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Monotonicity for the Chern-Moser-Weyl curvature tensor and CR embeddings

Dedicated to Professor ZHONG TongDe on the occasion of his 80th birthday

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Abstract We give, in this paper, a monotonicity formula for the Chern-Moser-Weyl curvature tensor under the action of holomorphic embeddings between Levi non-degenerate hypersurfaces with the same positive signature. As an application, we provide some concrete examples of algebraic Levi non-degenerate hypersurfaces with positive signature that are not embeddable into a hyperquadric of the same signature in a complex space of higher dimension.

Keywords: Chern-Moser-Weyl curvature, CR embedding, curvature decreasing property, several complex variables and analytic spaces

MSC(2000): 32H02, 32V05, 32V30

1 Introduction

This paper is motivated by a CR embedding problem in Several Complex Variables. This problem asks when a Levi non-degenerate hypersurface M_ℓ in $\mathbb{C}^{n+1}$ of signature ℓ with $0 \leqslant \ell \leqslant n/2$ can be embedded into a hyperqradric $\mathbf{H}_\ell^{N+1}$ in $\mathbb{C}^{N+1}$ of the same signature for $N \gg n$. By the general invariant theory and a Baire category argument, Forstneric[1] showed that most of such M_ℓ's are not smoothly embeddable into $\mathbf{H}_\ell^{N+1}$ (see also a recent paper of Zaitsev[2] on the related issue). On the other hand, 30 years ago, Webster in [3] showed that a Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ of signature ℓ, defined by a real polynomial, can always be embedded into the hyperquadric $\mathbf{H}_{\ell+1}^{n+2}$ of signature $\ell+1$ but in the $(n+2)$-complex space. This has then led to an interesting open problem to understand whether any algebraic Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ can be embedded into a hyperquadric of the same signature but in a much higher dimensional complex space.

In this paper, we give a checkable necessary condition whether M_ℓ can be embedded into $\mathbf{H}_\ell^{N+1}$ when $\ell \in (0, [n/2]]$. Our criterion is based on a monotonicity property for the Chern-Moser-Weyl tensor along the cone defined by tangent vectors of type (1,0) in the null space of the Levi form. Roughly speaking, our monotonicity property says that a CR embedding from a Levi non-degenerate hypersurface into another one with the same signature decreases the Chern-Moser-Weyl curvature along certain directions. This phenomenon may be compared

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with various monotonicity properties for (some type of) curvatures under the application of holomorphic maps, initiated from the classical Ahlfors-Pick-Schwarz lemma (see [4, 5], for instance). In the CR setting, the natural curvature tensor to be considered is the Chern-Moser-Weyl curvature tensor and the mappings to be involved are CR mappings. Unfortunately, there is no monotonicity phenomenon in general. Our crucial observation is that the monotonicity exists along directions in the null space of the Levi-form. Since the null space of the Levi-from may be regarded as the "largest" holomoprhic subset inside $T^{(1,0)}M$, our result may be considered as a generalization of those results on complex manifolds. Unfortunately, in our investigation, we have to exclude the important strongly pseudoconvex case: $\ell = 0$; for the null space of the Levi-form in this setting is the 0-space.

Since the hyperquarics have vanishing Chern-Moser-Weyl tensor, our criterion makes it possible to construct many algebraic Levi non-degenerate hypersurfaces which can not be embedded into a hyperquadric of the same signature $\ell > 0$ in a complex space of higher dimension. However, it still remains to be an open question to answer if any algebraic strongly pseudoconvex hypersurface M_ℓ can be embedded into $\mathbf{H}_\ell^N$ for some N with $\ell = 0$.

2 Chern-Moser-Weyl tensor on a Levi non-degenerate hypersurface

We use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. We always assume that $n \geqslant 2$. Let M be a smooth real hypersurface. We say that M is Levi non-degenerate at $p \in M$ with signature $\ell \leqslant n/2$ if there is a local holomorphic change of coordinates, that maps p to the origin, such that in the new coordinates, M is defined near 0 by an equation of the form:

$$r = v - |z|_\ell^2 + o(|z|^2 + |zu|) = 0. \tag{1}$$

Here, we write $u = \Re w, v = \Im w$ and $\langle a, \bar{b}\rangle_\ell = -\sum_{j\leqslant \ell} a_j\bar{b}_j + \sum_{j=\ell+1}^n a_j\bar{b}_j, |z|_\ell^2 = \langle z, \bar{z}\rangle_\ell$. When $\ell = 0$, we regard $\sum_{j\leqslant\ell} a_j = 0$.

Assume that M is Levi non-degenerate with the same signature ℓ at any point. A contact form θ over M is said to be appropriate if the Levi form $L_{\theta|_p}$ associated with θ at any point $p \in M$ has ℓ negative eigenvalues and $n - \ell$ positive eigenvalues. (See (2) for our definition of the Levi form.) Since our consideration in this paper is local, we only focus on a small piece of M with $0 \in M$ and M is defined by an equation as in (1). In particular, $\theta_0 = i\partial r$ is appropriate near 0. When $\ell < n/2$, a contact form θ is appropriate if and only if $\theta = k_0\theta_0$ with $k_0 > 0$.

Let θ be an appropriate contact form over M. Then from the Chern-Moser theory, there is a unique 4th order curvature tensor $\mathcal{S}_\theta$ associated with θ (see [6, 7]), which we call the Chern-Moser-Weyl tensor with respect to the contact form θ along M. $\mathcal{S}_\theta$ can be regarded as a section over $T^{*(1,0)}M \otimes T^{*(0,1)}M \otimes T^{*(1,0)}M \otimes T^{*(0,1)}M$. We write $\mathcal{S}_{\theta|_p}$ for the restriction of $\mathcal{S}_\theta$ at $p \in M$. For a basis $\{X_\alpha\}_{\alpha=1}^n$ of $T_p^{(1,0)}M$ with $p \in M$, write $(S_{\theta|_p})_{\alpha\bar{\beta}\gamma\bar{\delta}} = \mathcal{S}_{\theta|_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta)$. We then have the following symmetric properties:

$$(S_{\theta|_p})_{\alpha\bar{\beta}\gamma\bar{\delta}} = (S_{\theta|_p})_{\gamma\bar{\beta}\alpha\bar{\delta}} = (S_{\theta|_p})_{\gamma\bar{\delta}\alpha\bar{\beta}}, \quad \overline{(S_{\theta|_p})_{\alpha\bar{\beta}\gamma\bar{\delta}}} = (S_{\theta|_p})_{\beta\bar{\alpha}\delta\bar{\gamma}},$$

and the following trace-free condition:

$$\sum_{\beta,\alpha=1}^n g^{\bar{\beta}\alpha}(S_{\theta|_p})_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0.$$

