0)}\right)\right) z_h w^k \right|_\varrho \\ &\leq \sum_{k\geq 1, d-1\leq 1+2k\leq 2d-4} \frac{1}{2}\left(\sum_{j=2}^{h-1}\frac{2^n(k+2)^n\|E\|_r}{(2r^2)^{k+1}} + 2\frac{2^n(k+2)^n\|E\|_r}{(2r^2)^{k+1}}\right)\cdot 2^{-\frac{n-h}{2}}\varrho\cdot(2\varrho^2)^k \\ &\leq \sum_{k\geq 1, d-1\leq 1+2k\leq 2d-4} \frac{1}{2}\cdot\frac{n2^n(k+2)^n\|E\|_r}{(2r^2)^{k+1}}\cdot 2^k\varrho^{2k+1} \\ &\leq \sum_{k\geq 1, d-1\leq 1+2k\leq 2d-4} n2^n(2d)^n\|E\|_r\left(\frac{\varrho}{r}\right)^{2k+1} \\ &\leq \sum_{k\geq 1, d-1\leq 1+2k\leq 2d-4} n2^n(2d)^n\|E\|_r\cdot\frac{r}{r-\varrho}\left(\frac{\varrho}{r}\right)^{d-1} \\ &\leq \frac{n2^n(2d)^{2n}\|E\|_r}{r-\varrho}\cdot\left(\frac{\varrho}{r}\right)^{d-1}. \end{aligned}$$

For B_{h5}, we have

$$\begin{aligned}
|B_{h5}|_{\varrho} &= \left| \sum_{|I|\geq 1, d-1\leq |I|+2k+1\leq 2d-4} \left(\sum_{i=h+1}^{n} \overline{E_{(0,I)}^{(ke_1+e_i)}} - \overline{E_{(0,I)}^{(ke_1+e_h)}} \right) z^{I+e_h} w^k \right|_{\varrho} \\
&\leq \sum_{|I|\geq 1, d-1\leq |I|+2k+1\leq 2d-4} (R')^{I+e_h} (2\varrho^2)^k \cdot n \frac{2^n (k+2)^n \|E\|_r}{R^I \cdot (2r^2)^{k+1}} \\
&\leq \sum_{|I|\geq 1, d-1\leq |I|+2k+1\leq 2d-4} \frac{n 2^n (k+2)^n \|E\|_r}{2r} \left(\frac{\varrho}{r}\right)^{|I|+2k+1} \\
&\leq \frac{n 2^n (2d)^{2n} \|E\|_r}{r-\varrho} \left(\frac{\varrho}{r}\right)^{d-1}.
\end{aligned}$$

Here and in what follows, we write $R' = (2^{\frac{2-n}{2}}\varrho, 2^{\frac{2-n}{2}}\varrho, 2^{\frac{3-n}{2}}\varrho \ldots, 2^{-\frac{1}{2}}\varrho, \varrho)$. We have also used the fact that

$$\sharp\{(i_1, i_2, \ldots, i_n, k) \in \mathbb{Z}^{n+1} : i_h, k \geq 0 \text{ for } 1 \leq h \leq n, \sum_{h=1}^{n} i_h + 2k = 2d-1\} \leq (2d)^n.$$

Similarly, we have

$$|B_{h2}|_{\varrho}, |B_{h3}|_{\varrho}, |B_{h4}|_{\varrho} \leq \frac{n 2^n (2d)^{2n} \|E\|_r}{r-\varrho} \left(\frac{\varrho}{r}\right)^{d-1}.$$

Hence we get

$$|\hat{f}_h|_{\varrho} \leq \frac{n \cdot 2^{n+3} (2d)^{2n} \|E\|_r}{r-\varrho} \left(\frac{\varrho}{r}\right)^{d-1}.$$

