This is the Ph D Thesis of the author written in 1994 at Washington University in St. Louis. Some results contained in this Thesis have not been (and will not be) published. Since several unpublished results in this Thesis were quoted and used in the literature, we post it for convenience of some interested reader. For the current status of the open problems discussed in this Thesis, the reader should consult more updated references.

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Chapter 1. Introduction and open problems

In the present Chapter, we give a description and overview of topics to be treated in details in later chapters.

§1.1. **Mapping problems for algebraic real hypersurfaces in complex euclidean space**:

The study of this topic goes back to Henri Poincaré. In 1907 [Po], he showed that any biholomorphic mapping between parts of the sphere (boundary of the ball) in $\mathbf{C}^2$ is rational and extends to an automorphism of $\mathbf{B}_2$, the ball in $\mathbf{C}^2$. This striking result obviously fails in the setting of one complex variable and reveals strong rigidity properties of mappings in several variables. In 1960's, Poincaré's theorem was extended to the complex space $\mathbf{C}^n$ with $n \geq 2$ by Tanaka and later by Alexander [Al] in a more general setting. More recently, Webster [We1] investigated biholomorphic mappings between strongly pseudoconvex real algebraic hypersurfaces in the same complex space $\mathbf{C}^n$ with $n > 1$. He successfully applied the so-called Segre surfaces and proved the algebraicity for such mappings (a hypersurface is called algebraic if it is defined by a polynomial and a mapping is said to be algebraic if its graph is an open subset of an irreducible algebraic variety). Webster's idea was extensively used in later research, especially in the equi-dimensional case (see [DW], [DF1], etc). For the case of positive codimension, i.e, when the mapping is from a hypersurface in $\mathbf{C}^n$ to another one in $\mathbf{C}^{n+k}$ ($n \geq 2, k > 0$), the situation is much more subtle. In the spherical case, based on the fact that the Segre surfaces of a sphere are just hyperplanes, Forstneric [Fr1] was able to obtain the rationality of holomorphic mappings from the sphere in $\mathbf{C}^n$ to the sphere in $\mathbf{C}^{n+k}$ ($n > 1$). We mention here that special cases of Forstneric's theorem were previously obtained by Alexander [Al], Webster [We2], Faran [Fa1], and Cima-Suffridge [CS1].

In the first part of this thesis (Chapter 2), we will study the algebraic mapping problem for any codimension and will prove the following general algebraic mapping theorem (see §2.1 for relevant notation):

Theorem 1: Let $M_1 \subset \mathbf{C}^m$ and $M_2 \subset \mathbf{C}^{m+k}$ be strongly pseudoconvex algebraic real hypersurfaces with $m > 1$ and $k \geq 0$. Suppose that f is a holomorphic mapping from a neighborhood of M_1 to $\mathbf{C}^{m+k}$ such that $f(M_1) \subset M_2$. Then f is algebraic.

In many applications, it would be important to know the above mentioned results for mappings which are only holomorphic on one side of the hypersurface and have only a certain regularity up to the hypersurface. Since rationality and algebraicity are global properties, this leads one to the studies of reflection principles between pseudocoonvex real analytic hypersurfaces. In the equi-dimensional case, the situation is relatively clear by the work of Fefferman [Fe], Lewy [Le], Pinchuk [Pi], Diederich-Webster [DW], Baouendi-Bell-Rothschild [BBR], Baouendi-Rothschild [BR1] [BR2], Diederich-Fornaess [DF1] [DF2], etc. For positive codimension, the general question is open even in the spherical case. For example, is any C^1 smooth CR mapping from the sphere in $\mathbf{C}^2$ to the sphere in $\mathbf{C}^3$ real analytic? Equivalently, is any proper holomorphic mapping from $\mathbf{B}_2$ to $\mathbf{B}_3$, which admits a C^1 smooth extension up to the boundary, rational? Here we call a function defined on a strongly pseudoconvex hypersurface a CR function if it can be realized as the boundary value of some function holomorphic on one side of the hypersurface. We notice that, by the recent work on the existence of inner functions in several variables, one knows that there exists a proper holomorphic mapping from $\mathbf{B}_2$ to $\mathbf{B}_3$, which is continuous up to the boundary, but not C^2 at any boundary point. Thus some minimal smoothness assumption in the above question is necessary.

Indeed, with a little bit more regularity to begin, in the early 1980's, Webster [We2] already showed that any C^3-CR mapping from a strongly pseudoconvex real analytic hypersurface in $\mathbf{C}^n$ $(n > 2)$ to the sphere in $\mathbf{C}^{n+1}$ is real analytic almost everywhere. Webster's result was generalized in the subsequent work [Fa1], [CS1], [CKS], [Fa3], and [Fr1]. In Chapter 2 of this thesis, we will prove the following theorem by modifying the argument for the proof of Theorem 1:

Theorem 2: Every C^{k+1}-CR mapping from a strongly pseudoconvex real analytic hypersurface $M_1 \subset \mathbf{C}^n$ $(n > 1)$ into another strongly pseudoconvex real analytic hypersurface $M_2 \subset \mathbf{C}^{n+k}$ is real analytic (C^ω) on a dense open subset.

We mention that, in case the f in the above result is C^∞ smooth on M_1, then Theorem 2 is one of the main of themes of [Fr1]. However, reducing the infinite smoothness assumption to the finite smoothness assumption in this result requires new ideas and much more effort. Theorem 2 confirms a problem in [Fr1] and includes all previous results as special cases. Meanwhile, it also allows Theorem 1 to be formulated as follows:

Theorem 1$'$: Let $M_1 \subset \mathbf{C}^m$ and $M_2 \subset \mathbf{C}^{m+k}$ be two strongly pseudoconvex real algebraic hypersurfaces ($m > 1$, $k \geq 0$). Then every C^{k+1} smooth CR mapping from M_1 to M_2 is algebraic. That is, each component of the map can be annihilated by an irreducible polynomial.

A natural question which arises here is to ask if one can further conclude everywhere real analyticity in Theorem 2. As an application of Theorem 1$'$, we can obtain a solution in some special cases; while the general question is still open:

Corollary 1: Let M_1 and M_2 be two strongly pseudoconvex real algebraic hypersurfaces in (possibly different) complex spaces of dimension at least two. Then every C^∞ smooth CR mapping from M_1 to M_2 is real analytic on M_1.

We mention that Theorem 1' also holds if we replace M_1 and M_2 by two (Levi-) non degenerate algebraic hypersurfaces with the same signature under the assumption that f is an embedding with the Hopf Lemma property (see the remark in the end of §2.2.4). Moreover the degree of the maps can be bounded by a constant depending only on M_1 and M_2 (in fact, by the degree of M_1 and M_2).

When the CR mapping in the above result is only assumed to have certain minimal smoothness, one has the following problem:

Question:: Let M_1 and M_2 be two strongly pseudoconvex algebraic real hypersurfaces in (possibly different) complex spaces of dimension at least two. Is every continuous algebraic CR mapping from M_1 to M_2 real analytic on M_1?.

A theorem in this direction is due to Cima- Suffridge ([CS2]), who proved that any C^{k+1}-CR mapping from the sphere in $\mathbf{C}^n$ to the sphere in $\mathbf{C}^{n+k}$ is C^ω. In the last section of Chapter 2, we will present a partial solution to the above question by proving the following:

Proposition 1: Let $D \subset \mathbf{C}^2$ be a bounded smooth algebraic domain. Denote by t_p the type value of the type at $p \in \partial D$ in the sense of Kohn [Ko1] (or D'Angelo [Da]). Let $t = \max_{p\partial D} t_p$, which is finite. Suppose that f is a proper holomorphic mapping from D to $\mathbf{B}_3$ which admits a C^t smooth extension up to the boundary. Then f has a holomorphic extension across the boundary.

§1.2. Kobayashi extremal mappings and holomorphic self-mappings:

Let D be a bounded domain in complex euclidean space $\mathbf{C}^n$, and let $f \in \mathrm{Hol}(D, D)$ be a holomorphic self mapping of D. It is an old subject to investigate the asymptotic behavior of the sequence $\{f^k\}$ of iterates of f, defined inductively by $f^1 = f$ and $f^k = f^{k-1} \circ f$. In 1926, Denjoy [De] and Wolff [Wo] showed that in case D is the unit disk Δ in the complex plane, then the sequence $\{f^k\}$ converges uniformly on compact sets to a boundary point $p \in \partial D$ (which is viewed here as a constant map $h: D \to \mathbf{C}^n$ with $h(D) \equiv p$) if and only if f has no fixed point inside D. Since this pioneering work, much attention has been paid to extending iteration theory to domains in higher dimensions. To name a few of the recent results, we mention those on the ball in $\mathbf{C}^n$ [He] [Mac], on strongly convex domains [Ab2] and on bounded contractible strongly pseudoconvex domains in $\mathbf{C}^2$ [Ma]. For a very detailed account of the history and references, we refer the reader to the excellent book of Abate [Ab1].

One of the main themes of this thesis is to study holomorphic self mappings by making use of Kobayashi extremal disks. In Chapter 3 we will be concerned with iteration theory on strongly pseudoconvex domains in $\mathbf{C}^n$ for any $n \geq 1$. We will prove the following Theorem 3, which gives an exact description of the Denjoy-Wolff phenomenon for a large class of non-convex domains in $\mathbf{C}^n$ with $n \geq 1$ (see also [Ab3] for certain partial results in this regard). Theorem 3 answers a problem raised in [Ab3]:

Theorem 3: Let D be a (topologically) contractible, bounded strongly pseudoconvex domain in any dimension with C^3 boundary, and let $f \in \mathrm{Hol}(D, D)$ be a holomorphic self-mapping of D. Then $\{f^k\}$ converges to a boundary point uniformly on compact sets if and only if f has no fixed point in D.

As a corollary, we have the following:

Corollary 2: Let $D \subset\subset \mathbf{C}^n$ be a C^3 bounded strongly pseudoconvex domain ($n \geq 1$) that is homeomorphic to the unit ball in $\mathbf{C}^n$, and let $f \in \mathrm{Hol}(D, D)$ be a holomorphic self-mapping of D. Suppose that there exists $z_0 \in D$ so that $\{f^k(z_0)\}$ is a relatively compact subset of D. Then f fixes some point in D.

The key step toward proving Theorem 3 is to prove a fixed point theorem on lower dimensional holomorphic retracts of D. (Here we recall that a subset $E \subset D$ is called a holomorphic retract if there exists a holomorphic retraction $h \in \mathrm{Hol}(D, D)$ so that $h^2 = h$ and $E = h(D)$). In case D is strongly convex or strongly pseudoconvex in $\mathbf{C}^2$ with trivial topology, this can be achieved by making use of the property that the Kobayashi ball of a bounded convex domain is also convex in the euclidean metric [Ab2], or by using the Riemann mapping theorem and the classical Denjoy-Wolff theorem [Ma]. Since we will deal with a non-convex domain of any dimension, it does not seem that the aforementioned approaches can be adapted to our situation. The method presented here is based on a very careful investigation of the asymptotic behavior of Kobayashi extremal mappings near a strongly pseudoconvex point. Here, we recall that an extremal mapping ϕ of D is a holomorphic map from the unit disk Δ to D so that for any $\psi \in \mathrm{Hol}(\Delta, D)$ with $\psi(0) = \phi(0)$ and $\psi'(0) = \lambda\phi'(0)$ (where, as usual, λ denotes a real number), it holds that $|\lambda| \leq 1$. A holomorphic mapping from Δ to D is called a complex geodesic in the sense of Vesentini if it realizes the Kobayashi distance between any two points on its image (see [Ve]). We next present these technical results. Their statements require some preliminary notation. (for more definitions, see § 3.1).

Let D be a bounded domain in $\mathbf{C}^n$ with p a C^2 smooth boundary point. For any $z \in D$, close enough to p, there is a unique point nearest to z in ∂D, which is denoted by $\pi(z)$. For any complex vector $\xi \in \mathrm{T}^{(1,0)}D$, in what follows, we will use ξ_T and ξ_N to denote the complex tangential and complex normal components of ξ at $\pi(z)$, respectively.

We say that a bounded domain $D \subset \mathbf{C}^n$ has a Stein neighborhood basis if there exists a sequence of bounded domains $\{D_v\}$ in $\mathbf{C}^n$ such that $D \subset\subset \ldots \subset\subset D_3 \subset\subset D_2 \subset\subset D_1$ and the interior of $\cap_v D_v$ is D. A well known fact [Kr1] is that every bounded domain defined by a C^1 plurisubharmonic function (in particular, every bounded C^2-strongly pseudoconvex domain) has a Stein neighborhood basis.

Theorem 4: Let $D \subset\subset \mathbf{C}^n$ be either a pseudoconvex domain with a Stein neighborhood basis or a pseudoconvex domain with C^∞ boundary. Suppose that $p \in \partial D$ is a strongly pseudoconvex point of ∂D with at least C^3 smoothness. Then, for every open neighborhood U of p, there is a positive number ϵ such that for each extremal mapping ϕ of D, when $\|\phi(0) - p\| < \epsilon$ and $\|(\phi'(0))_N\| < \epsilon \|(\phi'(0))_T\|$, then ϕ is the complex geodesic of D and $\phi(\Delta) \subset U$.

Theorem 5: Let D be a bounded domain in $\mathbf{C}^n$ and $p \in \partial D$ a C^3 strongly pseudoconvex point. Then there is a small neighborhood U of p and a constant C depending only on U so that for any extremal mapping $\phi \in \mathrm{Hol}(\Delta, D)$ of D with $\phi(\Delta) \subset U \cap D$, it holds that $\|(\phi'(\tau))_N\| \leq \mathrm{C}\eta(\phi)\|(\phi'(\tau))_T\|$. Here $\|\cdot\|$ stands for the Euclidean norm in $\mathbf{C}^n$ and $\eta(\phi) = \max_{\xi \in \overline{\Delta}} \|\phi(\xi) - p\|$.

Corollary 3: Let D be a bounded domain in $\mathbf{C}^n$ and $p \in \partial D$ a C^3 strongly pseudoconvex point. Let $\{\phi_k\}$ be a sequence of extremal mappings of D and ϵ_0 a positive number so that $\{\phi_k(0)\}$ converges to p and $\|(\phi_k'(0))_N\| \geq \epsilon_0 \|(\phi_k'(0))_T\|$ for each k. Then the diameter of $\phi_k(\Delta)$ is greater than a fixed positive constant for every k.

Theorem 6: Let $D \subset \mathbf{C}^n$ be a bounded strongly pseudoconvex domain with C^k $(k \geq 3)$ smooth boundary and let f be a holomorphic self-mapping of D. Then the following holds:

(1): Every holomorphic retract of D with complex dimension greater than 1 is actually a closed complex sub-manifold with $C^{(k-1)-}$ smooth boundary.

(2): Suppose that $\{f^k\}$ is a precompact family and does not converge to a single point. Then there exists a unique holomorphic retract E, depending only on f, such that (a) $f|_E$ is an automorphism of E (moreover, when E has at least complex dimension 2 or it is a complex geodesic, then $f|_E$ admits a $C^{(k-1)-}$ smooth extension up to the boundary of E); (b) for each point $z_0 \in D$, the limit points of $\{f^\ell(z_0)\}$ stay in E.

The proofs of Theorem 4 and Theorem 5 are complicated and will be carried out in §3.2 and §3.3, respectively. However we remark that these two technical results, which are also of interest in their own right, can be used for many other purposes. For example, in §3.5, they will be used to prove the following two theorems. The first one was previously obtained in [CHL] by using Lempert's deformation theory in case the boundary is of class C^{14}, while the second result is an extension of the Burns-Krantz rigidity theorem (see [BK] [Hu2]).

Theorem 7: Let D be a bounded C^3 strongly convex domain in $\mathbf{C}^n$. For any given $p \in \partial D$ and complex vector $v \in \mathrm{T}^{(1,0)}\mathbf{C}^n$, but not in $\mathrm{T}_p^{(1,0)}\partial D$, there exists an extremal mapping ϕ so that $\phi(1) = p$ and $\phi'(1) = \lambda v$ for some real number λ (this ϕ must then be uniquely determined up to an automorphism of Δ according to Lempert [Lm1]).