Here

$$g_{\alpha\bar{\beta}} = L_{\theta|_p}(X_\alpha, X_\beta) := -i\langle d\theta|_p, X_\alpha \wedge \overline{X}_\beta\rangle = -\langle \partial\bar{\partial} r|_p, X_\alpha \wedge \overline{X}_\beta\rangle \tag{2}$$

is the Levi form of M associated with θ at $p \in M$ and $(g^{\bar{\beta}\alpha})$ is the inverse matrix of $(g_{\alpha\bar{\beta}})$. For a different contact form $\tilde{\theta} = \tilde{k}\theta$ smooth along M with $\tilde{k} > 0$, we have the following transformation formula:

$$\mathcal{S}_{\tilde{\theta}|_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta) = \tilde{k}\mathcal{S}_{\theta|_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta).$$

For a smooth vector field X, Y, Z, W of type $(1,0)$ and a smooth contact form along M, $\mathcal{S}_\theta(X, \overline{Y}, Z, \overline{W})$ is also a smooth function along M. One easy way to see this is to use the Webster-Chern-Moser-Weyl formula obtained in [8] through the curvature tensor of the Webster pseudo-Hermitian metric, whose constructions are done by only applying the algebraic and differentiation operations on the defining function of M.

$\mathcal{S}_\theta$ is described in terms of the normal coordinates for M as follows: First, by the Chern-Moser normal form theory[6], we can find a coordinate in which M is defined near 0 by an equation of the following form (see [6, (6.25), (6.30)]):

$$r = v - |z|_\ell^2 + \frac{1}{4}s(z,\bar{z}) + o(|z|^4) = v - |z|_\ell^2 + \frac{1}{4}\sum s_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \bar{z}_\beta z_\gamma \bar{z}_\delta + o(|z|^4) = 0. \tag{3}$$

Here $s(z,\overline{z}) = \sum s_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \bar{z}_\beta z_\gamma \bar{z}_\delta$, $i\partial r|_0 = \theta|_0$, $s_{\alpha\bar{\beta}\gamma\bar{\delta}} = s_{\gamma\bar{\beta}\alpha\bar{\delta}} = s_{\gamma\bar{\delta}\alpha\bar{\beta}}$, $\overline{s_{\alpha\bar{\beta}\gamma\bar{\delta}}} = s_{\beta\bar{\alpha}\delta\bar{\gamma}}$ and $\sum_{\alpha,\beta=1}^n s_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{\bar{\beta}\alpha} = 0$, where $g^{\bar{\beta}\alpha} = 0$ for $\beta \neq \alpha$, $g^{\bar{\beta}\beta} = 1$ for $\beta > \ell$, $g^{\bar{\beta}\beta} = -1$ for $\beta \leqslant \ell$. Then

$$s_{\alpha\bar{\beta}\gamma\bar{\delta}} = \mathcal{S}_{\theta|_0}\left(\frac{\partial}{\partial z_\alpha}\bigg|_0, \frac{\partial}{\partial \bar{z}_\beta}\bigg|_0, \frac{\partial}{\partial z_\gamma}\bigg|_0, \frac{\partial}{\partial \bar{z}_\delta}\bigg|_0\right).$$

Write $\triangle_\ell = -\sum_{j\leqslant\ell} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + \sum_{j=\ell+1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ and also write $s_{\theta|_0}(z,\bar{z})$ for $s(z,\overline{z})$. Then the trace-free condition above is equivalent to $\triangle_\ell s_{\theta|_0}(z,\bar{z}) \equiv 0$. Indeed, this follows from the following fact: Let $\Delta_H = \sum_{l,k=1}^n h^{l\overline{k}} \partial_l \overline{\partial}_k$ with $\overline{h^{l\overline{k}}} = h^{k\overline{l}}$ for any l,k. Then

$$\Delta_H s_{\theta|_0}(z,\overline{z}) = 4\sum_{\gamma,\delta=1}^n \sum_{\alpha,\beta=1}^n h^{\alpha\overline{\beta}} s_{\alpha\overline{\beta}\gamma\overline{\delta}} z_\gamma \overline{z_\delta}. \tag{4}$$

For the rest of this section, we assume that $\ell > 0$ and define $\mathcal{C}_\ell = \{z \in \mathbb{C}^n : |z|_\ell = 0\}$. Then $\mathcal{C}_\ell$ is a real algebraic variety of real codimension 1 in $\mathbb{C}^n$ with the only singularity at 0. For each $p \in M$, write $\mathcal{C}_\ell T_p^{(1,0)}M = \{v_p \in T_p^{(1,0)}M : \langle d\theta_p, v_p \wedge \bar{v}_p\rangle = 0\}$. Apparently, $\mathcal{C}_\ell T_p^{(1,0)}M$ is independent of the choice of θ. Let F be a CR diffeomorphism from M to M'. We also have $F_*(\mathcal{C}_\ell T_p^{(1,0)}M) = \mathcal{C}_\ell T_{F(p)}^{(1,0)}M'$. (We will explain this in details in the later discussion). Write $\mathcal{C}_\ell T^{(1,0)}M = \coprod_{p\in M} \mathcal{C}_\ell T_p^{(1,0)}M$ with the natural projection π to M. We say that X is a smooth section of $\mathcal{C}_\ell T^{(1,0)}M$ if X is a smooth vector field of type $(1,0)$ along M such that $X|_p \in \mathcal{C}_\ell T_p^{(1,0)}M$ for each $p \in M$. Later, we will see that $\mathcal{C}_\ell T^{(1,0)}M$ is a kind of cone bundle with each fiber isomorphic to $\mathcal{C}_\ell$ (see Remark 3.3).