Now letting $\tau = \frac{r+2\varrho}{3}$ and using the Cauchy estimates, we have the following estimate of the derivatives of $\hat{f}_h$:

$$\begin{aligned}
\left|\frac{\partial \hat{f}_h}{\partial z_1}\right|_{\varrho} &\leq \frac{|\hat{f}_h|_{\tau}}{2^{\frac{2-n}{2}}(\tau-\varrho)} \leq \frac{2^n \cdot n 2^{n+3}(2d)^{2n}\|E\|_r}{(\tau-\varrho)(r-\tau)} \left(\frac{\tau}{r}\right)^{d-1} \leq \frac{3^2 n \cdot 2^{2n+3}(2d)^{2n}\|E\|_r}{(r-\varrho)^2} \left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}}, \\
\left|\frac{\partial \hat{f}_h}{\partial z_i}\right|_{\varrho} &\leq \frac{|\hat{f}_h|_{\tau}}{2^{\frac{i-n}{2}}(\tau-\varrho)} \leq \frac{3^2 n \cdot 2^{2n+3}(2d)^{2n}\|E\|_r}{(r-\varrho)^2} \left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}}, \text{ for } 2 \leq i \leq n, \\
\left|\frac{\partial \hat{f}_h}{\partial w}\right|_{\varrho} &\leq \frac{|\hat{f}_h|_{\tau}}{2\tau^2 - 2\varrho^2} \leq \frac{3^2 n \cdot 2^{n+3}(2d)^{2n}\|E\|_r}{(r-\varrho)^2} \left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}}.
\end{aligned} \tag{4.16}$$

Here we have used the fact that

$$\left(\frac{\tau}{r}\right)^2 \le \frac{\varrho}{r} \text{ for } \frac{1}{2} < \varrho < \tau < r \le 1,\ \tau = \frac{r+2\varrho}{3}. \tag{4.17}$$

The inequality (4.16) shows that

$$|\nabla \hat{f}_h|_\varrho \le \frac{3^2 n(n+1)\cdot 2^{2n+3}(2d)^{2n}\|E\|_r}{(r-\varrho)^2}\left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}}.$$

The corresponding estimates on $\hat{f}_1$ and $\hat{g}$ can be achieved similarly.

We next estimate $E(z,\xi) - J^{2d-3}E(z,\xi)$.

$$\begin{aligned}
&\|E(z,\xi) - J^{2d-3}E(z,\xi)\|_\varrho \\
&= \left\| \sum_{|I|+|J|\ge 2d-2} a_{i_1\cdots i_n j_1\cdots j_n} z_1^{i_1}\cdots z_n^{i_n} {\xi_1}^{j_1}\cdots {\xi_n}^{j_n} \right\|_\varrho \\
&\le \sum_{|I|+|J|\ge 2d-2} \|E\|_r \left(\frac{R'}{R}\right)^{I+J} \\
&\le \sum_{|I|+|J|=2d-2,|K|,|L|\ge 0} \|E\|_r \left(\frac{\varrho}{r}\right)^{|I|+|J|}\cdot\left(\frac{\varrho}{r}\right)^{k_1}\cdots\left(\frac{\varrho}{r}\right)^{k_n}\cdot\left(\frac{\varrho}{r}\right)^{l_1}\cdots\left(\frac{\varrho}{r}\right)^{l_n} \\
&\le \sum_{|I|+|J|=2d-2} \|E\|_r \left(\frac{\varrho}{r}\right)^{2d-2}\cdot\left(\frac{1}{1-\frac{\varrho}{r}}\right)^{2n} \\
&\le \frac{(2d)^{2n}\|E\|_r}{(r-\varrho)^{2n}}\left(\frac{\varrho}{r}\right)^{2d-2}.
\end{aligned}$$

Here we have used the fact that

$$\sharp\{(i_1,\cdots,i_n,j_1,\cdots,j_n)\in\mathbb{Z}^{2n} : i_h, j_h \ge 0 \text{ for } 1\le h\le n, \sum_{h=1}^n (i_h+j_h) = k\} \le (k+1)^{2n}.$$

This finishes the proof of Proposition 4.3. ■

Proposition 4.4. Let $E, C(n)$ be as in Proposition 4.3 and let r', σ, ϱ, r be as in (4.13). Then there exists a small constant $\delta_0(n) > 0$ depending only on n (but independent of r, r', E) such that for

$$\frac{C(n)(2d)^{2n}\|E\|_r}{(r-\varrho)^2}\left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}} < \delta_0(n), \tag{4.18}$$

and

$$\frac{\|E\|_r}{(r-\varrho)^2} < \delta_0(n), \tag{4.19}$$

we have $\Psi(z', w') := \hat{H}^{-1}(z', w')$ is well defined in $\overline{\triangle_\sigma}$. Moreover, it holds that $\Psi(\triangle_{r'}) \subset \triangle_\sigma$, $\Psi(\triangle_\sigma) \subset \triangle_\varrho$, $E'(z,\xi)$ is holomorphic in $\triangle_\sigma$ and