Theorem 8: Let $D \subset\subset \mathbf{C}^n$ be either a simply connected smooth pseudoconvex domain or a simply connected taut domain with Stein neighborhood basis. Let $p \in \partial D$ be a strongly pseudoconvex point with at least C^3 smoothness. Suppose that $f \in \mathrm{Hol}(D, D)$ is a non-identical holomorphic self mapping of D so that $f(z) = z + o(\|z - p\|^k)$ as $z \to p$. Then the following hold:

(1) $k \leq 2$

(2) If $k = 1$, then either f fixes a holomorphic retract with positive dimension or $f^m \to p$. In case D is not biholomorphic to the ball, then f cannot be an automorphism.

(3) If $k = 2$, then f cannot be an automorphism of D and the sequence $\{f^m\}$ converges to p on compacta.

Corollary 4: Let D and p be as in Theorem 8. Suppose f is a holomorphic self mapping of D such that $f(z) = z + o(\|z - p\|^2)$ and f fixes some point in D. Then $f(z) \equiv z$.

We now state some open questions related to this work. First, in the proofs of Theorem 3, Theorem 4, Theorem 5, and Theorem 7 one can easily see that *our arguments actually work equally well (without any change) for the $C^{2+\alpha}$ ($\alpha > 0$) smoothness assumption of the boundary ∂D near the point p under study.* However, we do not know whether C^2 smoothness at p would be sufficient for them. A similar question is to ask for the optimal mapping theorem for biholomorphic maps between two strongly pseudoconvex domains with only C^2 boundaries. For example, one has the following

Conjecture: Let $D_1,\ D_2 \subset \mathbf{C}^n$ be two bounded strongly pseudoconvex domains with C^2 boundaries and let f be a proper holomorphic mapping from D_1 to D_2. Then f admits a C^1 smooth extension up to the boundary.

We remark that if Theorem 4 and Theorem 5 could be proved for C^2 smooth domains, then the answer to the above question is yes.

Another possible development of the present work is to generalize the Denjoy-Wollf theory to weakly pseudoconvex domains of finite type and then use it to answer the following question (see [HP1] and [HP2] for some approaches):

Question : Let $D \subset \mathbf{C}^n$ ($n > 1$) be a bounded smooth pseudoconvex domain of finite type and let f be a proper holomorphic self-mapping of D. Does it follow that f is an automorphism?

The extension of the Burns-Krantz theorem and Theorem 8 to weakly pseudoconvex domains is essentially unknown (see [H2]) and might require some completely new ideas and efforts. We formulate here the following:

Question : Let $D \subset\subset \mathbf{C}^n$ be a smooth pseudoconvex domains, and let $p \in \partial D$. Find optimal integers m_p and n_p such that for any holomorphic self-mapping f of D, if $f(z) = z + o(\|z-p\|^{m_p})$; or if $f(z) = z + o(\|z-p\|^{n_p})$ and f fixes some in D, one can then conclude that $f(z) \equiv z$.

§ 3. Local hull of holomorphy of a surface in $\mathbf{C}^2$

Given a subset $E \subset \mathbf{C}^2$, we let $\mathcal{O}(E)$ denote the collection of germs of holomorphic functions defined on E. By the hull of holomorphy of E, we mean the spectrum (i.e, maximal ideal space) of $\mathcal{O}(E)$. A remarkable fact in several complex variables is that a subset $E \subset \mathbf{C}^n$ may have a non-trivial hull of holomorphy, i.e, every holomorphic function defined on a small open neighborhood of E can be holomorphically extended to a fixed set $\widetilde{E}$ which is larger than E. In full generality, the determination of the holomorphic hull is a very hard problem and involves some global difficulties. In 1963, Bishop [Bis] first proposed the study of the local hull of a regular submanifold M in $\mathbf{C}^n$ by using analytic discs attached to M (it is known in general, however—see [STOLZ]—that a set may have a large hull of holomorphy that contains no analytic discs). In particular, when M is a two dimensional real submanifold of $\mathbf{C}^2$, Bishop classified the local study of the hull in terms of the local geometry of the base point $z_0 \in M$. Now it is understood that it is important to distinguish the case when the two dimensional tangent space $T_{z_0}M$ is a complex line from the case when $T_{z_0}M$ is totally real (that is, $T_{z_0}M \cap \sqrt{-1}T_{z_0}M = 0$). Points of the second type are of no interest for us because, by the work of Hörmander/Wermer [HOM], the local hull of holomorphy near such a point contains no new points. The situation in the first case is quite different.

Bishop [BIS] showed that, in the case that $z_0 \in M$ has a complex tangent and satisfies a non-degeneracy condition, then a holomorphic change of variables may be effected so that $z_0 = 0$ and the manifold M may be described in complex coordinates (z, w) by

$$w = h(z) = z\overline{z} + \lambda(z^2 + \overline{z}^2) + o(|z|^3)$$

with $0 \leq \lambda \leq +\infty$. Here the constant λ is a biholomorphic invariant of the manifold M. Now it is standard terminology to say that z_0 is an elliptic, parabolic, or hyperbolic point of M according to whether $\lambda \in [0, 1/2)$, $\lambda = 1/2$, or $\lambda > 1/2$ respectively.

In the elliptic case, Bishop obtained a family of analytic discs attached to M by using a Picard-style iteration scheme. In the later work of Bedford-Gaveau [BG] and Kenig-Webster [KW], it was shown that the local hull $\widetilde{M}$ of M is foliated by a family of embedded, pairwise disjoint analytic discs. Bedford-Gaveau proved that $\widetilde{M}$ is Lipschitz 1 continuous near z_0 provided that M itself is of class C^5 near z_0. Kenig-Webster proved that $\widetilde{M}$ is C^∞ near z_0 when M is of class C^∞ near z_0.

The analytic structure of $\widetilde{M}$ was studied in the papers of Moser-Webster [MOW] and Moser [MOS]. In [MOW], it was shown that $\widetilde{M}$ is real analytic up to z_0 when the original manifold M itself is real analytic at z_0, provided that $0 < \lambda < 1/2$. The case $\lambda = 0$ is not treated in [MOW]. Instead, in [MOS], Moser showed that a formal power series change of variables could be found in the case $\lambda = 0$ so that the manifold M is defined by an equation of the form

$$w = z\overline{z} + z^s + \overline{z}^s + \phi(z) + \overline{\phi(z)}.$$

Here $z_0 \leftrightarrow 0$ and s is a biholomorphic invariant of the surface M at z_0. Note also that ϕ is a formal power series in z beginning with terms of order at least $s+1$.

By using the rapidly convergent iteration technique, Moser was able to prove that, when $s = +\infty$, this formal coordinate change is also a convergent analytic coordinate change. However he left open the question of whether $\widetilde{M}$ is real analytic near z_0 when $s < \infty$.

In a very recent work with Krantz (see [HK2]), we settled the above open question of Moser. In the last Chapter of this thesis (Chapter 4), we will further develop the ideas which we used there and extend the results to a more general setting. To state the main result of Chapter 4, we first give the following definitions:

Let M be a smooth manifold in $\mathbf{C}^2$ and let p be an isolated complex tangent point of M. We call p a degenerate elliptic point of degree $2m$ if there exists a holomorphic change of variables so that p is mapped to the origin $0 \in \mathbf{C}^2$ and in the new coordinates M is given by an equation of the form:

$$w = h(z) = p_0(z) + h^*(z).$$

Here p_0 is a real valued homogeneous polynomial of degree $2m$ ($m \geq 1$) which is positive in the sense that $p_0(z) \geq \mathrm{C}|z|^{2m}$ and $\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} > \mathrm{C}|z|^{2m-2}$ for some positive constant C; $h^*(z) = o(|z|^{2m})$. We say that M can be flattened to order ℓ at p if, in the new coordinates mentioned above, one can make $\text{Im}(h^*(z)) = O(|z|^\ell)$. We remark that the degeneracy number $2m$ is a biholomorphic invariant.

Theorem 9: Let $M \subset \mathbf{C}^2$ be a real surface of class C^a, where $a = \infty$ or ω. Suppose that p is a degenerate elliptic point of M and suppose that M can be flattened to any order at p. Then the local hull of holomorphy of M at p is a C^a regular Levi flat hypersurface with C^a boundary M near p.

Notice that, in case $m = 1$, the above theorem reduces to the results of Kenig-Webster, Moser-Webster, Moser, Huang-Krantz.

As an application of Theorem 9, we have the following:

Theorem 10: Let M and p as in Theorem 9. Assume that M is of class C^ω, i.e, real analytic. Then there exists a biholomorphic mapping which sends M to a submanifold of the standard $\mathbf{R}^3 = \{(z, w) : \text{Im}w = 0\}$ in $\mathbf{C}^2$.

Contrary to the non-degenerate case, the assumption of arbitrary flatness of M at p in the aforementioned results is necessary as the following proposition shows:

Proposition 2: Let the real analytic surface M_n be defined by the equation

$$w = |z|^4 + |z|^{3+2n}(|z|^2 z + \sqrt{-1}).$$

Then the local hull of holomorphy of M at the degenerate elliptic point $0 \in M_n$ is only of class $C^{3/2+n}$ (at least when $n \geq 3$) and M_n can not be flattened to order $10 + 4n$ at 0.

To finish off the introduction, we present several open questions related to our work in Chapter 4. First, we notice that in the non-degenerate elliptic case, Moser-Webster derived a normal form for a surface near the complex tangent point, which in particular implies that any such surface is locally biholomorphic to an algebraic one. The following two questions have been asked also by Moser [MOS] (see [Go1] for some partial solutions):

Question: Let M be an analytic surface with a defining equation of the form $w = |z|^2 + o(|z|^2)$. Find all local biholomorphic invariants of M near 0.

Question: Let M be as above. Is M locally biholomorphic to an algebraic surface?

We say that a surface (M_0, p_0) can be approximated by the surface (M, p) (where $p_0 \in M_0$ and $p \in M$) if for any positive integer ℓ there exists a biholomorphic mapping near p so that p is mapped to p_0 and the image of M has an order of contact ℓ with M_0 at p_0.

Question: If (M_0, p_0) can be approximated by (M, p), can one then conclude that (M_0, p_0) is locally biholomorphic to (M, p)?

By [MOS], the answer to the last question is yes if p_0 an an elliptic point of M_0 with $\lambda = 0$. In case p_0 is a non-degenerate hyperbolic point of M_0 satisfying a certain Diophantine condition, then the affirmative answer to the last question can be found in the thesis of Gong [Go2].

Chapter 2: Mapping problems for algebraic real hypersurfaces in complex euclidean space

This chapter is devoted to the proofs of the theorems stated in §1.1. The organization is as follows: In §2.1, we give some notation and preparation. We prove Theorem 1 in §2.2 and Theorem 2 in §2.3. The proof of Proposition 1 can be found in the last section §2.4 .

Before proceeding, we remark that Theorem 1 and also Theorem 2 are false when $m = 1$. For example, Let $\Delta_\epsilon = \{\tau \in \mathbf{C}^1 : |\tau|^2 + \epsilon^2 |1 - e^\tau|^2 < 1\}$. Obviously, when $\epsilon \approx 0$, the domain Δ_ϵ is a strongly convex domain with analytic boundary. Let ϕ_ϵ be a conformal mapping from the unit disk Δ to Δ_ϵ, which is analytic on $\partial\Delta$ by the classical Schwarz reflection principle. Define f: $\Delta \to \mathbf{B}_2$(the unit two ball) by $f(\tau)$= $(\phi_\epsilon(\tau), \epsilon(1 - e^{\phi_\epsilon(\tau)}))$. Then f is proper and holomorphic on $\overline{\Delta}$, but is not algebraic.

§ **2.1 Preliminaries**:

The purpose of this section is to make some necessary preparations. In §2.1.1, we recall some definitions. In §2.1.2, we reformulate an immersion result of Pinchuk so that it can be easily applied to our situation (especially, to the proof of Theorem 2).

§ **2.1.1. Notation and an algebraic lemma**: Let $C(z)$ be the field of rational functions in the variable $z \in \mathbf{C}^n$. In this chapter, we call a function $\chi(z)$ holomorphic on an open subset $U \subset \mathbf{C}^n$ *algebraic* if there is a non-zero polynomial P with coefficients in $C(z)$ so that $P(\chi) = 0$, i.e., the field generated by adding χ to $C(z)$ is of finite extension. A mapping is called *algebraic* if each of its components is. For convenience, we collect here some facts about algebraic functions which will be used frequently in the later discussion:

Lemma 2.1: Let χ be a non trivial algebraic function in $z \in U \subset \mathbf{C}^n$. Then the following holds:

(1): For any fixed $(z^0_{k+1}, \cdots, z^0_n)$, the function $\chi(z_1, \cdots, z_k, z^0_{k+1}, \cdots, z^0_n)$ is algebraic in $(z_1, \cdots, z_k)$.

(2): Let $z_j = g(z_1, \cdots, \hat{z_j}, \cdots, z_n)$ be a holomorphic solution of $\chi = 0$ for some fixed j. Then z_j=$g(z_1, \cdots, \hat{z_j}, \cdots, z_n)$ is algebraic in $(z_1, \cdots, \hat{z_j}, \cdots, z_n) \in \mathbf{C}^{n-1}$.

(3): $\frac{\partial}{\partial z_j}\chi(z)$ is algebraic in $(z_1, \cdots, z_n)$ for each j.

(4): If $g(z)$ is also algebraic on $U \subset \mathbf{C}^n$, then so are $\chi \pm g$, χg, and χ/g (in its defining domain).

(5): Let $z_j = g_j(s)$ be algebraic in $s \in V \subset \mathbf{C}^{n'}$ for $j = 1, \cdots, n$ and let $g = (g_1, \cdots, g_n)$ map V into U. Then $\chi \circ g = \chi(g_1(s), \cdots, g_n(s))$ is algebraic on V.

(6): Let $N_1 \subset \mathbf{C}^{n_1}$ and $N_2 \subset \mathbf{C}^{n_2}$ be two open subsets. Suppose that $g(z^{(1)}, z^{(2)})$ is a function on $N_1 \times N_2$. Let g be (holomorphically) algebraic in $z^{(1)} \in N_1$ (respectively, in $z^{(2)} \in N_2$) when holding $z^{(2)}$ fixed (respectively, when holding $z^{(1)}$ fixed). Then g is algebraic.

(7): Let $g(z_1, \cdots, z_n)$ be a (holomorphic) algebraic function at 0 with $g(0) = 0$ and $g(0, \ldots, 0, z_n) \not\equiv 0$. Then the Weierstrass polynomial g^* of g, with respect to z_n, is also algebraic near 0.

Proof: The proofs of (1)-(4) are standard. The argument for (6) can be found, for example, in [BM] (pp. 199- 205). So we just say a few words about (5) and (7):

To prove the statement in (5), we assume that $\chi(g(s)) \not\equiv 0$ and $\chi(z)$ is annihilated by the irreducible polynomial $P(z, w) = \sum_0^N a_j(z)w^j$. Let $Q(s, w) = P(g(s), w)$. Noting $a_j(g(s))$ is algebraic in s by (4), we see the algebraicity of $Q(s, w)$ in (s, w). Hence, if it is not identically zero, then from (2) and the fact that $Q(s, \chi(g(s)) \equiv 0$ the algebraicity of $\chi \circ g$ follows. When $a_j(g(s)) \equiv 0$ for each j, we let $D_l = \frac{\partial}{\partial z_l}$ and apply it to the equation $P(z, \chi(z)) \equiv 0$. We then see that $\sum_j (D_l a_j) \circ g(s)(\chi \circ g)^j \equiv 0$. Thus we may redefine Q to be $\sum_j (D_l a_j) \circ g(s)w^j$. Arguing inductively, we then see the proof of (5).