We say that the Chern-Moser-Weyl curvature tensor $\mathcal{S}_\theta$ is pseudo semi-positive definite (or pseudo semi-negative definite) at $p \in M$ if $\mathcal{S}_{\theta|_p}(X, \overline{X}, X, \overline{X}) \geqslant 0$ for any $X \in \mathcal{C}_\ell T_p^{(1,0)}M$ (or $\mathcal{S}_{\theta|_p}(X, \overline{X}, X, \overline{X}) \leqslant 0$, respectively, for all $X \in \mathcal{C}_\ell T_p^{(1,0)}M$). We say that $\mathcal{S}_\theta$ is pseudo positive-definite (or pseudo negative-definite) at $p \in M$ if $\mathcal{S}_{\theta|_p}(X, \overline{X}, X, \overline{X}) > 0$ for all $X \in$

$\mathcal{C}_\ell T_p^{(1,0)}M \setminus 0$ (or $\mathcal{S}_{\theta|_p}(X,\overline{X},X,\overline{X}) < 0$, respectively, for all $X \in \mathcal{C}_\ell T_p^{(1,0)}M \setminus 0$). We use the terminology pseudo semi-definite to mean either pseudo semi-positive definite or pseudo semi-negative definite. We can similarly define the notion of pseudo definiteness.

$\mathcal{C}_\ell$ is obviously a uniqueness set for holomorphic functions. The following lemma shows that it is also a uniqueness set for the Chern-Moser-Weyl curvature tensor.

Lemma 2.1. (I). *Suppose that $H(z,\bar{z})$ is a real real-analytic function in $(z,\bar{z})$ near 0. Assume that $\triangle_\ell H(z,\bar{z}) \equiv 0$ and $H(z,\bar{z})|_{\mathcal{C}_\ell} = 0$. Then $H(z,\bar{z}) \equiv 0$ near 0.* (II). *Assume the above notation. If $\mathcal{S}_{\theta|_p}(X,\overline{X},X,\overline{X}) = 0$ for any $X \in \mathcal{C}_\ell T_p^{(1,0)}M$, then $\mathcal{S}_{\theta|_p} \equiv 0$.*

Proof. (I). Write $H(z,\bar{z}) = \sum_{m=1}^{\infty} H^{(m)}(z,\bar{z})$ with $H^{(m)}(z,\bar{z})$ homogeneous polynomials in $(z,\bar{z})$ of degree m. Then we easily see that $\triangle_\ell H(z,\bar{z}) \equiv 0$ if and only if $\triangle_\ell H^{(m)}(z,\bar{z}) \equiv 0$ for each m. For $p \in \mathcal{C}_\ell$, since $tp \in \mathcal{C}_\ell$ for $t \in \mathbb{R}$, we see that $H(tp,\overline{tp}) = \sum_{m=1}^{\infty} t^m H^{(m)}(p,\bar{p})$ and $H(tp,\overline{tp}) = 0$ for each $t \in \mathbb{R}$ if and only if $H^{(m)}(p,\bar{p}) = 0$ for each m. Hence we see that $H(z,\bar{z})|_{\mathcal{C}_\ell} = 0$ if and only if $H^{(m)}(z,\bar{z}) = 0$ along $\mathcal{C}_\ell$ for each m. Therefore, to prove Lemma 2.1, we can assume that $H(z,\bar{z})$ is already a homogeneous polynomial of degree m in $(z,\bar{z})$. Next, notice that

$$\mathrm{V} = \left\{(z,\xi) \in \mathbb{C}^n \times \mathbb{C}^n : \ \langle z,\xi\rangle_\ell = -\sum_{j=1}^{\ell} z_j\xi_j + \sum_{j=\ell+1}^{n} z_j\xi_j = 0\right\}$$

is a complex analytic variety defined by $\langle z,\xi\rangle_\ell = 0$ with $\langle z,\xi\rangle_\ell$ irreducible as an element in $\mathscr{O}_{(p,q)}$ for each $(p,q) \in V$. Hence, we easily see that $H(z,\xi) = h(z,\xi)\langle z,\xi\rangle_\ell$ for a certain holomorphic function $h(z,\xi)$ in $(z,\xi) \in \mathbb{C}^n \times \mathbb{C}^n$. Then it follows that $h(z,\xi)$ is a homogeneous polynomial of degree $m-2$. Now by a well-known argument in harmonic analysis (see [9, p. 140]), we can prove $H \equiv 0$ as follows:

First, write $H(z,\bar{z}) = \sum_{\alpha+\beta=m} a_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta$. Then

$$\sum_{\alpha+\beta=m} |a_{\alpha\bar{\beta}}|^2 \alpha!\beta! = H(\partial_z,\partial_{\bar{z}})(H(z,\bar{z})) = h(\partial_z,\partial_{\bar{z}})(\triangle_\ell(H(z,\bar{z}))) = 0.$$

Thus $H(z,\bar{z}) \equiv 0$.

(II): By the transformation law for the Chern-Moser-Weyl curvature tensor, we can assume that $p = 0$ and M near 0 is given in normal coordinates as in (3) with $\theta|_0 = i\partial r$. Write $X = \sum_{j=1}^{n} z_j(\frac{\partial}{\partial z_j}|_0)$. Then $X \in \mathcal{C}_\ell T_0^{(1,0)}M$ if and only if $|z|_\ell = 0$. Moreover $\mathcal{S}_{\theta|_0}(X,\overline{X},X,\overline{X}) = s_{\theta|_0}(z,\overline{z})$ with $\Delta_\ell s_{\theta|_0}(z,\overline{z}) \equiv 0$. Now, since $s_{\theta|_0}(z,\overline{z}) = 0$ for $|z|_\ell = 0$, we have, by Part (I) of the lemma, $s_{\theta|_0}(z,\overline{z}) = 0$ for any z. Namely, $\mathcal{S}_{\theta|_0}(X,\overline{X},X,\overline{X}) \equiv 0$. This then immediately shows that $\mathcal{S}_{\theta|_0} \equiv 0$. □