$$\|E'\|_{r'} \le C_d\|E\|_r^2 + \widetilde{C}_d\|E\|_r. \tag{4.20}$$

Here

$$C_d = \frac{(2n+1)\cdot 3^2 C(n)(2d)^{2n}}{(r-r')^2}\left(\frac{r'}{r}\right)^{\frac{d-1}{4}} + \left(\frac{r'}{r}\right)^{d-1} n\cdot\left(\frac{3C(n)(2d)^{2n}}{r-r'}\right)^2,$$

$$\widetilde{C}_d = \frac{3^{2n}\cdot(2d)^{2n}}{(r-r')^{2n}}\left(\frac{r'}{r}\right)^{d-1}. \qquad \square$$

Proof of Proposition 4.4. We need to show that for each $(z', w') \in \overline{\triangle_\sigma}$, we can uniquely solve the system:

$$\begin{cases} z' = z + \hat{f}(z,w), \\ w' = w + \hat{g}(z,w) \end{cases} \tag{4.21}$$

with $(z,w) \in \triangle_\varrho$. By (4.15) and (4.18), we can choose a fixed sufficiently small constant $\delta_0(n)$ depending only on n (and independent of r, r', E) such that

$$|\nabla \hat{f}|_\varrho + |\nabla \hat{g}|_\varrho < \frac{1}{2} \text{ and } |\hat{f}|_\varrho + |\hat{g}|_\varrho < 2^{\frac{2-n}{2}-1}\cdot\min(\varrho-\sigma, \sigma-r') = 2^{\frac{2-n}{2}-1}(\sigma-r'). \tag{4.22}$$

Here $|\nabla\hat{f}|_\varrho = \sum_{i=1}^n |\nabla \hat{f}_i|_\varrho$. Define $(z^{[1]}, w^{[1]}) = (z', w') \in \triangle_\sigma$ and $(z^{[j]}, w^{[j]})$ inductively by

$$\begin{cases} z^{[j+1]} = z' - \hat{f}\big(z^{[j]}, w^{[j]}\big) \\ w^{[j+1]} = w' - \hat{g}\big(z^{[j]}, w^{[j]}\big). \end{cases}$$

We next use the standard Picard iteration procedure to find a $(z,w) \in \triangle_\varrho$ satisfying $\hat{H}(z,w) = (z', w')$.

We first inductively show that $(z^{[j]}, w^{[j]}) \in \triangle_\varrho$ for all $j \ge 1$. Note that we already have $(z^{[1]}, w^{[1]}) = (z', w') \in \triangle_\sigma \subset \triangle_\varrho$. Suppose that $(z^{[i]}, w^{[i]}) \in \triangle_\varrho$ for $1 \le i \le j$. Then we have for all $2 \le k \le n$,

$$\left|\left(z^{[j+1]}\right)_1\right| \le |z'_1| + \left|\hat{f}_1\left(z^{[j]}, w^{[j]}\right)\right| \le 2^{\frac{2-n}{2}}\sigma + 2^{\frac{2-n}{2}-1}(\varrho - \sigma) \le 2^{\frac{2-n}{2}-1}(\sigma + \varrho) < 2^{\frac{2-n}{2}}\varrho,$$
$$\left|\left(z^{[j+1]}\right)_k\right| \le |z'_k| + \left|\hat{f}_k\left(z^{[j]}, w^{[j]}\right)\right| \le 2^{\frac{k-n}{2}}\sigma + 2^{\frac{2-n}{2}-1}(\varrho - \sigma) \le 2^{\frac{k-n}{2}-1}(\sigma + \varrho) < 2^{\frac{k-n}{2}}\varrho,$$
$$\left|w^{[j+1]}\right| \le |w'| + \left|\hat{g}\left(z^{[j]}, w^{[j]}\right)\right| < 2\sigma^2 + 2^{\frac{2-n}{2}-1}(\varrho - \sigma) < 2\sigma^2 + 2(\varrho^2 - \sigma^2) = 2\varrho^2.$$