Now, we turn to (7) For any $z'(= (z_1, \ldots, z_{n-1})) \approx 0$, by a standard argument (see [Kr1], for example), we obtain exactly (counting multiplicity) n' solutions of the equation $g(z', z_n) = 0$: $\{a_1(z'), \cdots, a_{n'}(z')\}$ (with n' fixed). The Weierstrass polynomial g^* of g is then expressed as $g^* = \prod_{j=0}^{n'}(z_n - a_j(z')) = z_n + \sum_{j=0}^{n'-1} s_j(z')z_n^j$, where $s_j = \sum(-1)^j a_{l_1} \cdots a_{l_j}$. By (6), to check that g^* is algebraic we have only to show that the s_j's are. But this follows easily from (2), (4), and the fact that for a generic point $z'_0 \approx 0$, the $a_j(z')$'s are holomorphic for $z' \approx z'_0$. ∎

Now let $M \subset \mathbf{C}^n$ be a real analytic hypersurface with (real analytic) $r(z, \overline{z}) = 0$ as its defining function. First, M is said to be strongly pseudoconvex if $r(z, \overline{z})$ is a strongly plurisubharmonic function. We call M *algebraic* if the complexification of r, i.e $r(z, \overline{\omega})$, is algebraic in $(z, \overline{\omega})$ for $(z, \overline{\omega}) \approx M \times \mathrm{Conj}(M)$, where we write $\mathrm{Conj}(M) = \{\overline{z} : z \in M\}$. Fix $p \in M$ and a small open neighborhood $\Omega \subset \mathbf{C}^n$ of p. When $\omega \approx p$ then the Segre surface Q_ω restricted to Ω is a complex manifold of dimension $n-1$. Here we recall that $Q_\omega = \{z \in \Omega : r(z, \overline{\omega}) = 0\}$ and the complexification of M is defined to be $M_c = \{(z, \omega) \in \Omega \times \Omega : r(z, \omega) = 0\}$, a complex manifold of dimension $2n-1$.

§ **2.1.2 Reformulation of a lemma of Pinchuk:** We now let M_1, M_2 and f be as in Theorems 1 and 2. Without loss of generality, we also let f be non-constant. For a given point $p \in M_1$, after making use of a suitable polynomial holomorphic change of variables (see [Fe]), we can assume that $p = 0, f(0) = 0$, and M_1, M_2 are locally defined by ρ_1 and ρ_2, respectively:

(2.1.1) $$\rho_1(z, \overline{z}) = z_m + \overline{z_m} + \sum_{j=1}^{m-1} |z_j|^2 + h_0(z, \overline{z});$$

(2.1.2) $$\rho_2(w, \overline{w}) = w_{m+k} + \overline{w_{m+k}} + \sum_{j=1}^{m+k-1} |w_j|^2 + h(w, \overline{w}).$$

Here $h_0(z, \overline{z}) = O(\|z\|^4)$ and $h(w, \overline{w}) = O(\|w\|^4)$.

From a result of Pinchuk ([Pi]), it follows that $\frac{\partial f_{m+k}}{\partial z_m}(0) \neq 0$ and

$$df : T_0^{(1,0)} M_1 \to T_0^{(1,0)} M_2$$

is injective. We write $\tilde{L}_j = \frac{\partial \rho_1}{\partial z_m}\frac{\partial}{\partial z_j} - \frac{\partial \rho_1}{\partial z_j}\frac{\partial}{\partial z_m}$ for $j = 1, \ldots, m-1$. Since $f(M_1) \subset M_2$ and $\tilde{L}_j \in T^{(1,0)}M_1$, we see that

$$(2.1.3)\quad f_{m+k}(z) + \overline{f_{m+k}(z)} + \sum_{j=1}^{m+k-1} |f_j(z)|^2 + h(f(z), \overline{f(z)}) = 0 \quad \text{for } z \in U \subset M_1.$$

Applying each $\tilde{L}_j$ to (2.1.3), we then obtain

$$(2.1.4)\qquad \tilde{L}_l f_{m+k}(z) + \sum_{j=1}^{m+k-1} \tilde{L}_l f_j(z)\overline{f_j(z)} + \sum_{j=1}^{m+k} \frac{\partial h}{\partial w_j} \tilde{L}_l f_j(z) = 0, \quad \text{for } z \in U.$$

Now, by letting $z = 0$ in the formula (2.1.4), we see that $\tilde{L}_j f_{m+k}(0) = 0$ for each j. On the other hand, since $\{\tilde{L}_1, \ldots, \tilde{L}_{m-1}\}$ consists of a local basis of $\mathrm{T}^{(1,0)}M$ near 0, we thus conclude that the rank of the matrix $(\tilde{L}_j f_l)_{\substack{1 \le j \le m-1 \\ 1 \le l \le m+k-1}}$ is $m-1$. Let S be the vector space spanned by

$$\{\tilde{L}_1 f(0), \ldots, \tilde{L}_{m-1} f(0)\}$$

and let $\{T_1, \ldots T_{m-1}\}$ be an orthonormal basis of S. Extend it to an orthonormal basis of $\mathbf{C}^{m+k-1}$: $\{T_1, \ldots, T_{m+k-1}\}$ and set

$$(\tilde{f}_1, \ldots, \tilde{f}_{m+k-1})^t = (\overline{T_1}, \ldots, \overline{T_{m+k-1}})^t (f_1, \ldots, f_{m+k-1})^t.$$

It then follows easily that $\tilde{L}_j \tilde{f}_l(0) = 0$ for $l = m, \ldots, m+k-1$ and that $(\tilde{f}, f_{m+k})$ still satisfies the equation (2.1.4) (up to a term that vanishes to 4^{th} order). Now, by choosing $(L_1, \ldots, L_{m-1})^t = (\tilde{L}_j \tilde{f}_l(0))^{-1} (\tilde{L}_1, \ldots, \tilde{L}_{m-1})^t$ and by making use of the identity (2.1.4) with $z = 0$, we obtain

$$(2.1.5)\qquad L_j \tilde{f}_l = \delta_j^l = \begin{cases} 0, & \text{if } j \ne l; \\ 1, & \text{if } j = l. \end{cases}$$

Consequently, to simplify the notation, we assume in what follows that M_1, M_2, f, and $\{L_1, \ldots L_{m-1}\}$ already have the properties in (2.1.1), (2.1.2), and (2.1.5).

§ 2.2 Proof of Theorem 1:

In this section, we present the proof of Theorem 1, which can be viewed as the main result of the whole chapter.

Theorem 1: Let $M_1 \subset \mathbf{C}^m$ and $M_2 \subset \mathbf{C}^{m+k}$ be strongly pseudoconvex algebraic real hypersurfaces with $m > 1$ and $k \geq 0$. Suppose that f is a holomorphic mapping from a neighborhood of M_1 to $\mathbf{C}^{m+k}$ such that $f(M_1) \subset M_2$. Then f is algebraic.

Our idea is to show that each component of the mapping f stays in the field generated by adding some algebraic elements (which are obtained from suitable operations on the defining functions of the hypersurfaces) to the rational functions field. For this purpose, we start by complexifying the identity: $\rho_2(f(z), \overline{f(z)}) = \lambda(z, \overline{z})\rho_1(z, \overline{z})$ and differentiate it along each Segre surface. Then we will obtain the algebraicity by a very careful case-by-case argument according to how degenerate the map is. Since the proof is long, we shall, for clarity, split it into 4 subsections and many small lemmas.

§ **2.2.1.** In this subsection, we concentrate on two major cases which we will study in detail in §2.2.3 and §2.2.4.

Let M_1 and M_2 be as in the main theorem. As we have discussed in the above section, we may let M_1, M_2, and f have the properties (2.1.1), (2.1.2), and (2.1.5) mentioned in §2.1.2. We first choose a small neighborhood $\Omega \subset \mathbf{C}^m$ of 0 so that f is holomorphic on this open subset and the Segre surfaces Q_ω of M_1 restricted to Ω are connected for any $\omega \approx 0$.

Now, since $f(M_1) \subset M_2$, we have the equation $\rho_2(f(z), \overline{f(z)}) = \lambda(z, \overline{z})\rho_1(z, \overline{z})$ with $\lambda(z, \overline{z})$ real analytic. By the standard complexification, we then see, for each $\omega \approx 0$, that $\rho_2(f(z), \overline{f(\omega)}) = \lambda(z, \overline{\omega})\rho_1(z, \overline{\omega})$. Thus $f(Q_\omega) \subset Q_{f(\omega)}$ for $\omega \approx 0$. Therefore we obtain the following identity:

$$(2.2.1) \qquad f_{m+k}(z) + \overline{f_{m+k}(\omega)} + \sum_{j=1}^{m+k-1} f_j(z)\overline{f_j(\omega)} + h(f(z), \overline{f(\omega)}) = 0,$$

where, $z \in Q_\omega$, i.e, $(z, \omega) \in M_{1c}$, the complexification of M_1. By (2) of Lemma 2.1 and the implicit function theorem, the above h can be changed to an algebraic function not involving the $\overline{f_{m+k}}(\omega)$ term. (h also has no harmonic term). Therefore, we can assume that $h(f,\overline{f}) = h(f,\overline{f_1(\omega)},\ldots,\overline{f_{m+k-1}(\omega)})$, where $h(w,y_1,\ldots,y_{m+k-1})$ is an algebraic holomorphic function on $O_w(0)\times O_{y_1}(0)\times\ldots\times O_{y_{m+k-1}}(0)$. Here and in what follows, we use the symbol $O_*(**)$ to denote a small neighborhood of $**$ in the $*$ variable, which may be different in different contexts.

Now we let L_j (for $j = 1,\ldots,m-1$) be the polarization of the previously defined operator L_j, i.e, L_j is a linear combination of the following operators

$$\left\{\frac{\partial\rho_1(z,\overline{\omega})}{\partial z_m}\frac{\partial}{\partial z_j} - \frac{\partial\rho_1(z,\overline{\omega})}{\partial z_j}\frac{\partial}{\partial z_m}\right\}_{j=1}^{m-1}.$$

Then for any ω fixed, $\{L_j(z,\omega)\}_{j=1}^{m-1}$ consists of a basis for the holomorphic vector fields of Q_ω.

Applying each L_l to (2.2.1), $l = 1\ldots,m-1$, we obtain

$$\text{(2.2.2)}\quad L_l f_{m+k}(z) + \sum_{j=1}^{m+k-1} L_l f_j(z)\overline{f_j(\omega)} + \sum_{j=1}^{m+k}\frac{\partial h}{\partial w_j}L_l f_j(z) = 0,\quad \text{for } z\in Q_\omega.$$

Let $V(z,\omega) = (v_{ij}(z,\omega))_{1\le i,j\le m-1}$ with $v_{ij}(z,\omega) = L_i f_j$. Moreover, define

$$\xi(z,\omega)\equiv V^{-1}(z,\omega)(L_1 f_{m+k},\ldots,L_{m-1}f_{m+k})^t,$$

and

$$\eta(z,\omega)\equiv V^{-1}\begin{pmatrix} L_1 f_m\ldots & L_1 f_{m+k-1}\\ \cdots & \\ \cdots & \\ L_{m-1}f_m\ldots & L_{m-1}f_{m+k-1}\end{pmatrix}.$$

Equation (2.2.2) can then be written in the following matrix form:

$$\xi(z,\omega) + \overline{F_0(\omega)} + \eta(z,\omega)\overline{F}(\omega) + (\mathrm{id},\eta(z,\omega),\xi(z,\omega))Dh(z,\omega) = 0\ \text{ for } z\in Q_\omega,$$

where $Dh(z,\omega) = (\frac{\partial h}{\partial w_1},\ldots,\frac{\partial h}{\partial w_{m+k}})^t(f(z),\overline{f(\omega)}) = O(\|z\|^3+\|\omega\|^3)\cap O(\|z\|\|\omega\|)$ as $(z,\omega)\to(0,0)$, $F_0 = (f_1,\ldots,f_{m-1})^t$, and $F = (f_m,\ldots,f_{m+k-1})^t$.

Again, by making use of the implicit function theorem and by shrinking Ω, we have, for some holomorphic vector function g, that

$$\text{(2.2.3)}\ \xi(z,\omega)+\overline{F}_0(\omega)+\eta(z,\omega)\overline{F}(\omega)+g(f(z),\xi(z,\omega),\eta(z,\omega),\overline{F}(\omega)) = 0\ \text{ on } Q_\omega.$$

Since the algebraic function field is closed under the application of the implicit function theorem (see (2) of Lemma 2.1), it follows that the function $g(w, a, b, y)$ in (2.2.3) is also algebraic and holomorphic on

$$O_w(0) \times O_a(\xi(0,0)) \times O_b(\eta(0,0)) \times O_{y_m}(0) \times \dots O_{y_{m+k-1}}(0),$$

where we identify the variables ξ, η, F with $a, b, Y = (y_m, \dots, y_{m+k-1})$, respectively. In fact, it is easy to see that g does not depend on f and is identically 0 when M_2 is the sphere. Set

$$H_0(f, \xi, \eta) = \xi + g(f, \xi, \eta, 0),$$

$$H_\alpha(f, \xi, \eta) = \eta_{\alpha^*} + \frac{\partial g}{\partial y_{\alpha^*}}(f, \xi, \eta, 0), \quad \text{with } \|\alpha\| = 1, \text{ and}$$

$$H^*(f, \xi, \eta, \overline{F})$$

$$= g(f, \xi, \eta, \overline{F}) - (g(f, \xi, \eta, 0) + \sum_{\|\alpha\|=1} \frac{\partial g}{\partial y_{\alpha^*}}(f, \xi, \eta, 0)\overline{f_{\alpha^*}}) \text{ for } (z, \overline{w}) \in M_{1c}.$$

Here and also in what follows, for a multi-index α with the j^{th} element 1 and all other components 0, we let $\alpha^* = m + j - 1$, and we let η_{α^*} denote the j^{th} column of the matrix η. Then (2.2.2) can be written as

(2.2.4)

$$H_0(f(z), \xi, \eta) + \overline{F}_0(\omega) + \sum_{\|\alpha\|=1} \overline{f_{\alpha^*}}(\omega) H_\alpha(f(z), \xi, \eta) + H^*(f(z), \xi, \eta, \overline{F}(\omega)) = 0,$$

for $(z, \overline{w}) \in M_{1c}$. Let $H^*(f, \xi, \eta, \overline{F}) = \sum_{\|\alpha\|=2}^{\infty} H_\alpha(f, \xi, \eta)\overline{F^\alpha}$. We will carry our discussion according to the following two possibilities:

(AA) $L_{l_0} H_{\alpha_0}(z_0, \overline{z}_0) \neq 0$ for some l_0, α_0, and $(z_0, \overline{z_0}) \in M_{1c}$ with $z_0 \approx 0$.

(BB) $L_l H_\alpha(z, \overline{z}) \equiv 0$ for all l, α, and $(z, \overline{z}) \in M_{1c}$ with $z \approx 0$.

§**2.2.2**: We present in this subsection two lemmas which will be useful in our later discussions.

Lemma 2.2: Let $\{H_\alpha\}$ be as above. If for some open subset $U \subset M_1$ of p, it holds that $L_l H_\alpha(z, \overline{z}) \equiv 0$ for all $z \in U$ and l, α, then there is an algebraic holomorphic function Ψ so that $F_0(z) = \Psi(z, F(z))$ for $z \approx p$.

We first observe that $U \times \text{Conj}(U) \subset M_{1c}$ is a totally real submanifold of maximal dimension, where the notation $\text{Conj}(U)$ is the same as that at the end of §1.1. So $U \times \text{Conj}(U)$ is a set of uniqueness for the holomorphic functions on M_{1c}. Thus, under the hypothesis of the lemma, it follows that $L_l H_\alpha(z, \overline{\omega}) \equiv 0$ for all l, α and $(z, \overline{\omega})(\subset M_{1c}) \approx (p, p)$.

Proof of Lemma 2.2: Since $\{L_1, \ldots, L_{m-1}\}$ is a basis for the collection of holomorphic vector fields on Q_ω and since $H_\alpha(f(z), \xi(z,\omega), \eta(z,\omega))$ is holomorphic for any fixed ω, it follows, from the just mentioned observation, that

$$H_\alpha(f(z), \xi(z,\omega), \eta(z,\omega))$$

is constant along any Q_ω.

Define

$$(2.2.5)\quad \Psi^*(z,\omega,Y) = -\overline{H_0(f(z),\xi,\eta)} - \sum_{\|\alpha\|=1} y_{\alpha^*}\overline{H_\alpha(f(z),\xi,\eta)} - \overline{H^*(f,\xi,\eta,\overline{Y})},$$

for $(z, \overline{\omega}) \in M_{1c}$ and $Y \approx 0$. We then can conclude that, for any fixed

$$Y = (y_m, \ldots, y_{m+k-1}) \in \mathbf{C}^k,$$

the function Ψ^* is constant on each Q_ω ($\omega \approx p$). Moreover, it can be seen that for any fixed (z,ω), Ψ^* is algebraic in Y since $\overline{H^*(f, \cdots, \overline{Y})}$ is. For any given $z(\in \Omega) \approx p$ and $Y \approx 0$, we define $\Psi(z,Y) = \Psi^*(\omega, z, Y)$ with $\omega \in Q_z$. This definition makes sense because Ψ^* is independent of the choice of $\omega \in Q_z$. By (2.2.4), it obviously holds that $F_0(z) = \Psi(z, F(z))$ for $z \approx 0$. We are now going to complete the proof of the lemma by showing that Ψ is algebraic in (z, Y).