Write $\mathbf{H}_\ell^{n+1} := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \Im w = \langle z,\bar{z}\rangle_\ell\}$ for the Levi non-degenerate real hyperquadric with signature $\ell > 0$. By the Chern-Moser theory, M is locally CR equivalent to $\mathbf{H}_\ell^{n+1}$ if and only if $\mathcal{S}_\theta \equiv 0$. Together with the above lemma, we have the following:

Lemma 2.2. *Let M be a Levi non-degenerate hypersurface of signature ℓ with $0 < \ell \leqslant \frac{n}{2}$. Then M is locally CR equivalent to the hyperquadric $\mathbf{H}_\ell^{n+1}$ of signature ℓ if and only if for any contact form θ and any vector $X_p \in \mathcal{C}_\ell T_p^{(1,0)}M$ with $p \in M$, it holds that $\mathcal{S}_{\theta|_p}(X_p,\overline{X}_p,X_p,\overline{X}_p)$ $= 0$.*

3 Monotonicity for the Chern-Moser-Weyl tensor and CR embeddings

Next, let $\widetilde{M} \subset \mathbb{C}^{N+1} = \{(z,w) \in \mathbb{C}^N \times \mathbb{C}\}$ be also a Levi non-degenerate smooth real hypersurface near 0 of signature $\ell \geqslant 0$ defined by an equation of the form:

$$\widetilde{r} = \Im \widetilde{w} - |\widetilde{z}|^2_\ell + o(|\widetilde{z}|^2 + |\widetilde{z}\widetilde{u}|) = 0. \tag{5}$$

Assume that $N \geqslant n$ and let $F := (\tilde{f}, g) = (f_1, \ldots, f_N, g) : M \to \widetilde{M}$ be a smooth CR map. We say that F is CR transversal at a point $p \in M$, the normal component of F has a non-vanishing normal derivative at p. Assume $F(0) = 0$. Then F is CR transversal at 0 if and only if $\frac{\partial g}{\partial w}|_0 \neq 0$.

In our setting here, namely, when M and $\widetilde{M}$ are both Levi non-degenerate hypersurfaces with the same signature, the CR transversality of F is equivalent to the local embeddability. In other words, F is CR transversal at p if and only if F is a CR embedding from a small neighborhood of p in M into $\widetilde{M}$. When F extends to a holomorphic map to a neighborhood of p in $\mathbb{C}^{n+1}$, which is automatically the case when $0 < \ell \leqslant n/2$ by the Lewy extension theorem, this is further equivalent to the property that F is a local holomorphic embedding from a neighborhood of p in $\mathbb{C}^{n+1}$ into $\mathbb{C}^{N+1}$. To see this, we can assume, without loss of generality, that $p = 0$. Since by the classical Hopf lemma, when $\ell = 0$, either F is a constant map or F is a local CR embedding at any point in M, we thus assume that $0 < \ell \leqslant n/2$. When F is CR transversal at $p = 0$, by the following (6), we easily see that F is a local embedding from a neighborhood of 0 in $\mathbb{C}^{n+1}$. Conversely, if F is not CR transversal at 0, then near 0, we have $g = O(|(z,w)|^2)$ and $\tilde{f} = zU + \vec{a}w + O(|(z,w)|^2)$, where U is an $n \times N$ matrix and $\vec{a} \in \mathbb{C}^N$. Since $F(M) \subset \widetilde{M}$, we have

$$\Im g = |\tilde{f}|^2_\ell + O(3), \quad (z,w) \in M.$$

We easily see that $U \cdot E_\ell \cdot \bar{U}^t = 0$. Here E_ℓ is the diagonal matrix with the first ℓ diagonal elements -1 and the rest diagonal elements 1. Hence, by [10, Lemma 4.2], the rank of U is bounded by ℓ. Thus the Jacobian matrix of F at 0 can at most have rank $\ell+1 < n+1$. Namely, F can not be a holomorphic embedding near 0 in $\mathbb{C}^{n+1}$.

Since the set of points where a holomorphic map fails to be local embedding is a complex analytic variety in a neighborhood of M where F is holomorphic, the above observation has an immediate consequence: When $0 < \ell < n/2$, either F fails to be CR transversal at any point in M or the set of CR non-transversal points of F in M is an intersection of a certain proper holomorphic variety with M and thus is a thin set in M. In particular, when M is real analytic, it has codimension at least 2 in M. Hence, in this situation, the complement of the set of the CR non-transversal points of F is dense and connected. (We assume $M, \widetilde{M}$ to be connected.)

Now, assume that F is CR transversal at 0. Then, as in [10, Chapter 2], we can write

$$\begin{aligned} \tilde{z} &= \tilde{f}(z,w) = (f_1(z,w), \ldots, f_N(z,w)) = \lambda z U + \vec{a}w + O(|(z,w)|^2), \\ \tilde{w} &= g(z,w) = \sigma \lambda^2 w + O(|(z,w)|^2). \end{aligned} \tag{6}$$

Here U can be extended to an $N \times N$ matrix $\widetilde{U} \in SU(N,\ell)$ (namely $\langle X\widetilde{U}, Y\overline{\widetilde{U}}\rangle_\ell = \langle X, Y\rangle_\ell$ for any $X, Y \in \mathbb{C}^N$). Moreover, $\vec{a} \in \mathbb{C}^N$, $\lambda > 0$ and $\sigma = \pm 1$ with $\sigma = 1$ for $\ell < \frac{n}{2}$. When $\sigma = -1$, by considering $F \circ \tau_{n/2}$ instead of F, where $\tau_{\frac{n}{2}}(z_1, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, \ldots, z_n, w) =$

$(z_{\frac{n}{2}+1},\dots,z_n,z_1,\dots,z_{\frac{n}{2}},-w)$, we can make $\sigma=1$. Hence, we will assume in what follows that $\sigma=1$.