Next, for $t \in [0, 1]$, let $P_{ti} = t(z^{[j]})_i + (1-t)(z^{[j-1]})_i$, $Q_t = t w^{[j]} + (1-t) w^{[j-1]}$. Then $(P_{ti}, Q_t) \in \triangle_\varrho$. Hence we have

$$\begin{aligned}
&\left|z^{[j+1]} - z^{[j]}\right| + \left|w^{[j+1]} - w^{[j]}\right| \\
&\quad = \left|\hat{f}\left(z^{[j]}, w^{[j]}\right) - \hat{f}\left(z^{[j-1]}, w^{[j-1]}\right)\right| + \left|\hat{g}\left(z^{[j]}, w^{[j]}\right) - \hat{g}\left(z^{[j-1]}, w^{[j-1]}\right)\right| \\
&\quad = \sum_{h,i=1}^{n} \int_0^1 \left(\left|\frac{\partial \hat{f}_h}{\partial z_i}(P_{ti}, Q_t)\right| \cdot \left|\left(z^{[j]}\right)_i - \left(z^{[j-1]}\right)_i\right| + \left|\frac{\partial \hat{f}_h}{\partial w}(P_{ti}, Q_t)\right| \cdot \left|w^{[j]} - w^{[j-1]}\right| \right) dt \\
&\qquad + \sum_{i=1}^{n} \int_0^1 \left(\left|\frac{\partial \hat{g}}{\partial z_i}(P_{ti}, Q_t)\right| \cdot \left|\left(z^{[j]}\right)_i - \left(z^{[j-1]}\right)_i\right| + \left|\frac{\partial \hat{g}}{\partial w}(P_{ti}, Q_t)\right| \cdot \left|w^{[j]} - w^{[j-1]}\right| \right) dt \\
&\quad \le (|\nabla \hat{f}|_\varrho + |\nabla \hat{g}|_\varrho) \cdot \left(\left|z^{[j]} - z^{[j-1]}\right| + \left|w^{[j]} - w^{[j-1]}\right|\right) \\
&\quad \le \frac{1}{2}\left(\left|z^{[j]} - z^{[j-1]}\right| + \left|w^{[j]} - w^{[j-1]}\right|\right).
\end{aligned}$$

By the fixed point theorem, we can solve (4.21) with $(z, w) \in \triangle_\varrho$. The uniqueness of the solution of (4.21) also follows, in a standard way, from the gradient estimate in (4.22).

Similarly, choosing a fixed small constant $\delta_0(n)$ (independent of r, r', E) such that (4.22) holds, then $\Psi(\triangle_{r'}) \subset \triangle_\sigma$. Hence we conclude that E' is holomorphic in $\triangle_\sigma$. Moreover,

$$\|E'(z', \xi')\|_{r'} \le \|Q\|_\sigma, \tag{4.23}$$

where

$$\begin{aligned}
Q = {} & (\hat{g}(z, \widetilde{\Phi}) - \hat{g}(z, \widetilde{u})) - \sum_{i=1}^{n} \xi_i(\hat{f}_i(z, \widetilde{\Phi}) - \hat{f}_i(z, \widetilde{u})) - \sum_{i=1}^{n} z_i(\overline{\hat{f}_i}(\xi, \hat{\Phi}) - \overline{\hat{f}_i}(\xi, \widetilde{u})) \\
& - \sum_{i=1}^{n} \hat{f}_i(z, \widetilde{\Phi}) \cdot \overline{\hat{f}_i}(\xi, \hat{\Phi}) + (E - J^{2d-3}E)(z, \xi), \text{ with } \widetilde{\Phi} = \Phi(z, \xi), \hat{\Phi} = \overline{\Phi}(\xi, z), \widetilde{u} = \sum_{i=1}^{n} z_i \xi_i.
\end{aligned} \tag{4.24}$$

By (4.19), we can choose δ_0 such that $\|E\|_r < 2(\varrho^2 - \sigma^2)$. Note that for $(z, \xi) \in D_\sigma$, we have

$$|\widetilde{\Phi}| = |\Phi(z,\xi)| \le \sum_{i=1}^{n} |z_i \xi_i| + |E(z,\xi)| < 2\sigma^2 + 2(\varrho^2 - \sigma^2) = 2\varrho^2.$$