First, we notice that, for any given z, Ψ is algebraic in Y by the above discussion. Thus, by (6) of Lemma 2.1, we have only to prove that Ψ is algebraic in z when holding Y fixed.

Fix $z_0 \approx 0$ and let $z \in Q_{z_0}$. Since z_0 is also contained in Q_z, we see by (2.2.5) that

$$\Psi(z,Y) = -\overline{H_0(f(z_0), \xi(z_0,z), \eta(z_0,z))}$$

$$- \sum_{\|\alpha\|=1} y_{\alpha^*} \overline{H_\alpha(f(z_0), \xi(z_0, z), \eta(z_0, z))} - \overline{H^*(f(z_0), \xi(z_0, z), \eta(z_0, z), \overline{Y})}.$$

Therefore it can be seen that $\Psi(z, Y)$ is holomorphic and algebraic along Q_{z_0} for any fixed Y, since H_α and H^* are algebraic in their separate variables and $\xi(z_0, z), \eta(z_0, z)$ are holomorphically algebraic in $\overline{z}$ (by the algebraicity of M_1 and M_2). Now the algebraicity of Ψ follows clearly from the following

Lemma 2.3: Let M be a piece of algebraic strongly pseudoconvex hypersurface. If g is a function defined near $p \in M$ which is holomorphic and algebraic on any Segre surface Q_z with $z \approx p$, then $g(z)$ is algebraic in z.

Proof of Lemma 2.3: By an algebraic change of variables, we can assume that $p = 0$ and M is defined by an equation $\rho(z, \overline{z}) = 0$ with a similar form to (2.1.1):

$$\rho(z, \overline{z}) = z_m + \overline{z_m} + \sum_{j=1}^{m-1} |z_j|^2 + h_0(z, \overline{z}) = 0.$$

Moreover, we can assume that $h_0(z, \overline{z})$ has no harmonic terms, i.e, $h_0(0, \omega) = h_0(z, 0) = 0$ (for otherwise, by noting that $h_0(z, 0)$ is algebraic, we can use $z_m + h_0(z, 0)$ as our new z_m-variable to get the required form). Also we can always assume h_0 contains no $\overline{z_m}$.

Let $\mathrm{e}_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the j^{th} position , and let $\tau (\neq 0) \in \mathbf{R}$ but close enough to 0. Obviously, we then have $\tau \mathrm{e}_j \in Q_0$ and thus $0 \in Q_{\tau \mathrm{e}_j}$ for $j = 1, \ldots, m-1$. Write $S_\tau = \cap_{j=1}^{m-1} Q_{\tau \mathrm{e}_j}$. We see that $0 \in S_\tau$. Define the map ϕ from $S_\tau \times Q_0$ to $\mathbf{C}^m$ by $\phi(0, 0) = 0$ and

$$\phi(s, t) = Q_s \cap \{\cap_{j=1}^{m-1} Q_{(\tau + t_j)\mathrm{e}_j}\},$$

where we write $t = (t_1, \cdots, t_{m-1}, 0)$.

Claim 1: When $\tau (\neq 0)$ is close enough to 0, then S_τ is a regular algebraic curve near 0 and $\overline{\phi}$ is an algebraic holomorphic map near $(0, 0)$. Moreover, the Jacobian of $\overline{\phi}$ is not identically zero near $(0, 0) \in S_\tau \times Q_0$.

Proof of Claim 1: Notice that S_τ is defined by the equations:

$$z_m + \tau z_j + h_0(z, \tau \mathrm{e_j}) = 0 \quad \text{for j} = 1, \ldots, \mathrm{m} - 1.$$

From the implicit function theorem and Lemma 2.1, it then follows easily that for $\tau \approx 0$, S_τ is a regular algebraic manifold of complex dimension 1 near (0,0), parametrized by:

$$z_j = -\frac{z_m}{\tau} + h_j(\frac{z_m}{\tau}, z_m, \tau) \quad j = 1, \ldots, m-1,$$

where $h_j(\cdot, \cdot, \cdot)$ is holomorphic and algebraic in its variables and

$$h_j(\frac{z_m}{\tau}, z_m, \tau) = o(\sum_{i+j+k=2, i+j\geq 1} \left(\frac{z_m}{\tau}\right)^i (z_m)^j (\tau)^k).$$

Now, let $z = \phi(s,t)$ with $s = (s_1, \ldots, s_m)$ and $t = (t_1, \ldots, t_{m-1}, 0)$. Then, by the above argument and the definition of ϕ, we see that

(i) $$s_j = -\frac{s_m}{\tau} + h_j(\frac{s_m}{\tau}, s_m, \tau) \quad j = 1, \ldots, m-1;$$

(ii) $$G_m = z_m + \overline{s_m} + \sum_{l=1}^{m-1} \overline{s_l} z_l + o(\|s\|^2 \|z\| + \|z\|^2 \|s\|) = 0; \quad \text{and}$$

(iii) $G_j = z_m + (\overline{t_j} + \tau) z_j + o(|\tau + \overline{t_j}|^2 \|z\| + \|z\|^2 |\tau + \overline{t_j}|) = 0$ for $j = 1, \ldots, m-1$.

Notice that the Jacobian of $G = (G_1, \cdots, G_m)$ with respect to z at the origin (i.e, z,s,t=0) takes the following form:

$$\begin{pmatrix} \tau + o(|\tau|^2) & o(|\tau|^2) & \ldots & o(|\tau|^2) & 1 + o(|\tau|^2) \\ o(|\tau|^2) & \tau + o(|\tau|^2) & \ldots & o(|\tau|^2) & 1 + o(|\tau|^2) \\ \ldots & \ldots & & & \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.$$

So, its determinant is $\tau^{m-1} + o(\tau^m)$ and thus not equal to 0 when $|\tau| (\neq 0) << 1$. Applying the implicit function theorem to (ii) and (iii) and combining with (i), we see that for a fixed small τ, there exist an open subset U_τ near 0 and a (holomorphic) algebraic function χ defined on U_τ such that when $(s_m, t) \in U_\tau$ and $z \approx 0$ (i), (ii), and (iii) can be uniquely solved as

$$z = \chi(\overline{s_m}, \overline{t}) \quad \text{with } \chi(0,0) = 0.$$

If we identify S_τ with a small neighborhood of 0 in $s_m \in \mathbf{C}^1$ through (i), it holds that $\phi = \chi$. Now, we will show that the Jacobian determinant of χ with respect to $\overline{s_m}$ and $\bar{t} \in \mathbf{C}^{m-1}$ does not vanish identically near $(0,0)$. For this purpose, we let $0 < \delta << |\tau|$ and $P_{\delta,\tau} = (\tau^2\delta, 0)$. We will study the small solution z of (i), (ii), and (iii) (when $(s_m, t) = P_{\delta,\tau}$).

Let $s_m = \tau^2\delta$. By (i), we then have

$$s_j(\tau,\delta) = -\frac{s_m}{\tau} + h_j(\frac{s_m}{\tau}, s_m, \tau) = -\tau\delta + o(\delta\tau^2).$$

Returning to (ii), we see that

$$(iv) \qquad z_m + \tau^2\delta + \sum_{l=1}^{m-1}(-\tau\delta)z_l + o(\|s\|^2\|z\| + \|s\|\|z\|^2 + \delta\tau^2\|z'\|) = 0,$$

where $z' = (z_1, \cdots, z_{m-1})$. Using the implicit function theorem and solving for z_m in (iv), we get

$$(v) \qquad z_m = -\tau^2\delta + \tau\delta\sum_1^{m-1} z_l + o(\delta\tau\|z'\| + \delta\tau^2\|z'\|^2 + \tau^2\delta^2).$$

Substituting the right hand side for z_m in (iii), we obtain

$$-\tau^2\delta + \tau\delta\sum_1^{m-1} z_l + \tau z_j + o(\tau\|z'\|^2 + \tau^2\|z'\| + \tau^2\delta) = 0.$$

Thus, it follows that

$$-\tau\delta + \delta\sum_1^{m-1} z_l + z_j + o(\|z'\|^2 + \tau\|z'\| + \tau\delta) = 0 \quad j = 1, \cdots, m-1.$$

From these equations, one can easily see that

$$-\tau\delta + (1 + (m-1)\delta)z_j + o(\|z'\|^2 + \tau\|z'\| + \tau\delta) = 0, \quad j = 1, \cdots, m-1.$$

Now, we shrink τ (and thus also δ) so that the above equations can be solved near 0 for z'. Therefore, we have

$$z_j = \tau\delta + o(\tau\delta), \quad j = 1, \cdots, m-1.$$

Returning to (v), we see that $z_m = -\tau^2\delta + o(\tau^2\delta^2)$. Denote by P^* the uniquely solved z as above We next consider the Jacobian of G with respect to $(\overline{s_m}, \overline{t})$ at $P = (P_{\delta,\tau}, P^*)$. First, we notice that at P, it holds that $\frac{\partial G_k}{\partial \overline{t_j}} = z_k \delta_k^j + o(\tau^2\delta)$, where $\delta_k^j = 1$ if $k = j$ and 0 otherwise;

$$\frac{\partial G_k}{\partial \overline{s_m}} = 0; \quad \frac{\partial G_m}{\partial \overline{t_j}} = 0;$$

and

$$\frac{\partial G_m}{\partial \overline{s_j}} = 1 + \sum \overline{\frac{\partial s_l}{\partial s_m}} z_l + \frac{\partial}{\partial \overline{s_m}}(\|s\|^2\|z\| + \|s\|\|z\|^2) = 1 + o(\delta),$$

where we need to use the fact that $\frac{\partial s_l}{\partial s_m} = -\frac{1}{\tau} + o(1)$. Hence, we see that

$$J\left(\frac{G_1, \cdots, G_m}{\overline{s_m}, \cdots, \overline{t_{m-1}}}\right) = \begin{pmatrix} \tau\delta + o(\tau\delta) & o(\tau\delta) & \dots & o(\tau\delta) & 0 \\ o(\tau\delta) & \tau\delta + o(\tau\delta) & \dots & o(\tau\delta) & 0 \\ \dots & \dots & & & \\ 0 & \dots & 0 & 0 & 1 + o(\delta) \end{pmatrix}.$$

So, its determinant has the magnitude $(\tau\delta)^{m-1}(1 + o(1))$ and thus does not vanish when τ and δ are smaller than certain number.

We are now ready to complete the proof of the claim by arguing as follows: First, we let τ be very small (but fixed) and then shrink δ so that we can assume that $P_{\delta,\tau} \in U_\tau$ and the Jacobian of G with respect to $(\overline{s_m}, \cdots, \overline{t})$ is not zero at P. Thus by applying the implicit function theorem to (i), (ii), and (iii) again, we have an algebraic holomorphic solution $(\overline{s_m}, \overline{t})$ with respect to z near P, which is obviously the inverse function of χ. Hence the Jacobian of ϕ with respect to $(\overline{s_m}, \overline{t})$ is not zero at $P_{\delta,\tau}$ and thus is not identically zero near 0. ■

Now, by the way that ϕ was constructed and by the hypotheses of Lemma 2.2, we see that $g \circ \overline{\phi(s,t)}$ is holomorphic and algebraic on s (respectively, t) when holding t (respectively, s) fixed. From (6) of Lemma 2.1, it thus follows the algebraicity of g on an open subset of the defining domain (which is always assumed to be connected). Notice that algebraicity is a global property, we see that the proof of Lemma 2.3 is now complete. ■

§**2.2.3** : We now suppose that (AA) occurs. Then we will obtain the algebraicity of f when $k = 1$, or reduce the situation to a lower codimensional case when $k > 1$.

We first fix some notation. In what follows, we use the symbol $\hat{*}_l$ to denote the tuple obtained by deleting the element with index l from the vector $*$. For example, according to this convention, $\hat{F}_m$ means the vector function $(f_{m+1}, \ldots, f_{m+k-1})^t$, since $F = (f_m, \ldots, f_{m+k-1})^t$; and $\hat{F}_{m,m+1}$ is the vector function $(f_{m+2}, \ldots, f_{m+k-1})^t$.

Let us choose an integer n in the following way (the existence of such an integer can be seen by the condition in (AA)):

(i) If for some $p_0 \approx 0$, j_0, and l_0, it holds that $L_{l_0} g_{j_0}(p_0, p_0) \neq 0$, then we let $n = m$. Here $g_0(f(z), \xi(z, \omega), \eta(z, \omega), \hat{\overline{F}}_m(\omega)) = H_0 + \sum_{\|\alpha\|=1, \alpha^* \neq m} \overline{f_{\alpha^*}} H_\alpha$, $g_1(f, \xi, \eta, \hat{\overline{F}}_m) = H_{(1,0,\cdots,0)}$, and g_j for $j > 1$ is determined by

$$H^*(f, \xi, \eta, \overline{F}) = \sum_{j=2}^{\infty} g_j(f, \xi, \eta, \hat{\overline{F}}_m) \overline{f_m}^j.$$

(ii) If (i) does not hold, we then let n be the smallest integer such that for each j, in the following expansion with respect to $\overline{f}_m, \ldots, \overline{f}_{n-1}$:

$$g_j = \sum_{\alpha} \phi_{j,\alpha}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n-1}) \overline{f_m^{\alpha_1}} \cdots \overline{f_{n-1}^{\alpha_{n-m-1}}},$$

it holds that $L_l \phi_{j,\alpha}(z, \overline{z}) \equiv 0$ on $(0 \in) U' \subset M_1$ for all l and $(n-m-1)$-degree multi-index α. But, for the expansion of some ϕ_{j_0,α_0} with respect to $\overline{f_n}$, there exist some l_0 and i_0 so that $L_{l_0} \phi_{j_0,i_0,\alpha_0}(z, \overline{z}) \not\equiv 0$ on any small neighborhood of 0, where

$$\phi_{j_0,\alpha_0}(f, \xi, \eta, \hat{\overline{F}}_{m,\ldots,n}) = \sum_i \phi_{j_0,i,\alpha_0}(f, \xi, \eta, \hat{\overline{F}}_{m,\ldots,n}) \overline{f}_n^i.$$

Lemma 2.4: Let n as above. There is an algebraic function Φ and an open neighborhood Ω^* of $p \in M_1 \cap \Omega$ in $\mathbf{C}^m$ such that for any $(z, \overline{\omega}) \in (\Omega^* \times \mathrm{Conj}(\Omega^*)) \cap M_{1c}$ it holds that

$$\overline{f_n}(\omega) = \Phi(f(z), f^{(1)}(z, \omega), f^{(2)}(z, \omega), \hat{\overline{F}}_n(\omega)). \qquad (2.2.6)$$

Here the $f^{(j)}$'s are certain type of derivatives of f. This notation will be explained below.

Proof of Lemma 2.4: We first assume that $n = m$. Then for some j_0, l_0, $p_0 \approx 0$, and the e^{th} element $g_{j_0}^{\text{e}}$ of the vector function g_{j_0}, it holds that

$$L_{l_0} g_{j_0}^{\text{e}}(f(p_0), \xi(p_0, p_0), \eta(p_0, p_0), \hat{\overline{F}}_m(p_0)) \neq 0.$$

We write

$$f^{(1)} = (L_1 f_1, L_1 f_2, \ldots, L_1 f_{m+k}, \cdots, L_{m-1} f_{m+k}),$$

$$f^{(2)} = (L_1 f^{(1)}, L_2 f^{(1)}, \ldots, L_{m-1} f^{(1)}),$$

$$f^{(3)} = (L_1 f^{(2)}, L_2 f^{(2)}, \ldots, L_{m-1} f^{(2)}),$$

$$\cdots.$$

By the way that ξ and η were constructed, we see that they are rational functions of $f^{(1)}$. Hence, it follows that H_α and H^* are algebraic in $(f, f^{(1)}, \hat{\overline{F}}_m)$ for each α. Define

$$A(f, f^{(1)}, \overline{F}) = H_0^e(f, \xi, \eta) + \sum_{\|\alpha\|=1} \overline{f_{\alpha^*}} H_\alpha^e(f, \xi, \eta, \overline{F}) + H^{*e}(f, \xi, \eta, \overline{F}),$$

where H_α^{e} and $H^{*\ \text{e}}$ are the e^{th} elements of H_α and H^*, respectively. By (3) and (4) of Lemma 2.1, we obtain the algebraicity of $L_{l_0} A(f, f^{(1)}, \hat{\overline{F}}_m)$(for simplicity, we denote it by $A^{(1)}(f, f^{(1)}, f^{(2)}, \hat{\overline{F}})$) in $(f, f^{(1)}, f^{(2)}, \hat{\overline{F}})$. This is so because

$$A^{(1)}(f, f^{(1)}, f^{(2)}, \hat{\overline{F}}) = \sum_j \frac{\partial A}{\partial w_j} L_{j_0} f_j + \sum_{i,j} \frac{\partial A}{\partial w_{i,j}^{(1)}} L_{j_0}(L_i f_j),$$

where we identify the variable $w_{i,j}^{(1)}$ with $L_i f_j$ in $f^{(1)}$. Meanwhile, by (2.2.4) and the definition of g_j, we notice that

$$A^{(1)}(f, f^{(1)}, f^{(2)}, \hat{\overline{F}}) = \sum_j L_{l_0} g_j^{\text{e}} \overline{f_m}^j.$$

Thus, by the choice of p_0, we see that

$$A^{(1)}(f(p_0), f^{(1)}(p_0, p_0), f^{(2)}(p_0, p_0), \hat{\overline{F_m}}(p_0), y_m) \not\equiv 0$$

for $y_m \approx \overline{f}_m(p_0)$.