Write $r_0=\frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde z,\tilde w)=1+2i\langle \tilde z,\lambda^{-2}\overline{\vec a}\rangle_\ell+\lambda^{-4}(r_0-i|\vec a|^2_\ell)\tilde w$,

$$T(\tilde z,\tilde w)=\frac{(\lambda^{-1}(\tilde z-\lambda^{-2}\vec a\tilde w)\widetilde U^{-1},\lambda^{-2}\tilde w)}{q(\tilde z,\tilde w)}. \tag{7}$$

Then

$$F^\sharp(z,w)=(\tilde f^\sharp,g^\sharp)(z,w):=T\circ F(z,w)=(z,0,w)+O(|(z,w)|^2) \tag{8}$$

with $\Re\{g^{\sharp''}_{ww}(0)\}=0$.

Assume that $\widetilde M$ is also defined in the Chern-Moser normal form up to the 4th order:

$$\tilde r=\Im\tilde w-|\tilde z|^2_\ell+\frac{1}{4}\tilde s(\tilde z,\bar{\tilde z})+o(|\tilde z|^4)=0. \tag{9}$$

Then $M^\sharp=T(\widetilde M)$ is defined by

$$r^\sharp=\Im w^\sharp-|z^\sharp|^2_\ell+\frac{1}{4}s^\sharp(z^\sharp,\bar{z^\sharp})+o(|z^\sharp|^4)=0 \tag{10}$$

with $s^\sharp(z^\sharp,\bar{z^\sharp})=\lambda^{-2}\tilde s(\lambda z^\sharp\tilde U,\overline{\lambda z^\sharp\tilde U})$.

One can verify that

$$\Bigg(-\sum_{j=1}^{\ell}\frac{\partial^2}{\partial z^\sharp_j\partial\bar z^\sharp_j}+\sum_{j=\ell+1}^{N}\frac{\partial^2}{\partial z^\sharp_j\partial\bar z^\sharp_j}\Bigg)s^\sharp(z^\sharp,\overline{z^\sharp})=0. \tag{11}$$

Therefore (10) is also in the Chern-Moser normal form up to the 4th order. Now we assign the weight of $z,\bar z$ to be 1, and that of w to be 2. We use the standard notation $h^{(k)}$ and $o_{wt}(k)$ to denote terms in function h of weighted degree k and terms vanishing to the weighted degree higher than k, respectively. Write $F^\sharp(z,w)=\sum_{k=1}^\infty F^{\sharp(k)}(z,w)$. Since $F^\sharp$ maps M into $M^\sharp=T(\widetilde M)$, we get the following

$$\begin{aligned}&\Im\Bigg\{\sum_{k\geqslant 2}g^{\sharp(k)}(z,w)-2i\sum_{k\geqslant 2}\langle f^{\sharp(k)}(z,w),\bar z\rangle_\ell\Bigg\}\\&\qquad=\sum_{k_1,\ k_2\geqslant 2}\langle f^{\sharp(k_1)}(z,w),\overline{f^{\sharp(k_2)}(z,w)}\rangle_\ell+\frac{1}{4}(s(z,\bar z)-s^\sharp((z,0),\overline{(z,0)}))+o_{wt}(4)\end{aligned} \tag{12}$$

over $\Im w=|z|^2_\ell$. Here, we write $F^\sharp(z,w)=(\tilde f^\sharp(z,w),g^\sharp(z,w))=(f^\sharp(z,w),\phi^\sharp(z,w),g^\sharp(z,w))$.

Collecting terms of weighted degree 3 in (12), we get

$$\Im\{g^{\sharp(3)}(z,w)-2i\langle f^{\sharp(2)}(z,w),\bar z\rangle_\ell\}=0\ \text{ on }\ \Im w=|z|^2_\ell.$$

By [11], we get $g^{\sharp(3)}\equiv 0, f^{\sharp(2)}\equiv 0$.

Collecting terms of weighted degree 4 in (12), we get

$$\Im\{g^{\sharp(4)}(z,w)-2i\langle f^{\sharp(3)}(z,w),\bar z\rangle_\ell\}=|\phi^{\sharp(2)}(z)|^2+\frac{1}{4}(s(z,\bar z)-s^\sharp((z,0),\overline{(z,0)})).$$

Similarly to the argument in [11] and making use of the fact that $\Re\{\frac{\partial^2 g^{\sharp(4)}}{\partial w^2}(0)\} = 0$, we get the following:

$$\begin{aligned} g^{\sharp(4)} \equiv 0, \quad f^{\sharp(3)}(z,w) = \frac{i}{2}a^{(1)}(z)w,\\ \langle a^{(1)}(z), \bar{z}\rangle_\ell |z|_\ell^2 = |\phi^{\sharp(2)}(z)|^2 + \frac{1}{4}(s(z,\bar{z}) - s^\sharp((z,0),\overline{(z,0)})). \end{aligned} \tag{13}$$

We assume in the following (except in Proposition 3.1 and Remark 3.2) that $\ell > 0$. Letting $z \in \mathcal{C}_\ell$, we get

$$\begin{aligned} 4|\phi^{\sharp(2)}(z)|^2 &= s^\sharp((z,0),\overline{(z,0)}) - s(z,\bar{z}) = \lambda^{-2}\widetilde{s}((\lambda z,0)\widetilde{U}, \overline{(\lambda z,0)\widetilde{U}}) - s(z,\bar{z})\\ &= \lambda^2 \widetilde{s}((z,0)\widetilde{U}, \overline{(z,0)\widetilde{U}}) - s(z,\bar{z}). \end{aligned} \tag{14}$$

We claim that, for $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$, $F_*(v_p) \in \mathcal{C}_\ell T_{F(p)}^{(1,0)}\widetilde{M}$ and $F_*^\sharp(v_p) \in \mathcal{C}_\ell T_{F^\sharp(p)}^{(1,0)}M^\sharp$. Indeed, to see this, we need only to notice that for any contact form $\tilde{\theta}$ along $\widetilde{M}$, $F^*(\tilde{\theta})$ is also a contact form of M and

$$\langle d(F^*(\tilde{\theta}))|_p, v_p \wedge \bar{v}_p\rangle = \langle d\tilde{\theta}_{F(p)}, F_*(v_p) \wedge \overline{F_*(v_p)}\rangle.$$

Thus, if $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$, then $\langle d\tilde{\theta}_{F(p)}, F_*(v_p) \wedge \overline{F_*(v_p)}\rangle = 0$ and hence $F_*(v_p) \in \mathcal{C}_\ell T_{F(p)}^{(1,0)}\widetilde{M}$. Next, if we identify z with the $(1,0)$ vector $v = \sum z_j(\frac{\partial}{\partial z_j}|_0)$, then $(\lambda z, 0)\widetilde{U}$ is identified with the vector $F_*(v)$. Moreover, $z \in \mathcal{C}_\ell$ if and only if $v \in \mathcal{C}_\ell T_0^{(1,0)}M$.