This implies that $(z, \widetilde{\Phi}) \in \triangle_\varrho$. Moreover, for $t \in [0,1]$, the line segment $(z, t\widetilde{u} + (1-t)\widetilde{\Phi}) \in \triangle_\varrho$; for $\triangle_\varrho$ is convex. Similarly, we have $(\xi, \hat{\Phi}) \in \triangle_\varrho$, $(\xi, t\widetilde{u} + (1-t)\hat{\Phi}) \in \triangle_\varrho$. Hence

$$\begin{aligned} |\hat{g}(z,\widetilde{\Phi}) - \hat{g}(z,\widetilde{u})|_\sigma &\le |\nabla \hat{g}|_\varrho \cdot \|E\|_r \le \frac{C(n)(2d)^{2n}\|E\|_r^2}{(r-\varrho)^2}\left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}} \\ &\le \frac{3^2 C(n)(2d)^{2n}\|E\|_r^2}{(r-r')^2}\left(\frac{r'}{r}\right)^{\frac{d-1}{4}}. \end{aligned} \tag{4.25}$$

Here we have used the fact that $(\frac{\varrho}{r})^2 < \frac{r'}{r}$. (This can be achieved by the same token as for (4.17).) Similarly, we have

$$|\hat{f}_i(z,\widetilde{\Phi}) - \hat{f}_i(z,\widetilde{u})|_\sigma, \ |\overline{\hat{f}_i}(\xi,\hat{\Phi}) - \overline{\hat{f}_i}(\xi,\hat{u})|_\sigma \le \frac{3^2 C(n)(2d)^{2n}\|E\|_r^2}{(r-r')^2}\left(\frac{r'}{r}\right)^{\frac{d-1}{4}} \quad \text{for } 1 \le i \le n. \tag{4.26}$$

We also have

$$\begin{aligned} \left|\sum_{i=1}^{n} \hat{f}_i(z,\widetilde{\Phi}) \cdot \overline{\hat{f}_i}(\xi,\hat{\Phi})\right|_\sigma &\le |\hat{f}|_\varrho^2 \le n \cdot \left(\frac{C(n)(2d)^{2n}\|E\|_r}{r-\varrho}\left(\frac{\varrho}{r}\right)^{d-1}\right)^2 \\ &\le n \cdot \left(\frac{3C(n)(2d)^{2n}\|E\|_r}{r-r'}\right)^2 \cdot \left(\frac{r'}{r}\right)^{d-1}, \\ \|(E - J^{2d-3}E)(z,\xi)\|_\sigma \le \frac{(2d)^{2n}\|E\|_r}{(r-\sigma)^{2n}}\left(\frac{\sigma}{r}\right)^{2d-2} &\le \frac{3^{2n}(2d)^{2n}\|E\|_r}{(r-r')^{2n}} \cdot \left(\frac{r'}{r}\right)^{d-1}. \end{aligned} \tag{4.27}$$

By (4.24)–(4.27), we obtain:

$$\begin{aligned} \|E'\|_{r'} \le &\left\{\frac{(2n+1)\cdot 3^2 C(n)(2d)^{2n}}{(r-r')^2}\left(\frac{r'}{r}\right)^{\frac{d-1}{4}} + \left(\frac{r'}{r}\right)^{d-1} n \cdot \left(\frac{3C(n)(2d)^{2n}}{r-r'}\right)^2\right\}\|E\|_r^2 \\ &+ \frac{3^{2n}\cdot(2d)^{2n}}{(r-r')^{2n}}\left(\frac{r'}{r}\right)^{d-1}\|E\|_r. \end{aligned}$$

This completes the proof of Proposition 4.4. ■

Now we turn to the proof of Theorem 1.1. Set $r_\upsilon, \varrho_\upsilon, \sigma_\upsilon$ as follows:

$$r_\upsilon = \frac{1}{2}\left(1 + \frac{1}{\upsilon+1}\right), \qquad \varrho_\upsilon = \frac{1}{3}(2r_\upsilon + r_{\upsilon+1}), \qquad \sigma_\upsilon = \frac{1}{3}(2r_\upsilon + \varrho_\upsilon).$$

We will apply the previous estimates with $r = r_\upsilon, \varrho = \varrho_\upsilon, \sigma = \sigma_\upsilon, r' = r_{\upsilon+1}, \Psi = \Psi_\upsilon, \ldots,$ with $\upsilon = 0, 1, \ldots,$. Then we have the following (see [21, (4.5)]):

$$(r_\upsilon - r_{\upsilon+1})^{-1} = 2(\upsilon+1)(\upsilon+2), \qquad \frac{r_{\upsilon+1}}{r_\upsilon} = 1 - \frac{1}{(\upsilon+2)^2}. \tag{4.28}$$