The proof of Lemma 2.4 in this case therefore follows from Claim 2 with $I = A^{(1)}$:

Claim 2: Let $p \in U$ and let $I(\overline{z}, w, w^{(1)}, \cdots, w^{(i)}, \hat{Y}_j, y_j)$ be holomorphically algebraic on $O = O_{\overline{z}}(\overline{p}) \times O_w(f(p)) \times \cdots \times O_{y_j}(\overline{f_j}(p))$. Suppose that $(\overline{z}, f(z), \cdots, \hat{\overline{F}}_j(z), \overline{f_j}(z))$ is a zero point of I for every $z \in \Omega \cap U$ and suppose that $I(\overline{p}, f(p), \cdots, \hat{\overline{F_j}}(p), y_j) \not\equiv 0$ for $y_j \approx \overline{f_j}(p)$. Then there exists an open subset U' of U so that, for $(z, \overline{\omega})(\approx U') \in M_{1c}$, it holds that

$$\overline{f_j}(\omega) = \Phi(\omega, f(z), f^{(1)}(z, \omega), \ldots, f^{(i)}(z, \omega), \hat{\overline{F}}_j(\omega))$$

for some algebraic holomorphic function Φ.

Proof of Claim 2: From the given hypothesis, the Weierstrass preparation theorem, and (7) of Lemma 2.1, it follows that the equation $I(\overline{z}, w, w^{(1)}, \cdots, y_j) = 0$ is locally equivalent to the following algebraic equation:

$$(y_j - y_0)^{n^*} + \sum_{j=0}^{n^*-1} \lambda_j(\overline{z}, w, w^{(1)}, \cdots, \hat{Y}_j)(y_j - y_0)^j = 0, \tag{2.2.7}$$

with $y_0 = \overline{f_j}(p)$. Let D_1 be the variety associated with (2.2.7), defined by

$$n^*(y_j - y_0)^{n^*-1} + \sum_{j=1}^{n^*-1} j\lambda_j(\overline{z}, w, w^{(1)}, \cdots, \hat{Y}_j)(y_j - y_0)^{j-1} = 0. \tag{2.2.8}$$

If for $z \approx p$, the vector $(\overline{z}, f(z), f^{(1)}(z, z), \cdots, \hat{\overline{F}}_j(z), \overline{f_j}(z))$ also satisfies (2.2.8), whose degree, with respect to y_j, is smaller than that of (2.2.7), we then pass to the study of the variety associated with D_1. Otherwise, by the implicit function theorem, (2.7) tells us, for $z \approx p'(\approx p)$, that

$$\overline{f_j(z)} = \Phi(\overline{z}, f(z), f^{(1)}, \cdots, \hat{\overline{F}}_j(z)), \tag{2.2.9}$$

for some algebraic, holomorphic function $\Phi(\overline{z}, w, w^{(1)}, \cdots, \hat{Y}_j)$ on

$$O_{\overline{z}}(\overline{p'}) \times O_w(f(p')) \times \cdots \times O_{\hat{Y}_j}(\hat{\overline{F_j}}(p'))$$

(here, we need to apply (6) of Lemma 2.1 to obtain the algebraicity of Φ). Complexifying (2.2.9) and noting again the maximal total reality of $U' \times \mathrm{Conj}(U')$ in M_1c, we thus obtain

$$\overline{f_j(\omega)} = \Phi(\omega, f(z), f^{(1)}(z,\omega), \cdots, \hat{\overline{F}}_j(\omega)),$$

for $(z,\omega)(\approx (p',p')) \in M_{1c}$. We now use an inductive argument with respect to n^* and notice that (2.2.8) will eventually reduce to the equation: $n^*(y_j - y_0) + \lambda_{n^*-1} = 0$. We then conclude the existence of the Φ in the claim. This completes the proof of Claim 1. ■

Now let $n > m$. We then have, for each l and α, that

$$L_l \phi_{j,\alpha}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n-1}) \equiv 0 \text{ on some } U_1 \subset U,$$

where $\phi_{j,\alpha}$ is defined as before and

$$\phi_{j,\alpha} = \sum_{i=0}^{\infty} \phi_{j,i,\alpha}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n}) \overline{f_n^i}. \tag{2.2.10}$$

But for some $p_0^* \approx 0$, α_0, i_0 j_0, and l_0, it holds that

$$L_{l_0} \phi_{j_0,i_0,\alpha_0}(p_0^*) \neq 0$$

Let $L_{l_0}\phi_{j_0,\alpha_0} = \psi_{j_0,\alpha_0}(f, f^{(1)}, f^{(2)}, \overline{f}_{n+1}, \ldots, \overline{f_{m+k-1}}, \overline{f_n})$. We claim that ψ_{j_0,α_0} is algebraic in $(f, f^{(1)}, f^{(2)}, \cdots, \overline{f_{m+k-1}})$. In fact, since ϕ_{j_0,α_0} is the Taylor coefficient of the algebraic function g_{j_0}, we can thus see the algebraicity of ϕ_{j_0,α_0}, by Taylor's formula and by inductively using (3) of Lemma 2.1. Again from Lemma 2.1, we determine the algebraicity of ψ_{j_0,α_0} (see the argument for the algebraicity of $A^{(1)}$). Now it is easy to check that Claim 2 can be applied to the equation $\psi_{j_0,\alpha_0}(w, w^{(1)}, w^{(2)}, y_{n+1}, \ldots, y_{m+k-1}, y_n) = 0$ for solving $\overline{f}_n$. So the proof of Lemma 2.4 is complete. ■

An immediate consequence of Lemma 2.4 is that, in case the codimension $k = 1$, then f_m is algebraic. The reason for this is similar to the proof of Lemma 2.2. In fact, let $z_0 \approx U'$ and $z \in Q_{z_0}$. Since $z_0 \in Q_z$, we see by Lemma 2.4 that

$$f_n(z) = \overline{\Phi(z_0, f(z_0), f^{(1)}(z_0,z), f^{(2)}(z_0,z))}.$$

Notice that $f^{(j)}(z_0, z)$ is algebraic in $\overline{z}$ and Φ is algebraic in its variables. We thus conclude the algebraicity of f_m along each Q_{z_0}. From Lemma 2.3, we may then conclude the global algebraicity of f_m.

Now, returning to (2.2.4) with $\alpha = 1$, we get

$$F_0(\omega) = -\overline{H_0(f(z), \xi(z, \omega), \eta(z, \omega))}$$

$$-f_m(\omega)\overline{H_1(f(z), \xi(z, \omega), \eta(z, \omega))} - \overline{H^*(f(z), \xi, \eta, \overline{f_m(\omega)})},$$

where $z \in Q_\omega$. Notice the algebraicity of $f_m(\omega)$, $\overline{\xi(z, \omega)}$, and $\overline{\eta(z, \omega)}$ with respect to the variables ω and the algebraicity of H_0, H_1, and H^* with respect to their own variables. From the above argument, we therefore also see the algebraicity of $F_0(z)$ along each Q_{z_0} $(z_0 \approx U)$. Thus $F_0(z) = (f_1(z), \ldots, f_{m-1}(z))$ is algebraic in z. By the same token, we can prove the algebraicity of f_{m+1} by using the equality (2.1.1) and the results just obtained. So we have

Lemma 2.5: When $k = 1$, then f is algebraic in case (AA).

For the general codimension $k(> 1)$, we have

Lemma 2.6: Under the assumptions in Lemma 2.4, there exist a small open subset U'' of U' and an algebraic holomorphic function Ψ so that

$$f_n(z) = \Psi(z, \hat{F}_n(z)) \text{ for } z \in U'', \tag{2.2.11}$$

where n is as in Lemma 2.4.

The proof of Lemma 2.6 follows easily from the following slightly more general assertion:

Claim 3: Let $p \in U \subset M_1$ and Ψ^* an algebraic holomorphic function on $O_{\overline{z}}(\overline{p}) \times O_w(f(p)) \times \ldots \times O_{w^{(k')}}(f^{(k')}(p,p)) \times O_{\hat{Y}_j}(\hat{\overline{F}}_j(p))$ $(j > m-1)$ so that for some $i \geq 1$, it holds that

$$\overline{f_i(z)} = \Psi^*(\overline{z}, f(z), f^{(1)}(z,z), \cdots, f^{(k')}(z,z), \hat{\overline{F}}_j(z)), \quad z(\approx p) \in U. \tag{2.2.12}$$

Then there is a holomorphically algebraic function Ψ, defined on $O_z(p^*) \times O_{\hat{F}_j(p^*)}$ with $p^*(\approx p) \in U$, such that

$$f_i(z) = \Psi(z, \hat{F}_j(z)), \quad \text{for } z(\approx p^*) \in U.$$

Proof of Claim 3: We proceed by induction on the number of the variables $\overline{f_l}$'s in the formula of Ψ^*. First, if Ψ^* involves no $\overline{f_l}$ terms ($l \geq m$), then Claim 3 follows immediately from the argument presented to prove Lemma 2.5 (in this situation, the complexification of (2.2.12) is:

$$\overline{f_i(\omega)} = \Psi^*(\overline{\omega}, f(z), f^{(1)}(z,\omega), \cdots, f^{(k')}(z,\omega)), \quad z \in Q_\omega).$$

In the general case, to simplify the notation, we let $j = m$ and expand Ψ^* as follows:

$$\Psi^*(\overline{z}, f(z), f^{(1)}(z,z), \ldots, \hat{\overline{F}}_m(z)) = \phi_0(\overline{z}, f(z), \ldots, f^{(k')}(z,z))$$

$$+ \sum_{\|\alpha\|=1, \alpha^* \neq m} \phi_\alpha(\overline{z}, f, \ldots, f^{(k')})(\overline{f_{\alpha^*}}(z) - \overline{f_{\alpha^*}}(p)) + \phi^*(\overline{z}, \ldots, \hat{\overline{F}}_m),$$

where

$$\phi^* = \sum_{\|\alpha\| \geq 2} \phi_\alpha(\overline{z}, f_1, \ldots, f^{(k')})(\hat{\overline{F}}_m(z) - \hat{\overline{F}}_m(p))^\alpha.$$

(1): In case $L_l \phi_\alpha(z) \equiv 0$ for all α, l, and $z(\approx p) \in U$, we may complete the proof of the claim by applying Lemma 2.2 in the following way:

Since $L_l \phi_\alpha(\overline{\omega}, f(z), f^{(1)}(z,\omega), \ldots, f^{(k')}(z,\omega))$ is holomorphic in $(z, \overline{\omega})$, by the observation which we made before the proof of Lemma 2.2, we know that the given hypothesis implies that

$$\phi_\alpha(\overline{\omega}, f(z), f^{(1)}(z,\omega), \ldots, f^{(k')}(z,\omega))$$

is constant along each Segre surface Q_ω. Set

$$\widetilde{\Psi}(z, \omega, \hat{Y}_m) = \overline{\phi_0(\overline{\omega}, f(z), f^{(1)}(z,\omega), \ldots, f^{(k')}(z,\omega))}$$

$$+ \overline{\sum_{\|\alpha\|=1, \alpha^* \neq m} \phi_\alpha(\overline{\omega}, f(z), \ldots, f^{(k')}(z,\omega))}(y_{\alpha^*} - f_{\alpha^*}(p)) + \overline{\phi^*(\overline{\omega}, \ldots, \hat{Y}_m)},$$

where $z \in Q_\omega$. Thus we similarly see that $\Psi(z, \hat{Y}_m) = \widetilde{\Psi}(z^*, z, \hat{Y}_m)$ with $z^* \in Q_z$ is well defined. Moreover, the same argument as in Lemma 2.2 shows that Ψ is holomorphically algebraic in its variables. So, the proof of Claim 3 in this case is complete; for it obviously holds that $f_i(z) = \Psi(z, \hat{F}_m(z))$.

(2): Now, we assume that (1) does not occur. We then define a natural number n' in a fashion similar to that used for n (the existence of such an n' can also be seen from the hypotheses):

(a): If for some $p'(\approx p)$, α_0, and l_0, it holds that $L_{l_0}(\psi_{\alpha_0}(p')) \neq 0$, we then let $n' = m + 1$. Here

$$\psi_0 = \phi_0 + \sum_{\|\alpha\|=1, \alpha^* \neq m, m+1} \phi_\alpha \left(\overline{f_{\alpha^*}(z)} - \overline{f_{\alpha^*}(p)} \right),$$

$\psi_1 = H_{(0,1,0,\ldots,0)}$, and ψ_j for $j > 1$ are determined by

$$\phi^*(\overline{z}, f, \ldots, \hat{\overline{F}}_m) = \sum_{j \geq 2} \psi_j(\overline{z}, \ldots, f^{(k')}) \left(\overline{f_{m+1}(z)} - \overline{f_{m+1}(p)} \right)^j .$$

(b): When (a) does not hold, we let n' be the smallest integer such that: for each j, in the expansion of ψ_j with respect to $(\overline{f_{m+1}}(z) - \overline{f_{m+1}(p)}), \ldots, (\overline{f_{n'-1}}(z) - \overline{f_{n'-1}(p)})$, all coefficients are annihilated by the operators $\{L_l\}$; but at least for one coefficient of certain ψ_{j_0}, say $b_{j_0}(\overline{z}, f, \ldots, \hat{\overline{F}}_{m,\cdots,n'-1})$, there exist some l_0, i_0 with $L_{l_0} b_{j_0,i_0}(z) \not\equiv 0$ on a small neighborhood p in U. Here

$$b_{j_0}(\overline{z}, f, \ldots, \hat{\overline{F}}_{m,\cdots,n'-1}) = \sum_i b_{j_0,i}(\overline{z}, f, \ldots, \hat{\overline{F}}_{m,\cdots,n'})(\overline{f_{n'}}(z) - \overline{f_{n'}(p)})^i.$$

We now apply the argument in Lemma 2.4 with g_j 's there being replaced by the ϕ_j' (in case (a)), or with ϕ_{j_0,α_0} and ϕ_{j_0,i,α_0} being replaced by b_{j_0} and $b_{j_0,i}$, respectively (in case (b)). We then obtain an algebraic holomorphic function Ψ^{**} so that

$$\text{(2.2.12)}' \qquad \overline{f_{n'}(z)} = \Psi^{**}(\overline{z}, f(z), \cdots, f^{(k')}(z, z), \hat{\overline{F}}_j(z)),$$

where z is in a small open subset U of p. Substitute $(2.2.12)'$ into the $\overline{f_{n'}}$ variable in the formula of Ψ^*. Since the number of $\overline{f_l}$'s is now decreased by 1,we can thus conclude the proof of Claim 3 by the induction hypothesis. ■

Replace $\overline{f_n}(z)$ in (2.2.4) by $\overline{\Psi(z, \hat{F}_n(z))}$ obtained in Lemma 2.6. Then we have, for each $i \leq m-1$, that

$$\overline{f}_i(z) = g_i^*(\overline{z}, f, f^{(1)}, \hat{\overline{F}}_n) \text{ on } U^{(2)} \subset U'', \tag{2.2.13}$$

where g_i^* is holomorphic and algebraic on $O_{\overline{z}}(\overline{p''}) \times O_w(f(p'')) \times \cdots \times O_{\hat{Y}_j}(\hat{\overline{F_j}}(p''))$ with p'' being some point in U''. From Claim 3, it follows that on some $U^{(3)} \subset U''$, there exist algebraic holomorphic functions $\{\Psi_1, \cdots, \Psi_{m-1}\}$ so that it holds for each $i \leq m-1$ that

$$f_i(z) = \Psi_i(z, \hat{F}_n(z)) \ \text{ for } z \in U^{(3)}. \tag{2.2.14}$$

Similarly, by substituting (2.2.11) and (2.2.14) to (2.2.1), we obtain

$$\overline{f}_{m+k}(z) = g_{m+k}^*(\overline{z}, f(z), \hat{\overline{F}}_n(z)) \ \text{ on } U^{(4)} \subset U^{(3)}, \tag{2.2.15}$$

with g_{m+k}^* holomorphic and algebraic in $(f, \overline{z}, \hat{\overline{F}}_n)$. Thus it can be seen, after shrinking $U^{(4)}$, that we have

$$f_{m+k}(z) = \Psi_{m+k}(z, \hat{F}_n(z)) \ \text{ on } U^{(4)}, \tag{2.2.16}$$

for some algebraic holomorphic function Ψ_{m+k}. Combining all these formulas, we now obtain:

Lemma 2.7: There are a small neighborhood $\Omega^* \subset \mathbf{C}^m$ of some $p \in U^{(4)}$ and a nonsingular algebraic complex variety $M^* \subset \mathbf{C}^{m+k}$, which contains $f(p)$, so that $f(\Omega^*) \subset M^*$.