Set $\theta = i\partial r$ and $\tilde{\theta} = i\partial\tilde{r}$. Then

$$F^*(\tilde{\theta})|_0 = \frac{1}{2}dg|_0 = \lambda^2\theta|_0.$$

Write $F^*(\tilde{\theta}) = k\theta$, then $k(0) = \lambda^2$. Hence (14) can now be written as:

$$\begin{aligned} &\tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) = \lambda^2 \mathcal{S}_{\theta|_0}(v,\bar{v},v,\bar{v}) + 4\lambda^2|\phi^{\sharp(2)}(z)|^2,\\ &v = \sum_{j=1}^n z_j\left(\frac{\partial}{\partial z_j}\bigg|_0\right) \in T_0^{(1,0)}M,\\ \text{or}\quad &\tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) = \mathcal{S}_{F^*(\tilde{\theta})|_0}(v,\bar{v},v,\bar{v}) + 4\lambda^2|\phi^{\sharp(2)}(z)|^2. \end{aligned} \tag{15}$$

Summarizing the above, we have the following: (In Proposition 3.1 and Remark 3.2, ℓ can be 0.)

Proposition 3.1. *Let M and $\widetilde{M}$ be defined by* (3) *and* (9)*, respectively. Let*

$$F = (\widetilde{z}, \widetilde{w}) = (\widetilde{f}(z,w), g(z,w)) = (f_1(z,w), \ldots, f_{n-1}(z,w), g(z,w))$$

be a smooth CR map sending M into $\widetilde{M}$, satisfying the normalization in (6) *with $\sigma = 1$. Let T be given as in* (7) *and write $F^\sharp = T \circ F = (\widetilde{f}^\sharp, g^\sharp)$ as in* (8)*. Then, for any $v = \sum_{j=1}^n z_j(\frac{\partial}{\partial z_j}|_0) \in T_0^{(1,0)}M$, the follows holds:*

$$\begin{aligned} &g^{\sharp(2)} - w = g^{\sharp(3)} = g^{\sharp(4)} \equiv 0, \quad f^{\sharp(2)} = 0, \quad f^{\sharp(3)}(z,w) = \frac{i}{2}a^{(1)}(z)w,\\ &4\langle a^{(1)}(z), \bar{z}\rangle_\ell |z|_\ell^2 = 4|\phi^{\sharp(2)}(z)|^2 - \lambda^{-2}\tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) + \mathcal{S}_{\theta|_0}(v,\bar{v},v,\bar{v}). \end{aligned} \tag{16}$$

Remark 3.2. (1). We notice that when $N = n$, $\phi^{\sharp(2)}(z) \equiv 0$. Since the left-hand side of the second equation in (16) is divisible by $|z|_\ell^2$ and the right-hand side of the second equation in (16) is annihilated by Δ_ℓ, we conclude that both sides have to be identically zero and thus we have

$$\tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) = \mathcal{S}_{F^*(\tilde{\theta}|_0)}(v, \bar{v}, v, \bar{v}) \text{ for any } v \in T_0^{(1,0)}M. \tag{17}$$

This is the Chern-Moser invariant property (or the biholomoprhic transformation law) of the Chern-Moser Weyl tensor in the case of $N = n$.

(2). Our proof of the above proposition uses basically the same argument as what first appeared in [11], where a certain version of Proposition 3.1 was first obtained. We repeated it here due to the reason that we have to trace precisely how the tangent vectors of type (1,0) and others are transformed when we normalize the map, which will be crucial for our later application. Indeed, as in [11], in the case of $\ell = 0$, we can just assume that the map F is only a C^2-smooth CR map.

Notice that when $\tilde{\theta}$ is an appropriate contact form along $\widetilde{M}$, then $F^*(\tilde{\theta})$ is also an appropriate contact form. From (15), we get the following monotonicity property for the Chern-Moser-Weyl curvature tensor under a CR embedding:

Theorem 3.3. *Let $M \subset \mathbb{C}^{n+1}$ and $\widetilde{M} \subset \mathbb{C}^{N+1}$ be two Levi non-degenerate smooth real hypersurfaces with the same signature $0 < \ell < \frac{n}{2}$. Suppose that $F : M \to \widetilde{M}$ is a CR transversal mapping (or, equivalently, a local holomorphic embedding). For an appropriate contact form $\tilde{\theta}$ along $\widetilde{M}, p \in M$ and $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$, we have*

$$\mathcal{S}_{F^*(\tilde{\theta})|_p}(v_p, \bar{v}_p, v_p, \bar{v}_p) \leqslant \tilde{\mathcal{S}}_{\tilde{\theta}|_{F(p)}}(F_*(v_p), \overline{F_*(v_p)}, F_*(v_p), \overline{F_*(v_p)}).$$

When $\ell = \frac{n}{2}$, after replacing M by $\tau_{\frac{n}{2}}(M)$ and F by $F \circ \tau_{\frac{n}{2}}$ (to make $F^*(\tilde{\theta}) = \tilde{k}\theta$ with $\tilde{k} > 0$) if necessary, we also have the same statement as above. Here $\tau_{\frac{n}{2}}(z_1, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, \ldots, z_n, w) = (z_{\frac{n}{2}+1}, \ldots, z_n, z_1, \ldots, z_{\frac{n}{2}}, -w)$.