Define a sequence of real analytic submanifolds:

$$M_k \ : \ w = |z|^2 + E_k(z, \bar{z})$$

by $M_0 = M$, $M_{\upsilon+1} = \Psi_\upsilon^{-1}(M_\upsilon)$ for all $\upsilon = 0, 1, 2, \ldots$, where Ψ_υ is the biholomorphic mapping taking $\triangle_{\sigma_\upsilon}$ into $\triangle_{\varrho_\upsilon}$. And let

$$d_\upsilon = \mathrm{Ord}(E_\upsilon), \ \Phi_\upsilon = \Psi_0 \circ \Psi_1 \circ \cdots \circ \Psi_\upsilon.$$

Since $s = \infty$, we find that

$$\mathrm{Ord}(E_\upsilon) = d_\upsilon \geq 2^\upsilon + 2 \text{ for } \upsilon \geq 0.$$

We next state the following elementary fact.

Lemma 4.5. Suppose that there is a constant $C > 0$ and number $a > 1$ such that $d_\upsilon \geq Ca^\upsilon$. Then for any integer $m_1, m_2, m_3 > 0$,

$$\lim_{\upsilon\to\infty} \upsilon^{m_3} d_\upsilon^{m_1}\left(1 - \frac{1}{\upsilon^{m_2}}\right)^{d_\upsilon} = 0. \qquad \square$$

Now we are in a position to verify that the hypothesis in (4.18) and (4.19) holds for all $\upsilon \geq 0$, by choosing $\eta_0^* = \|E_0\|_{r_0}$ sufficiently small. Indeed, we can even have

$$\frac{\|E_\upsilon\|_{r_\upsilon}}{(r_\upsilon - \varrho_\upsilon)^2} \leq \epsilon \cdot 2^{-\upsilon} \qquad \text{and} \qquad \frac{C(n)(2d_\upsilon)^{2n}\|E_\upsilon\|_{r_\upsilon}}{(r_\upsilon - \varrho_\upsilon)^2}\left(\frac{\varrho_\upsilon}{r_\upsilon}\right)^{\frac{d_\upsilon - 1}{2}} \leq \epsilon \cdot 2^{-\upsilon}$$

for all $\upsilon \geq 0$ and any given $0 < \epsilon < 1$.

Set

$$\epsilon_\upsilon = \frac{\|E_\upsilon\|_{r_\upsilon}}{(r_\upsilon - \varrho_\upsilon)^2}, \qquad C^*_{d_\upsilon} = C_{d_\upsilon} \cdot \frac{(r_\upsilon - \varrho_\upsilon)^4}{(r_{\upsilon+1} - \varrho_{\upsilon+1})^2},$$
$$\widetilde{C}^*_{d_\upsilon} = \widetilde{C}_{d_\upsilon} \cdot \left(\frac{r_\upsilon - \varrho_\upsilon}{r_{\upsilon+1} - \varrho_{\upsilon+1}}\right)^2, \qquad \hat{C}^*_{d_\upsilon} = C(n) \cdot (2d_\upsilon)^{2n} \cdot \left(\frac{\varrho_\upsilon}{r_\upsilon}\right)^{\frac{d_\upsilon - 1}{2}}.$$

One sees, by using (4.28) and Lemma 4.5, that

$$\lim_{\upsilon\to\infty} C^*_{d_\upsilon} = 0\,, \qquad \lim_{\upsilon\to\infty} \widetilde{C}^*_{d_\upsilon} = 0, \qquad \lim_{\upsilon\to\infty} \hat{C}^*_{d_\upsilon} = 0.$$

Hence $C^*_{d_\upsilon}$, $\widetilde{C}^*_{d_\upsilon}$, and $\hat{C}^*_{d_\upsilon}$ are bounded. Assume that $C^*_{d_\upsilon}, \widetilde{C}^*_{d_\upsilon}, \hat{C}^*_{d_\upsilon} < C$, where C is a fixed positive constant. Choose N large enough such that $C^*_{d_\upsilon}, \widetilde{C}^*_{d_\upsilon} \le \frac{1}{4}$ when $\upsilon \ge N$. Suppose $C > 1$ and choose E_0 such that

$$\epsilon_0 = \frac{\|E_0\|_{r_0}}{(r_0 - \varrho_0)^2} \le \epsilon(2C)^{-2N} < 1.$$