Proof of Lemma 2.7: Let Ψ and n be as in Lemma 2.6. Consider the equation $w_n = \Psi(z, \hat{w_n^*})$, where $w = (w_*, w^*, w_{m+k})$ with $w_* = (w_1, \ldots, w_{m-1})$ and

$$w^* = (w_m, \ldots, w_{m+k-1}).$$

Set $\chi_1(z, w) = w_n - \Psi(z, \hat{w_n^*})$. If χ_1 does not involve any z terms, then we define $M^* \subset \mathbf{C}^{m+k}$ by the equation $w_n = \Psi(p, \hat{w_n^*})$. Obviously, M^* is a regular algebraic manifold near $f(p)$ and $f(U^{(5)}) \subset M_2$, where $U^{(5)} \subset U^{(4)} \subset U''$ is a small neighborhood of p in M_1 (see Lemma 2.6). Since $U^{(5)}$ is a set of uniqueness for the holomorphic function f, it follows that $f(\Omega^*) \subset M^*$ for some small neighborhood of $U^{(5)}$ in $\mathbf{C}^m$. So, without loss of generality, we assume that the Taylor expansion of Ψ at $(p, \hat{F}_n(p))$ does have z terms. After a rotation around p in $\mathbf{C}^m$ (if necessary), we may assume that $\frac{\partial^j \Psi}{\partial z_1^j}(p) \neq 0$ for some $j \geq 1$. Notice that $\chi_1(p, f(p)) = 0$. By the Weierstrass preparation theorem, the equation $\chi_1 = 0$ is therefore equivalent to

$$(2.2.17) \qquad (z_1 - p^1)^{n^*} + \sum_{j=0}^{n^*-1} a_j(w^*, z_2, \cdots, z_m)(z_1 - p^1)^j = 0$$

with $n^* \geq 1$, where $p = (p^1, \cdots, p^m)$ and a_j $'s$ are algebraic. Arguing as in Claim 2, we can conclude that $z_1 = \chi_1^*(z_2, \ldots, z_m, F(z))$ for $z(\in U^{(5)}) \approx p^*$ (here we may have to shrink $U^{(5)}$). Now, substituting this into (2.2.14), we obtain, for $i < m$ that

$$(2.2.17)' \qquad f_i(z) = \Psi_i^{(1)}(z_2, \cdots, z_m, F(z)) \text{ for } z \in U^{(5)},$$

where $\Psi_i^{(1)} = \Psi_i(\chi_1^*(z_2, \cdots, z_m, w^*), z_2, \cdots, z_n, \hat{w^*}_n)$. Consider in particular the equation:

$$\chi_2(z_2, \ldots, z_m, w) = w_1 - \Psi_1^{(1)}(z_2, \cdots, z_m, w^*) = 0.$$

By the same token, if the above equation is independent of $(z_2, \ldots, z_m)$, then the M^* in the lemma can be defined by $w_1 = \Psi_1^{(1)}(p_2^*, \cdots, p_m^*, w^*)$, where $(p_1^*, \cdots, p_m^*)$ is a fixed point in $U^{(5)}$. Otherwise, after a rotation at $(p_2^*, \cdots, p_m^*)$ with respect to the variables $(z_2, \cdots, z_m)$, we can also assume that $\frac{\partial^j \Psi_1^{(1)}}{\partial z_2^j}(p) \neq 0$ for some $j \geq 1$. Then it follows similarly that there exists an algebraic holomorphic function $\chi_2^*(z_3, \cdots, z_m, w_1, w^*)$ with $z_2 = \chi_2^*(z_3, \cdots, z_m, f_1(z), F(z))$ for z in a small open subset of $U^{(5)}$. Now, substitute this again into $(2.2.17)'$ and consider the equation:

$$(2.2.17)'' \qquad w_2 - \Psi_2^{(2)}(z_3, \cdots, z_m, w_1, w^*) = 0,$$

where $\Psi_2^{(2)} = \Psi_2^{(1)}(\chi_2^*(z_3, \cdots, z_m, w_1, w^*), z_3, \cdots, w^*)$. Repeating what we just did, we see that either we complete the proof of the lemma, or we can solve $(2.2.17)''$ to determine that $z_3 = \chi_3^*(z_4, \cdots, z_m, w_1, w_2, w^*)$ with

$$f_3(z) = \chi_3^*(z_4, \cdots, z_m, f_1(z), f_2(z), F(z))$$

for z in a small subset of $U^{(5)}$. Arguing inductively in this way, we then either come up with the proof of Lemma 2.7, or we obtain algebraic holomorphic functions

$$\chi_j^*(z_{j+1}, \cdots, z_m, w_1, w_2, \cdots, w_{j-1}, w^*) \ (j = 1, \ldots, m)$$

so that $z_j = \chi_j^*(z_{j+1}, \cdots, f_1(z), \cdots, f_{j-1}(z), F(z))$ for z on a small open subset of $U^{(5)}$. Here we understand w_0 and z_{m+1} as 0. In the latter case, we can easily obtain an algebraic vector function $\chi(w_*, w^*)$ with $z = \chi(F_0(z), F(z))$ for z in a small open subset of $U^{(5)}$. Meanwhile, combining this equality with (2.2.16), we also see that the algebraic manifold M^*, defined by the equation $w_{m+k} = \Psi_{m+k}(\chi(w_*, w^*), \hat{w}_n^*)$, completes the proof of the lemma. ∎

Lemma 2.8: Let M^* be as in Lemma 2.7. Then $M^* \cap M_2$ is an algebraic strongly pseudoconvex hypersurface of M_2.

Proof of Lemma 2.8: Through a linear change of variables, we may assume that $p = 0$ and that the complex tangent space of M_2 is defined by $w_{m+k} = 0$. Since $f(\Omega^*) \subset M^*$ and since $f(\Omega^*)$ is transversal to $T_0^{(1,0)} M_2$ (see §1.1.2), it follows that $T_0^{(1,0)} M^* \neq \{w_{m+k} = 0\}$. Thus, by the implicit function theorem, we see that M^* can be locally expressed by the equation: $w_l = \phi(\hat{w}_l)$ for some $l \neq m+k$. Now it is easy to see that $\rho_2^* = \rho_2(w_1, \cdots, \phi(\hat{w}_l), \cdots)$ is a non degenerate real algebraic defining function of $M^* \cap M_1$, which is obviously strongly plurisubharmonic at 0. ∎

§**2.2.4**: We are now in a position to study the main theorem in case (BB). We will either reduce to the situation (AA) or obtain the algebraicity of f.

By Lemma 2.2, we have an algebraic function Φ so that

$$F_0 = \Phi(z, F) \tag{2.2.18}$$

for $z \approx 0$ (we remark that $\frac{\partial \Phi}{\partial F}(0,0) = 0$). Substituting this equation into (2.2.1) (here we assume that the h in (2.2.1) does not contain any $\overline{f_{m+k}}(\omega)$ term), we obtain

$$f_{m+k}(z) + \overline{f_{m+k}(\omega)} + \sum_{j=1}^{m+k-1} f_j(z)\overline{f_j(\omega)} + h^*(\overline{\omega}, f(z), \overline{F(\omega)}) = 0 \tag{2.2.19}$$

for $z \in Q_\omega$, or, $(z,\omega) \in M_{1c}$ with $h^*(\overline{\omega}, f(z), \overline{F(\omega)}) = h(f(z), \overline{\Phi(\omega, F(\omega))}, \overline{F(\omega)})$. We now repeat what we did at the beginning of this section. Then we obtain also an equation with a form similar to (2.2.2):

$$\xi + \overline{F}_0 + \eta\overline{F} + g(\overline{\omega}, f, \xi, \eta, \overline{F}) = 0, \quad \text{for } (z,\overline{\omega}) \in M_{1c}, \tag{2.2.20}$$

where $g(\overline{\omega}, f, \xi, \eta, \overline{F}) = (\mathrm{id}, \eta, \xi)Dh^*$. Define similarly the new H_α functions in terms of (2.2.20). We also consider the conditions (AA) and (BB), respectively. By §2.2.3, in case (AA) occurs, then either we obtain the algebraicity ($k = 1$) or we can transform the problem to the case of codimension $k - 1$.

Assume that we are still in case (BB). So the new $H'_\alpha s$ are also constant along each Segre surface. We note that

$$H_0 = \xi + (id, \eta, \xi)Dh^*|_{Y=0} \tag{2.2.21}$$

and

$$H_\alpha = \eta_{\alpha^*} + (id, \eta, \xi)\frac{\partial}{\partial y_{\alpha^*}}Dh^*|_{Y=0} \text{ for } \|\alpha\| = 1. \tag{2.2.22}$$

From (2.2.21) and the definition of η and ξ, it follows, for any l, that

$$L_l(\sum_{j=1}^{m-1} H_0^j(z,\omega)f_j) = L_l(f_{m+k}) + L_l h^*,$$

where $H_0 = (H_0^1, \cdots, H_0^{m-1})^t$. Thus if we set

$$E_0 = \sum_{j=1}^{m-1} H_0^j(z,\omega)f_j - f_{m+k} - L_l h^*, \tag{2.2.23}$$

then $E_0(z,\omega)$ is constant on Q_ω for every $\omega \approx 0$. Similarly, (2.2.22) tells that

$$E_j = \sum_{i=1}^{m-1} H^i_{(0,\cdots,1(j^{\text{th}}),\cdots,0)}(z,\omega)f_i - f_j - \frac{\partial h^*}{\partial y_j} \tag{2.2.24}$$

is also constant along Q_ω for $j = m, \cdots, m+k-1$. Notice $H_\alpha(0,0) = 0$. Applying the implicit function theorem to (2.2.18), (2.2.23), and (2.2.24), we then obtain

$$f(z) = G(z, \overline{\omega}, E, H_\alpha), \quad (z,\omega) \in M_{1c},$$

where $E = (E_0, E_m, \cdots, E_{m+k-1})$ and $\|\alpha\| \leq 1$. From Lemma 2.1, it follows that G is also algebraic in its variables. By Lemma 2.3, to show that f is algebraic in z, it suffices for us to prove that $f(z)$ is algebraic along any Q_ω. However, this follows immediately from the fact that E and H are constant along each Segre surface.

§ **2.2.5.** Summarizing all the above discussion, we can conclude the algebraicity of the mapping when $k = 1$. In case $k > 1$, we either obtain the algebraicity of f (see §2.2.4) or we may reduce to a problem with smaller codimension (see §2.2.3 and §2.2.4). Thus, by a simple induction argument, the proof of our main theorem follows.

Remark: From the proof, it is easy to see that *when f is apriori assumed to be an immersion from M_1 to M_2 with the Hopf Lemma being held, then M_1 and M_2 in the main theorem can be relaxed to (Levi-) non-degenerate algebraic hypersurfaces which have the same number ℓ of negative Levi eigenvalues, which we call the signature of M_1 and M_2.* Here, we arrange the number of negative Levi eigenvalues to be no bigger than the number of positive Levi eigenvalues. (However, the main theorem is obviously false if M_1 and M_2 are allowed to be Levi flat). In fact, the only difference from the strongly pseudoconvex case is that when we use a nice coordinate system like (2.1.2) we might have -1 for the eigenvalues of the Levi form. In this case, when we define F_0 and F as in §2.2.1, we need to place a negative sign before a component if the corresponding eigenvalue at that position is negative. Notice that after the complexification, the signs will play no role at all. In our proof, the strong pseudo-convexity (instead of Levi non-degeneracy) is only used to get two things: (a). The Hopf lemma property for the map (the Pinchuk lemma), which is now stated as part of the assumption. (b). The only other place where the strong pseudoconvexity is used is in the proof of Lemma 2.8 which guarantees the induction process would go through. With the assumption that the Hopf lemma holds and the signature of M_1 and M_2 are the same, we can also show that $M^* \cap M_2$ defined as in Lemma 2.8 is a *Levi non-degenerate hypersurface with the same signature ℓ* as follows:

Lemma 2.8′: Assume that V is a smooth complex analytic hypersurface in a neighborhood of 0 in $\mathbf{C}^{n+1}$. Assume that M' is a Levi non-degenerate hypersurface of signature $\ell > 0$ at 0 and $T_0^{(1,0)}M' \neq T_0^{(1,0)}V$. Assume that $M' \cap V$ contains a Levi non-degenerate submanifold of hypersurface type with signature ℓ near 0. Then $M' \cap V$ is a Levi non-degenerate hypersurface of signature ℓ in V near 0. (In our application, M is simply taken to be the image of the source manifold under the local (CR) transversal embedding).

Proof of Lemma 2.8′: The case $\ell = 1$ is easy. We assume that $\ell > 1$. Choose a holomorphic coordinate $(z_1, \cdots, z_n, z_{n+1})$ with $z_{n+1} = u + iv$ such that M' is defined near 0 by an equation of the form:

$$v = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n} |z_j|^2 + o(|z|^2).$$

By the transversality of V with M', we can assume that M' near 0 is defined by an equation of the form: $z_{j_0} = \sum_{j=1, j\neq j_0, n+1}^{n+1} a_j z_j + O(z_{n+1}) + o(z)$ with $j_0 \neq n+1$. We argue in two steps:

Step 1. We assume that $j_0 < \ell$. For simplicity of notation, we can assume that $j_0 = 1$. We first find an $(\ell-1)\times(\ell-1)$ unitary matrix U_I such that $(a_2, \cdots, a_\ell)\overline{U_I} = (\alpha, 0, \cdots, 0)$ with $\alpha \geq 0$. We also find an $(n-\ell)\times(n-\ell)$ unitary matrix U_{II} such that $(a_{\ell+1}, \cdots, a_n)\overline{U_{II}} = (\beta, 0, \cdots, 0)$ with $\beta \geq 0$. Define $(z_2', \cdots, z_\ell') = (z_2, \cdots, z_\ell)\cdot U$, $(z_{\ell+1}', \cdots, z_n') = (z_\ell, \cdots, z_n)\cdot U$ and $z_{n+1}' = z_{n+1}$. Then $M' \cap V$ is defined in the $(z_2' \cdots, z_{n+1}')$-coordinates by an equation of the following form:

$$v' = -|\alpha z_2' + \beta z_{\ell+1}'|^2 - \sum_{j=2}^{\ell} |z_j'|^2 + \sum_{j=\ell+1}^{n} |z_j'|^2 + o(|z'|^2).$$

Now consider the quadric form: $-|\alpha z_2' + \beta z_{\ell+1}'|^2 - |z_2'|^2 + |z_{\ell+1}'|^2$, whose corresponding Hermitian metric is as follows:

$$A = \begin{pmatrix} -1-|\alpha|^2 & -\alpha\beta \\ -\alpha\beta & 1-|\beta|^2 cr \end{pmatrix}$$

Now. By the assumption, it should have exactly two negative eigenvalues. Hence $M' \cap V$ is a Levi non-degenerate hypersurface in the $(z_2' \cdots, z_{n+1}')$-coordinates with signature ℓ, for it has exactly ℓ-negative Levi eigenvalues and $(n-\ell-1)$-positive Levi eigenvalues.