Now, assume that F is a holomorphic mapping from a domain $U \subset \mathbb{C}^{n+1}$ into $\mathbb{C}^{N+1}$. F is called to be totally degenerate if F fails to be a local holomorphic embedding at any point inside U, namely, if the rank of the Jacobian matrix of F is less than $n+1$ at any point $p \in U$. Hence, F is not totally degenerate over U if and only if it is a local holomorphic embedding away from a proper holomorphic variety. Now, let $M, \widetilde{M}$ be as above with $M \subset U$, $F \in \mathrm{Hol}(U, \mathbb{C}^{N+1})$ and $F(M) \subset \widetilde{M}$. If F is not totally degenerate, then we apparently have $F(U) \not\subset \widetilde{M}$. Conversely, in case $M, \widetilde{M}$ are real analytic, if $F(U) \not\subset \widetilde{M}$, by a result of Baouendi-Ebenfelt-Rothschild[12] (see already the paper of Baouendi-Huang[10] for a related investigation), F is not totally degenerate over U and thus is CR transversal over a dense open subset of M.

As the first application of Theorem 3.3, we have the following:

Corollary 3.4. *Let $M \subset \mathbb{C}^{n+1}$ be a smooth Levi non-degenerate hypersurface of signature ℓ. Suppose that F is not a totally degenerate holomorphic mapping defined in a neighborhood U of M in $\mathbb{C}^{n+1}$ that sends M into $\mathbf{H}_\ell^{N+1} \subset \mathbb{C}^{N+1}$. Then when $0 < \ell < \frac{n}{2}$, the Chern-Moser-Weyl curvature tensor with respect to any appropriate contact form θ is pseudo semi-negative. When $\ell = \frac{n}{2}$, along any contact form θ, $\mathcal{S}_\theta$ is pseudo semi-definite.*

Proof. By the observation above, since F is not totally non-degenerate, F is CR transversal over an open dense subset E_F of M. Without loss of generality, we assume that $\ell < \frac{n}{2}$. Since the Chern-Moser-Weyl pseudo-conformal curvature tensor for the hyperquadric $\mathbf{H}_\ell^{N+1}$ vanishes, by the previous theorem, we have for $p \in E_F$, $\mathcal{S}_{F^*(\tilde{\theta})|_p}(v_p, \bar{v}_p, v_p, \bar{v}_p) \leqslant 0$, when $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$ and $\tilde{\theta}$ is an appropriate contact form of $\mathbf{H}_\ell^{N+1}$ near $F(p)$. This implies that $\mathcal{S}$ is pseudo semi-negative definite at each point $p \in E_F$.

When $p \notin E_F$, let θ be an appropriate contact form at p and $X_1, \ldots, X_n$ an orthonormal basis of $T^{(1,0)}M$ with respect to L_θ on some neighborhood of p, say U_p. Indeed, $\forall p \in M$, choose $X_1(p), \ldots, X_n(p)$ to be an orthonormal basis of $T_p^{(1,0)}M$ with respect to $L_{\theta|_p}$, i.e.,

$$\langle X_j(p), X_k(p)\rangle_{L_{\theta|_p}} = \begin{cases} -1, & \text{if } j = k \leqslant \ell; \\ 1, & \text{if } j = k > \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Applying Gram-Schmidt process if necessary, one can always extend $\{X_j(p)\}_{j=1}^n$ to an orthonormal basis $\{X_j\}_{j=1}^n$ (with respect to the Levi form L_θ) of $T^{(1,0)}M$ on some small neighborhood U_p of p. Moreover, a straightforward computation shows that for any vector-valued smooth function $\vec{a}(q) = (a_1(q), \ldots, a_n(q))$ along M near p,

$$\sum_{j=1}^n a_j X_j \ \text{ is a smooth section of } \ \mathcal{C}_\ell T^{(1,0)}U_p \Leftrightarrow |\vec{a}(q)|_\ell^2 = 0 \text{ for all } q \in U_p.$$

Now for the above $p \notin E_F$ and any $v_p = \sum_{j=1}^n a_j X_j|_p \in \mathcal{C}_\ell T_p^{(1,0)}M$ with $a_j \in \mathbb{C}$, take a sequence $\{q_k\}_{k=1}^\infty \in E_F$ converging to p. By the previous argument, $\sum_{j=1}^n a_j X_j|_{q_k} \in \mathcal{C}_\ell T_{q_k}^{(1,0)}M$ and $\mathcal{S}_{\theta|_{q_k}}(v_{q_k}, \bar{v}_{q_k}, v_{q_k}, \bar{v}_{q_k}) \leqslant 0$ for any k. Moreover, $\mathcal{S}_{\theta|_q}$ depends smoothly on q as we mentioned before. Letting $k \to \infty$, we then obtain the desired inequality at p. □

Remark 3.5. From the above, we see the following fact: For any point $p \in M$, there is an open neighborhood U_p of p in M and a smooth frame $\{X_1, \ldots, X_n\}$ of $T^{(1,0)}U_p$ such that the diffeomorphism Ψ from $T^{(1,0)}U_p$ to $U_p \times \mathbb{C}^n$ defined by $\Psi(\sum_{j=1}^n a_j X_j|_q) = (q, (a_1, \ldots, a_n))$ maps $\mathcal{C}_\ell T_q^{(1,0)}U_p$ to $\{q\} \times \mathcal{C}_\ell$ for each $q \in U_p$.