Next we inductively prove that if

$$\epsilon_\upsilon \le \epsilon \cdot 2^{-\upsilon},\ \epsilon_\upsilon \cdot \hat{C}^*_{d_\upsilon} \le \epsilon' \cdot 2^{-\upsilon},\ \text{with } \epsilon' = C \cdot \epsilon, \tag{4.29}$$

then we also have

$$\epsilon_{\upsilon+1} \le \epsilon \cdot 2^{-\upsilon-1},\ \epsilon_{\upsilon+1} \cdot \hat{C}^*_{d_{\upsilon+1}} \le \epsilon' \cdot 2^{-\upsilon-1},\ \text{with } \epsilon' = C \cdot \epsilon.$$

First, we get by (4.29) and Proposition 4.4 that

$$\|E_{\upsilon+1}\|_{r_{\upsilon+1}} \le C_{d_\upsilon} \cdot \|E_\upsilon\|^2_{r_\upsilon} + \widetilde{C}_{d_\upsilon} \cdot \|E_\upsilon\|_{r_\upsilon},$$

which is obviously equivalent to

$$\epsilon_{\upsilon+1} \le C^*_{d_\upsilon} \cdot \epsilon_\upsilon^2 + \widetilde{C}^*_{d_\upsilon} \cdot \epsilon_\upsilon.$$

(1) When $\upsilon < N$, we have

$$\epsilon_{\upsilon+1} \le C(\epsilon_\upsilon + 1)\epsilon_\upsilon \le 2C \cdot \epsilon_\upsilon \le (2C)^{\upsilon+1}\epsilon_0 \le \epsilon(2C)^{\upsilon-2N+1} \le \epsilon 2^{-N},$$
$$\hat{C}^*_{d_{\upsilon+1}} \cdot \epsilon_{\upsilon+1} \le C\epsilon_{\upsilon+1} \le C\epsilon 2^{-N} = \epsilon' 2^{-N}.$$

(2) When $\upsilon \geq N$, we have

$$\epsilon_{\upsilon+1} \leq \frac{1}{4}(\epsilon_\upsilon + \epsilon_\upsilon^2) \leq \frac{1}{4} \cdot 2 \cdot \epsilon_\upsilon \leq (\frac{1}{2})^{\upsilon-N+1}\epsilon_N \leq \epsilon 2^{-\upsilon-1},$$
$$\hat{C}^*_{d_{\upsilon+1}} \cdot \epsilon_{\upsilon+1} \leq C\epsilon_{\upsilon+1} \leq C\epsilon 2^{-\upsilon-1} \leq \epsilon' 2^{-N}.$$

Now, choose ϵ sufficiently small. Then it follows from (4.15) and Proposition 4.4 that $1 - C_0\epsilon 2^{-\upsilon} \leq \|d\Psi_\upsilon^{-1}\|_{\triangle_{\varrho_\upsilon}} \leq 1 + C_0\epsilon_\upsilon \leq 1 + C_0\epsilon 2^{-\upsilon}$ for some constant C_0. Note that Ψ_υ maps $\triangle_{\sigma_\upsilon}$ into $\triangle_{\varrho_\upsilon}$. By Cramer's rule, we have $1 - \epsilon C_0' 2^{-\upsilon} \leq \|d\Psi_\upsilon\|_{\triangle_{\sigma_\upsilon}} \leq 1 + \epsilon C_0' 2^{-\upsilon}$ for some constant C_0'. Now the convergence of Φ_υ in $\triangle_{\frac{1}{2}}$ follows from the fact that $\Phi_\upsilon(0) = 0$ and

$$0 < \prod_{\upsilon=0}^{\infty}(1 - \epsilon C_0' 2^{-\upsilon}) \leq \prod_{\upsilon=0}^{\infty}\|d\Psi_\upsilon\|_{\triangle_{\sigma_\upsilon}} \leq \prod_{\upsilon=0}^{\infty}(1 + \epsilon C_0' 2^{-\upsilon}) < \infty,$$

which completes the proof of Theorem 1.1. ■

Remark 4.6. We note that formal maps in Theorem 1.1 sending $(M, 0)$ to its quadric $(M_\infty, 0)$ may not be convergent as $\mathrm{aut}_0(M_\infty)$ contains many nonconvergent elements. This is quite different from the setting for CR manifolds, where formal maps are always convergent under certain not too degenerate assumptions. We refer the reader to the survey article [1] for discussions and references on this matter. □

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