Step 2: $j_0 > \ell$. For simplicity, let us assume that $j_0 = \ell + 1$.

We first find a $\ell \times \ell$ unitary matrix U_I such that $(a_1, \cdots, a_\ell)\overline{U_I} = (\alpha, 0, \cdots, 0)$ with $\alpha \geq 0$. We also find a $(n-\ell-1) \times (n-\ell-1)$ unitary matrix U_{II} such that

$$(a_{\ell+2}, \cdots, a_n)\overline{U_{II}} = (\beta, 0, \cdots, 0)$$

with $\beta \geq 0$. Define $(z_1', \cdots, z_\ell') = (z_1, \cdots, z_\ell) \cdot U$, $(z_{\ell+2}', \cdots, z_n') = (z_\ell + 2, \cdots, z_n) \cdot U$ and $z_{n+1}' = z_{n+1}$. Then $M' \cap V$ is defined in the $(z_1' \cdots, z_\ell', z_{\ell+2}', \cdots, z_{n+1}')$-coordinates by an equation of the following form:

$$v' = -\sum_{j=1}^{\ell} |z_j'|^2 + |\alpha z_1' + \beta z_{\ell+2}'|^2 + \sum_{j=\ell+2}^{n} |z_j'|^2 + o(|z'|^2).$$

Now consider the quadric $|\alpha z_1' + \beta z_{\ell+2}'|^2 - |z_1'|^2 + |z_{\ell+2}'|^2$, whose corresponding Hermitian metric is as follows:

$$A = \begin{pmatrix} |\alpha|^2 - 1 & \alpha\beta \\ \alpha\beta & 1 + |\beta|^2 \end{pmatrix}$$

By the assumption, we must have at least one negative eigenvalue from A. Since $Tr(A) = |\alpha|^2 + |\beta|^2 > 0$. It must also have a positive eigenvalue. Thus the Levi form of $M' \cap V$ at 0 has exactly ℓ-negative eigenvalues and $(n-\ell-1)$-positive eigenvalues, namely, $M' \cap V$ is Levi non-degenerate. This completes the proof of the Lemma. ∎

To see that the degree of the mapping is bounded by a constant depending only on the degree of M_1 and M_2, we just notice that the bound of the degree of the resulting functions in the operations of Lemma 2.1 depends only on the degree of the known functions. Here we recall that the degree of an algebraic function f is defined as the degree of the irreducible polynomial which annihilates f. According to these observations and the proof of Corollary 1 in the next section, we may state the following:

Corollary: Let M_1 and M_2 be two algebraic hypesurfaces in (possibly different) complex spaces of dimension at least 2. Assume that M_1 is minimal (thus, any CR function defined on M_1 can be extended to certain side of M_1 at each point). Suppose that f is a smooth CR mapping from M_1 into M_2 so that at least there is a non degenerate point $p \in M_1$ with signature $\ell \geq 0$ where f is an immersion with the Hopf lemma property being held, and $f(p)$ is also a non degenerate point in M_2 with the same signature. Then f is the restriction of some holomorphic algebraic map to M_1. Moreover, the vanishing order of each non-zero component of f is bounded by a constant depending only on the degree of M_1 and M_2.

§ **2.3 Proof of Theorem 2**:

The purpose of this section is to prove Theorem 2 by modifying the previous argument.

Theorem 2: Every C^{k+1}-CR mapping from a strongly pseudoconvex real analytic hypersurface $M_1 \subset \mathbf{C}^n$ $(n > 1)$ into another strongly pseudoconvex real analytic hypersurface $M_2 \subset \mathbf{C}^{n+k}$ is real analytic (C^ω) on a dense open subset.

We first notice that, by Lewy's extension result, we may assume that the map f in Theorem 2 can be extended holomorphically to the pseudoconvex side. We still start with the equation $\rho_2(f(z), \overline{f(z)}) = \lambda(z, \overline{z})\rho_1(z, \overline{z})$. Since we do not know the existence of the complexification in the present setting, we will differentiate the equation along M_1. Then we will come up with a new equation similar to (2.2.3), which also enables us to divide the discussions according to how degenerate the map f is: In a sort of the totally degenerate case (analogous to (BB)), we will reduce the analytic extendibility to the analytic hypoellipticity of a differential equation by making use of the CR-extension results. In the other situations, we will similarly obtain the analyticity of f (in case k=1) or get a reduction with respect to the codimension.

For the sake of brevity, we retain most of the notation in §2.2.

§**2.3.1** We now let M_1, M_2, and L_l as in §1.2. To prove Theorem 2, we proceed by seeking a point $q \in U$, where U is an arbitrarily fixed small neighborhood of 0 in M_1, so that f has an analytic extension at q.

By the properness of f (i.e., the fact: $f(M_1) \subset M_2$), we see that

$$(2.3.1)\quad f_{m+k}(z) + \overline{f_{m+k}(z)} + \sum_{j=1}^{m+k-1} |f_j(z)|^2 + h(f(z), \overline{f(z)}) = 0 \ \text{ for } z \in U \subset M_1.$$

As we did in §2.2.1, by using the implicit function theorem, we can assume that $h(f, \overline{f}) = h(f, \overline{f_1}, \ldots \overline{f_{m+k-1}})$, where $h(w, y_1, \ldots, y_{m+k-1})$ is a holomorphic function on $O_w(0) \times O_{y_1}(0) \times \ldots \times O_{y_{m+k-1}}(0)$.

Applying L_l to (2.3.1) for each l, we obtain

$$(2.3.2)\qquad L_l f_{m+k}(z) + \sum_{j=1}^{m+k-1} L_l f_j(z)\overline{f_j(z)} + \sum_{j=1}^{m+k} \frac{\partial h}{\partial w_j} L_l f_j(z) = 0, \quad \text{for } z \in U.$$

Let V, ξ, and η as defined in §2.1, except replacing $\overline{\omega}$ by $\overline{z}$. Equation (2.3.2) can then be written as

$$\xi(z) + \overline{F_0(z)} + \eta(z)\overline{F}(z) + (\mathrm{id}, \eta(z), \xi(z))Dh(z) = 0 \ \text{ for } z \in U,$$

where $Dh(z) = (\frac{\partial h}{\partial w_1}, \ldots, \frac{\partial h}{\partial w_{m+k}})^t(z) = O(\|z\|^3)$ as $z \to 0$, $F_0 = (f_1, \ldots, f_{m-1})^t$, and $F = (f_m, \ldots, f_{m+k-1})^t$.

Again, by making use of the implicit function theorem and by shrinking U, we have that

$$(2.3.3)\qquad \xi + \overline{F}_0 + \eta\overline{F} + g(f, \xi, \eta, \overline{F}) = 0 \quad \text{on} \quad U.$$

Here g is holomorphic in its variables and if

$$g(f, \xi, \eta, \overline{F}) = \sum_{\alpha} g_\alpha(f, \xi, \eta)\overline{F^\alpha},$$

then

$$\frac{\partial g_\alpha}{\partial \xi}, \frac{\partial g_\alpha}{\partial \eta} \to 0$$

as $z(\in U) \to 0$ for $\|\alpha\| \le 1$ (by (2.1.2) and (2.1.3)).

We now expand g with respect to f_m:

$$g = \sum_{j=0}^{\infty} g_j(f, \xi, \eta, \overline{\hat{F}}_m)\overline{f_m^j},$$

where $g_j(w, \xi, \eta, y_{m+1}, \cdots, y_{m+k-1})$ is holomorphic on

$$O_1 = O_w(0) \times O_\xi(0) \times O_\eta(0) \times \cdots \times O_{y_{m+k-1}}(0)$$

and there exists a number $R >> 1$ so that

$$|g_j(w, \xi, \eta, \cdots y_{m+k-1})| \le R^j$$

for each j and for every $(w, \xi, \eta, \cdots, y_{m+k-1}) \in O_1$.

Set

$$H_0 = \xi + \hat{\eta}_m \overline{\hat{F}}_m + g_0(f, \xi, \eta, \overline{\hat{F}}_m),$$

$$H_1 = \eta_m + g_1(f, \xi, \eta, \overline{\hat{F}}_m),$$

and

$$H_j = g_j(f, \xi, \eta, \overline{\hat{F}}_m) \quad \text{for } j = 2, \cdots, .$$

Here, as we defined before, we write $\eta = (\eta_m, \ldots, \eta_{m+k-1})$ and

$$\hat{\eta}_m = (\eta_{m+1}, \ldots, \eta_{m+k-1}).$$

Now (2.3.3) reads as

$$H_0 + \overline{F}_0 + \overline{f_m} H_1 + \sum_{j=2}^{\infty} \overline{f_m^j} H_j = 0 \quad \text{on } U_0 \subset U, \tag{2.3.4}$$

where U_0 is a small neighborhood of $0 \in M_1$.

§**2.3.2**: In this subsection, we study a situation similar to (BB) in §2.2.2. We will obtain the analyticity by using CR-extension and partial differential equations results.

From (2.3.4), we now define Ind(1) = 0 if, for each k and j, $L_l(H_j) = 0$ on a small neighborhood $U_1 \subset U$ of 0. Otherwise, we define Ind(1) = 1. In case Ind(1) = 0, we then let, for each j_0,

$$(2.3.5) \qquad H_{j_0}(z) = \sum_{j=0}^{\infty} \phi_{j_0,j}(f, \xi, \eta, \hat{\overline{F}}_{m,m+1})\overline{f}_{m+1}^{j}.$$

Applying L_l to (2.3.5) for each l, we see that

$$(2.3.6) \qquad \sum_j L_l(\phi_{j_0,j}(f, \xi, \eta, \hat{\overline{F}}_{m,m+1}))\overline{f}_{m+1}^{j} = 0, \quad \text{on } U_1.$$

Define Ind(2) = 0 if $L_l(\phi_{j_0,j}) = 0$ for all l, j_0, and j in a small neighborhood $U_2(\subset U_1)$ of 0; otherwise let Ind(2) = 1.

If it still happens that Ind(2) = 0, we expand ϕ_{j_0,j_1}, for every j_0 and j_1, with respect to $\overline{f}_{m+2}$. Then we can similarly define the value of Ind(3) $\cdots$. Arguing inductively, if it always happens that Ind(j) = 0 for $j = 1, \cdots, k$, we then easily see, for any index i, that

$$H_i(f, \xi, \eta, \hat{\overline{F}}_m) = \sum_{\|\alpha\|=0}^{\infty} h_{i,\alpha}(f, \xi, \eta)\hat{\overline{F}}_m^{\alpha} \quad \text{for } z \in U_k(\text{ a small neighborhood of } 0),$$

where $L_l h_{i,\alpha}(f, \xi, \eta) \equiv 0$ for all indices i, l, α, and $h_{i,\alpha}(w, a, b)$ is holomorphic on $O_2 = O_w(0) \times O_a \times O_b(0)$ with $|h_{i,\alpha}(w, a, b)| < R^{\|\alpha\|}$ for each (k-1)-multi-index α and for some $R >> 1$. Returning to (2.3.4), we then obtain the expansion:

$$(2.3.7) \qquad \overline{F_0} + \sum_{\|\alpha\|=0}^{\infty} H_{\alpha}^{*}\overline{F^{\alpha}} = 0,$$

with $L_l H_{\alpha}^{*} \equiv 0$, for all l and k-multi-index α, on U' (a neighborhood of 0). By the uniqueness of the power series of a holomorphic function and by combining the upper bound (Cauchy) estimates of $h_{i,\alpha}$ with those of g_j, it therefore follows that

$$(2.3.7)' \qquad |H_{\alpha}^{*}(w, \xi, \eta)| \leq R^{\|\alpha\|}$$

for some $R >> 1$ and that

$$H_0^{*} = \xi + g_0(f, \xi, \eta, 0),$$

$$H^*_{(1,0,\ldots,0)} = \eta_m + g_1(f,\xi,\eta,0),$$

and

$$H^*_\alpha = \eta_{\alpha^*} + \frac{\partial g_0}{\partial y_{\alpha^*}}(f,\xi,\eta,0) \tag{2.3.8}$$

where the notation α^* is the same as before.

Lemma 2.9: Under the above circumstances, we have, for each α, a holomorphic extension $A_\alpha(z)$ of $\overline{H}^*_\alpha$ on Ω^* (a small neighborhood of U' in $\mathbf{C}^m$). Moreover, it holds that $\max_{z\in\Omega^*}|A_\alpha(z)| < R^\alpha$.

Proof of Lemma 2.9: First, we note that $\overline{H^*_\alpha}(z)$ is a CR-function on U_k for each α, since $\overline{L_l}(\overline{H^*_\alpha})(z) = 0$ for all l. Consequently, by the Lewy extension theorem, we have, for each $\overline{H^*_\alpha}$, a holomorphic extension $A^+_\alpha(z)$ defined on some open subset $\Omega'(\subset \Omega)$ whose size depends only on U'. Since the analytic discs with their boundaries attached to U_k sweep out an open subset of Ω, so after shrinking Ω', the maximal principle then implies that $\max_{\Omega'}|A^+_\alpha| = \max_{U'}|\overline{H^*_\alpha}| < R^\alpha$ (by (2.3.7)$'$).

Now, let ϕ: $V \subset \mathbf{C}^1 \to \mathbf{C}^m$ be an embedding such that $\phi(V\cap\Delta) \subset \Omega'$, $\phi(1) \approx 0$, $\phi(\partial(\Delta\cap V)) \subset M_1$, and $\phi(\Delta^c \cap V) \subset \mathbf{C}^m - \Omega'$ (where Δ denotes the unit disk in $\mathbf{C}^1$). Since M_1 is real analytic, we can extend $f\circ\phi(\tau)$, $\xi\circ\phi(\tau)$, and $\eta\circ\phi(\tau)$ holomorphically into $\Delta\cap V'$ (where $V' \subset V$ is a small neighborhood of $\partial\Delta\cap V$ and symmetric with respect the unit circle). For $\tau \in \Delta^c\cap V'$, we define

$$A^-_\alpha(\tau) = \overline{H^*_\alpha(f\circ\phi(1/\overline{\tau}), \xi\circ\phi(1/\overline{\tau}), \eta\circ\phi(1/\overline{\tau}))}.$$

Then A^-_α is holomorphic on $\Delta^c\cap V'$ and $A^-_\alpha(\tau) = A^+_\alpha(\tau)$ for $\tau\in\partial\Delta\cap V'$. It thus follows from the Hartogs theorem that A^+_α has a holomorphic extension A_α, on an open subset Ω^* near 0, which does not depend on α. By the construction of A_α and (2.3.7)$'$, it obviously holds that $\max_{\Omega^*}|A_\alpha| < R^\alpha$ (we may have to shrink Ω^* here). This completes the proof of Lemma 2.9. ∎

Now by (2.3.7), we have that

$$F_0(z) + \sum_\alpha A_\alpha(z)F^\alpha(z) = 0. \tag{2.3.9}$$

Let $J(z, w^*) = -\sum_{\|\alpha\|=0}^{\infty} A_\alpha(z) w^{*\alpha}$. For $(z, w^*) \in \Omega^* \times O_{w^*}(0)$, since

$$\max_{\Omega^*} |A_\alpha(z)| < R^\alpha$$

for some $R >> 1$,we see that $J(z, w^*)$ is holomorphic on $\Omega^* \times O_{w^*}(0)$ (where we may have to shrink the domains). On the other hand, by making use of the formulas in (2.3.8) and the implicit function theorem, we have that

$$\xi = H_0^* + G_0(f, H_0^*, H_\alpha^*)$$

and

$$\eta_{\alpha^*} = H_\alpha^* + G_\alpha(f, H_0^*, H_\alpha^*).$$

Here G_0 and G_α ($\|\alpha\| = 1$)) are holomorphic in their variables and have no linear terms. Applying $\mathbf{L} = (L_1, \cdots, L_{m-1})$ to (2.3.9), we obtain

$$V = \mathbf{L}_z J + V \times (H_0^* + G, H_\alpha^* + G_\alpha) \times \frac{\partial J}{\partial w},$$

where $\|\alpha\| \leq 1$ and $\mathbf{L}_z J$ is the partial differential operator $\mathbf{L}$ applied to J while holding w^* fixed. So it follows easily that

$$V = (\mathbf{L}J)(\mathrm{id} - (H_0^* + G, H_\alpha^* + G_\alpha)\frac{\partial J}{\partial w})^{-1} = G_1^*(z, f),$$

where $G_1^*(z, w)$ is real analytic on $O_z(0) \times O_w(0)$ for H_0^* and H_α^* are real analytic on U_k by Lemma 2.9. Combining this with the formulas for ξ and η, we therefore conclude that $L_i f_j = G_{ij}^{**}(z, f)$ with G_{ij}^{**} real analytic in z and f (when $z \approx 0$).