In Theorem 3.3, we only assume that F is not a totally degenerate holomorphic map in a neighborhood U of M. Then F is CR transversal along a dense open subset of M. As observed at the beginning of this section, the complement of non-CR transversal points of F in M is actually a dense open subset of M. Assume that F fails to be CR transversal at $p \in M$. Choose a sequence of points $\{q_j\} \subset M$ with $q_j \to p$, where the CR transversality holds. Apply a standard procedure to normalize M and $\widetilde{M}$ at $q \in M$ and $F(q)$ up to the 4th order, respectively, for any $q \approx p$. Notice that we can make the normalizations to depend continuously on q and $F(q)$, respectively. Now, we can similarly define $\lambda(q)$ as in (6). Then $\lambda(q)$ depends continuously on q and thus converges to 0 as $q \to p$, by the assumption that F is not CR transversal at p. Now, applying (14) with $q = q_j$ and then letting $q_j \to p$, we see the following:

$$\mathcal{S}_{\theta|_p}(v_p, \overline{v_p}, v_p, \overline{v_p}) \leqslant 0, \quad \text{for } v_p \in \mathcal{C}_\ell T_p^{(1,0)}M. \tag{18}$$

Here when $\ell < n/2$, we have assumed that θ is appropriate and when $\ell = n/2$, we have assumed that $F^*(\tilde{\theta}|_{F(q_j)}) = \tilde{k}(q_j)\theta|_{q_j}$ with $\tilde{k}(q_j) > 0$ for a certain choice of the sequence $q_j \to p$. Hence, we get another application of Theorem 3.3:

Corollary 3.6. *Let $M \subset \mathbb{C}^{n+1}$ and $\widetilde{M} \subset \mathbb{C}^{N+1}$ be two smooth Levi non-degenerate hypersurfaces with the same signature $0 < \ell \leqslant \frac{n}{2}$. Suppose that F is not a totally degenerate holomorphic map defined over a neighborhood U of M in $\mathbb{C}^{n+1}$ with $F(M) \subset \widetilde{M}$. Let $p \in M$. If F fails to be CR transversal at p (or, equivalently, if F fails to be a local holomorphic embedding near p), then the following holds:* (i) *If $0 < \ell < n/2$, then the Chern-Moser-Weyl tensor at p with respect to any appropriate contact form is pseudo semi-negative definite.* (ii) *If $\ell = n/2$, then the Chern-Moser-Weyl tensor of M (with respect to any contact form) at p is pseudo semi-definite.*

Corollary 3.4 can be used to construct many examples which fail to be embeddable into hyperquadrics. Here we provide one example as follows.

Example 3.7. (1). Suppose that $P(z,\overline{z})$ is a real-valued homogeneous polynomial of bidegree $(2,2)$ for $z \in \mathbb{C}^n$ $(n \geqslant 3)$ and $P(z,\overline{z}) > 0$ for $z \neq 0$. Let $0 < \ell < n/2$. Let $M \subseteq \mathbb{C}^{n+1}$ be defined by

$$\Im w = |z|^2_\ell - N_4(z,\bar{z}) \tag{19}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, where N_4 is obtained from the following decomposition

$$P(z,\overline{z}) = N_4(z,\bar{z}) + N_2(z,\bar{z})|z|^2_\ell$$

with $\Delta_\ell N_4(z,\bar{z}) = 0$. Then M cannot be CR embedded into $\mathbf{H}^N_\ell$ for any N.
(2). Suppose that $P(z,\overline{z})$ is a real-valued homogeneous polynomial of bidegree $(2,2)$ for $z \in \mathbb{C}^n$ $(n = 2k \geqslant 4)$ and $P(z,\overline{z})$ does not have a fixed sign for $|z|_\ell = 0$. (namely, neither $P \geqslant 0$ for all $|z|_\ell = 0$ nor $P \leqslant 0$ for all $|z|_\ell = 0$.) Let $0 < \ell = k$. Let $M \subseteq \mathbb{C}^{n+1}$ be defined by

$$\Im w = |z|^2_\ell - N_4(z,\bar{z}) \tag{20}$$

as above. Then M cannot be CR embedded into $\mathbf{H}^N_\ell$ for any N.

Indeed, (19) and (20) are already of the Chern-Moser normal form near the origin and their corresponding Chern-Moser-Weyl curvature tensor $\mathcal{S}_{\theta|_0}(z,\overline{z}) = 4N_4(z,\bar{z})$. Moreover, by the construction of N_4, it is pseudo positive-definite in (19) and not pseudo semi-definite in (20). Corollary 3.4 then directly implies that M cannot be CR embedded into $\mathbf{H}^N_\ell$. In particular, the following two real hypersurfaces M_1 and M_2 can not be CR embedded into real hyperquadrics of the same signature in any $\mathbb{C}^N$:

$$\begin{aligned} &M_1 \subset \mathbb{C}^4 : \Im w = |z|^2_\ell - \frac{1}{2}(|z_1|^4 + |z_2|^4 + |z_3|^4 + 2|z_1z_2|^2 + 2|z_1z_3|^2 - 2|z_2z_3|^2), \quad \ell = 1; \\ &M_2 \subset \mathbb{C}^5 : \Im w = |z|^2_\ell - \frac{1}{3}(|z_1|^4 - |z_3|^4 - 2|z_1z_2|^2 + 2|z_1z_4|^2 - 2|z_2z_3|^2 + 2|z_3z_4|^2), \quad \ell = 2. \end{aligned} \tag{21}$$

One may verify that, for M_1, the corresponding $P(z,\bar{z}) = |z_1|^4 + |z_2|^4 + |z_3|^4$ and $N_4(z,\bar{z}) = P(z,\bar{z}) - \frac{1}{2}|z|^4_\ell$, which falls into Case (1); while for M_2, the corresponding $P(z,\bar{z}) = |z_1|^4 - |z_3|^4$ and $N_4(z,\bar{z}) = P(z,\bar{z}) + \frac{2}{3}(|z_1|^2 + |z_3|^2)|z|^2_\ell$, which falls into Case (2).

We conclude this paper with the following two open problems related to Corollary 3.4, Example 3.7 and Crollary 3.6:

Question 3.8. Let M be a strongly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$ with $n \geqslant 1$ defined by a real polynomial. For any $p \in M$, does there exist a sufficiently large positive integer

N, whihc may depend on p, such that a small piece of M near p can be embedded into the Heisenberg hypersurface $\mathbf{H}_0^{N+1}$ (with signature 0)?

Question 3.9. Let M and $\widetilde{M}$ be smooth Levi non-degenerate hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively, with $N > n$. Assume that both M and $\widetilde{M}$ have the same signature ℓ with $0 < \ell < n/2$. Let U be a (connected) neighborhood of M in $\mathbb{C}^{n+1}$. Suppose that F is not a totally degenerate holomorphic map from U into $\mathbb{C}^{N+1}$ with $F(M) \subset \widetilde{M}$. Is then F a local holomorphic embedding along M?

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