Let $T = [L_1, \overline{L_1}]$. Then $Tf_j = \overline{L_1}(L_1 f_j) = \overline{L_{1z}}(G_{1j}^{**}) + \sum \frac{\partial G_{1j}^{**}}{\partial \overline{w_l}} \overline{G_{1l}^{**}}$, which is also real analytic on z and f for $j = 1, \cdots, m + k$. Since $\{\mathbf{L}, \overline{\mathbf{L}}, T\}$ consists of a local analytic basis of TM_1, we can conclude that $f \in C^\infty(U_k)$. Now, from Lemma 2.10 or Theorem 1 of [Fr1], we thus have the real analyticity of f on U_k.

Lemma 2.10: Let $f \in C^\infty(B_n, R^m)$ be such that $f(0) = 0$ and

$$\frac{\partial f}{\partial x} = G(x, f).$$

Here B_n stands for the ball in $\mathbf{R}^n$ and $G(x, f)$ is real analytic in x and f . Then f is real analytic at 0.

Proof of Lemma 2.10: Th proof is a straightforward application of the Cauchy-Kovalevsky theorem. In fact, when $n = 1$, it follows immediately from the uniqueness of the solution near the origin and the Cauchy-Kovalevsky theorem. Suppose the lemma is proved for $n \leq p - 1$ and assume $n = p$. Write

$$(x_1, \cdots, x_{p-1}, x_p) = (y, t)$$

with $t = x_p$. By the induction, we see that $g(y) = f(y, 0)$ $(g(0) = 0)$ is real analytic for $y \approx 0$. Consider now the following Cauchy problem:

$$\frac{\partial f(y,t)}{\partial t} = G(y, t, f) \text{ with } f(y, 0) = g(y).$$

Notice that the above equation also has a unique solution for $(y, t) \approx 0$. Again by the Cauchy-Kovalevsky theorem, we see the real analyticity of f near 0. ■

§**2.3.3** This subsection is very similar to §2.2.3. We will directly show the analyticity of f in case $k = 1$ and obtain a reduction in case $k > 1$.

By the argument in § 2.3.2, to complete the proof of Theorem 2 it now suffices for us to assume that there is an $n \geq m$ so that $\mathrm{Ind}(j) = 0$ for $j \leq n - m$, but $\mathrm{Ind}(n - m + 1) = 1$. This similarly implies the following:

Lemma 2.11: There exist an open subset $U' \subset U$ and a holomorphic function Φ so that it holds that $\overline{f_n}(z) = \Phi(f(z), f^{(1)}(z), f^{(2)}(z), \hat{\overline{F}}_n(z))$ for $z \in U'$.

Proof of Lemma 2.11: We first assume that $n = m$. Then for some j_0, l_0, $p_0^* \approx p_0$, and the e^{th} element $H_{j_0}^e$ of the vector function H_{j_0}, it holds that

$$L_{l_0} H_{j_0}^e (f, \xi, \eta, \hat{\overline{F}}_m)(p_0^*) \neq 0.$$

Then it is easy to see that for each j, $L_l H_j^e = \psi_j(f, f^{(1)}, f^{(2)}, \overline{f_{m+1}}, \cdots, \overline{f_{m+k-1}})$ for some ψ_j that is holomorphic in its variables and satisfies the corresponding Cauchy estimates.

Define

$$I_1(w, w^{(1)}, w^{(2)}, y_{m+1}, \cdots, y_{m+k-1}, u)$$

$$= \sum \psi_j(w, w^{(1)}, w^{(2)}, y_{m+1}, \cdots, y_{m+k-1})(u - u_0)^j$$

with $u_0 = \overline{f_m}(p_0^*)$. Obviously, by (2.3.4), I_1 is holomorphic on

$$O_w(f(p_0^*)) \times O_{w^{(1)}}(f^{(1)}(p_0^*)) \cdots \times O_u(0).$$

We note that $(f(z), f^{(1)}(z), f^{(2)}(z), \overline{f_{m+1}}(z), \cdots, \cdots, \overline{f_{m+k-1}}(z), \overline{f_m}(z) + u_0)$ satisfies the equation $I_1 = 0$ and $I_1(f(p_0^*), \cdots, \overline{f_{m+k-1}}(p_0^*), u) \not\equiv 0$ for $u \approx 0$. The proof in this case thus follows from Claim 2 of §2.2.

Now let $n > m$. We then have, for each l and α, that

$$L_l \phi_\alpha(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n-1}) = 0 \quad \text{on some } U_1 \subset U,$$

where α is an $(n-1)$-multi-index, ϕ_α is defined as in §3.2.2, and

$$\phi_\alpha = \sum_{j=0}^{\infty} \phi_{\alpha,j}(f, \xi, \eta, \hat{\overline{F}}_{m,\cdots,n}) \overline{f_n^j}. \tag{2.3.10}$$

But for some $p_0^* \approx 0$, α_0, j_0, and l_0, it holds that

$$L_{l_0} \phi_{\alpha_0, j_0}(p_0^*) \neq 0.$$

Let $L_l \phi_{\alpha_0,j} = \psi_{\alpha_0,j}(f, f^{(1)}, f^{(2)}, \cdots, \overline{f_{m+k-1}})$ and define

$$I_2(w, w^{(1)}, w^{(2)}, y_{m+1}, \cdots, y_{m+k-1}, u) = \sum \psi_{n,\alpha_0,j}(w, w^{(1)}, \cdots, y_{m+k-1})(u - u_0)^j$$

with $u_0 = \overline{f_n}(p_0)$. Then it is easy to see that Claim 2 can be applied to the equation $I_2 = 0$ for solving for $\overline{f}_n$. So the proof of Lemma 2.2.11 is complete. ∎

When the codimension $k = 1$, Lemma 2.11 tells that f_m admits a holomorphic extension to U'; for the formula of Φ involves no conjugate holomorphic terms (see the proof of Lemma 2.9 or [Pi] for details on this matter). Returning to (2.3.3) with $k = 1$, we see that

$$\overline{F_0}(z) = A(z, \xi(z), \eta(z), f(z)) \text{ for } z(\approx p_0) \in U',$$

where

$$A(z, \xi, \eta, f) = -\xi - \eta \overline{f_m(z)} - g(f, \xi, \eta, \overline{f_m(z)}).$$

We claim that this also implies the analyticity of $F_0(z) = (f_1(z), \cdots, f_m(z))$ near p_0. In fact, from the analyticity of $f_m(z)$, it follows that A is holomorphic in $(\overline{z}, \xi, \eta, f)$. So, let ϕ and V be as constructed in Lemma 2.9. Then $A(\phi(\tau), \xi(\phi(\tau)), \eta(\phi(\tau)), f(\phi(\tau)))$ admits a (uniform) holomorphic extension to Δ near $1 \in \partial\Delta$ (see the proof of Lemma 2.9 for more details concerning this matter). Notice that $\overline{F_0}(\phi(\tau))$ allows a uniform holomorphic extension to the outside of Δ near 1 and coincides with $A(\phi(\tau), \xi(\phi(\tau)), \eta(\phi(\tau)), f(\phi(\tau)))$ on part of the circle $\partial\Delta$ near 1. So, by using the Hartogs theorem, we can conclude the claim (see the proof of Lemma 2.9). By the same token, with these results at our disposal and returning to (2.3.1), we obtain the analyticity of f_{m+1}. For general codimension, we have

Lemma 2.12: Under the assumptions of Lemma 2.11, there exist a small open subset U'' of U' and a holomorphic function Ψ so that

$$f_n(z) = \Psi(z, \hat{F}_n(z)) \text{ for } z \in U'', \tag{2.3.11}$$

where n is as in Lemma 2.11.

Proof of Lemma 2.12: Make use of the assumption that f is of class C^{k+1} and copy the proof for Lemma 2.6 (Claim 3). ■

§**2.3.4**: Now we replace $\overline{f_n}$ in (2.3.4) by $\overline{\Psi(z, \hat{F}_n)}$. Then we have, for each $i < m-1$, that

$$\overline{f}_i = g_i^*(\overline{z}, f, f^{(1)}, \hat{\overline{F}}_n) \text{ on } U^{(2)} \subset U'', \tag{2.3.12}$$

where g_i^* is holomorphic on $O_z(p'') \times O_w(f(p'')) \times \cdots \times O_{\hat{Y}_j}(\hat{F}_j(p''))$ with p'' being some point in U''. By a slight modification of Lemma 2.12, it follows that on some $U^{(3)} \subset U^{(2)}$ there exist holomorphic functions $\{\Psi_1, \cdots, \Psi_{m-1}\}$ so that it holds for each i that

$$f_i(z) = \Psi_i(z, \hat{F}_n(z)) \quad \text{for } z \in U^{(3)}. \tag{2.3.13}$$

Similarly, by substituting (2.3.12) and (2.3.13) to (2.3.1), we obtain

$$\overline{f}_{m+k} = g_{m+k}^*(\overline{z}, f, \hat{\overline{F}}_n) \quad \text{on } U^{(4)} \subset U^{(3)}, \tag{2.3.14}$$

with g^*_{m+k} holomorphic in $(f, z, \overline{\hat{F}}_n)$. Thus it can be seen, after shrinking $U^{(4)}$, that we have

$$f_{m+k} = \Psi_{m+k}(z, \hat{F}_n) \text{ on } U^{(4)}, \tag{2.3.15}$$

for some holomorphic function Ψ_{m+k}. Combining all these formulas, we have

Lemma 2.13: There are a small neighborhood $\Omega^* \subset \mathbf{C}^m$ of some $p \in U^{(4)}$ and a nonsingular complex variety $M^* \subset \mathbf{C}^{m+k}$, which contains $f(p)$, so that $f(\Omega^*) \subset M^*$.

Proof of Lemma 2.13: Copy that for Lemma 2.7. ■

Lemma 2.14: The mapping f admits a holomorphic extension on some point near p. Thus the proof of Theorem 2 is complete.

Proof of Lemma 2.14: First, summarizing the argument in §2.3.2 and the argument following Lemma 2.11, we note that Lemma 2.14 is true in case $k = 1$. For $k > 1$, we also see that either f has a holomorphic extension at some point on $U^{(4)}$ or f has no analytic extension at any point on $U^{(4)}$ but Lemma 2.13 holds. In the latter case, similar to Lemma 2.8, we conclude that there is a complex manifold M^* of dimension $m + k - 1$ so that $f(\Omega) \subset M^*$ and $f(U^{(4)})$ is contained in some strongly pseudoconvex real analytic hypersurface of M^* (here we may have to shrink $U^{(4)}$). By making use of a local coordinates chart of M^*, we then see that Theorem 2 is false in the case of codimension $k - 1$. Inductively, this would result in a contradiction with the situation of $k = 1$. ■

§2.4 Proof of Proposition 1

In this section, we present the proofs of Proposition 1 and Corollary 1. In fact, we will first prove the following slightly stronger local result:

Proposition 1′: Let $M \subset \mathbf{C}^2$ be an algebraic pseudoconvex hypersurface which bounds D on its pseudoconvex side. Let f be a non-trivial holomorphic mapping from D into $\mathbf{C}^3$ that is continuous up to M and maps M to $\partial \mathbf{B}_3$ (the boundary of the unit 3-ball). For any point $p \in M$, denote by $t(p)$ the type value of M at p (in the sense of Kohn [Ko1] or D'Angelo [Da]). If f is of class $C^{t(p)}$ at p with $t(p) < \infty$, then f admits a holomorphic extension at p.

Proof of Proposition 1′: We first note that the set of all strongly pseudoconvex points of M is dense in M; for M is of finite type. Thus, from Theorem 1, the algebraicity of f follows easily. After making a suitable algebraic change of variables, we assume that $p = 0$, $f(p) = 0$, M is locally defined by

$$\rho(z, \overline{z}) = z_2 + \overline{z_2} + \sum_{i+j=m} c_{ij} z_1^i \overline{z_1^j} + O(|z_1|^{m+1} + |z_2 z_1|) = 0$$

with $m = t(p)$, and S_3 is defined by

$$w_3 + \overline{w_3} + \sum_{j=1}^{2} |w_j|^2 = 0.$$

Let

$$L = \frac{\partial}{\partial z_1} - \frac{\rho'_{z_2}}{\rho'_{z_1}} \frac{\partial}{\partial z_2}$$

be the (conjugate) Cauchy-Riemann operator of M.

Lemma 2.15: $\sum_{k=1}^{m-1} (|L^k f_1| + |L^k f_2|)(0) \neq 0$.

Proof of Lemma 2.15: Seeking a contradiction, suppose that $L^j f_l(0) = 0$ for all $j \leq m-1$ and $l = 1, 2$. Notice that

$$(2.4.1) \qquad f_3(z) + \overline{f_3(z)} + \sum_{j=1}^{2} |f_j(z)|^2 = 0, \quad \text{for } z \in M(\approx 0).$$

Applying the operator $\overline{L^{k_1}} L^{k_2}$ to Equation (2.4.1) and letting $z = 0$, we then see that

$$\overline{L^{k_1}} L^{k_2} f_3(0) = -\sum_{j=1}^{2} \sum_{i=0}^{k_1} \binom{k_1}{i} \overline{L^i} L^{k_2}(f_j(0)) \overline{L^{k_1 - i} f_j(0)}.$$

Thus, by the assumption, it holds that

$$(2.4.1)' \qquad \overline{L^{k_1}} L^{k_2} f_3(0) = 0,$$

whenever k_1 or k_2 is less than m.

Now, since 0 is a point of type m of ∂M, we can conclude that, for some choice of the operator N: $N = [\cdots, [L, \overline{L}], \cdots, L(\text{or } \overline{L})]$ (where the Lie bracket length is m), $\{L, \overline{L}, N\}$ consists of a basis of $T_0 M$ (see [Ko1]). Meanwhile, it is easy to see that $(2.4.1)'$ implies $N(f_3)(0) = 0$. Consequently, we obtain $Tf_3(0) = 0$ for any $T \in T_0 M$. In particular, it follows that

$$\left(\frac{\partial}{\partial z_2} - \frac{\partial}{\partial \overline{z_2}}\right) f_3(0) = 0.$$

Hence, $\frac{\partial}{\partial z_2} f_3(0) = 0$. This is a contradiction, because by the pseudoconvexity of M and a standard application of the Hopf lemma, we have that $f \equiv 0$. ∎

From now on, we let, without loss of generality, $L^k f_1(0) \neq 0$ for some $k \leq t_0 - 1$. Applying L^k to (3.4.1), we obtain

$$L^k f_3 + \sum L^k f_j \overline{f_j} = 0 \text{ for } z(\approx 0) \in M. \tag{2.4.2}$$

Similar to what we did before, we consider the following two cases:

(AAA): $L(H_j) \equiv 0$ for $z(\approx 0) \in M$, and $j = 1, 2$.
(BBB): $L(H_1) \not\equiv 0$ on any small neighborhood of 0 in M.

Here $H_j = \frac{L^k f_{j+1}(z)}{L^k f_1(z)}$ for $j = 1, 2$.

§**2.4.1**: We first assume (AAA). Then by Lemma 2.9, we know that $\overline{H_j}$ is holomorphic near 0, in particular H_j is real analytic on M near 0. In the meantime, we also see that $L^k(f_{j+1} - f_1(z)H_j(z)) \equiv 0$. Again by Lemma 2.9, we therefore have some analytic $H_{j,1}$ such that $L^{k-1}(f_{j+1} - f_1(z)H_j(z)) = H_{j,1}$ and $LH_{j,1} \equiv 0$. Notice that $L^{k-1}(z_1^{k-1}/(k-1)!) = 1$. It follows that $L^{k-1}(f_{j+1} - f_1 H_j - H_{j,1} z_1^{k-1}/(k-1)!) \equiv 0$. Hence, by inductively using Lemma 2.9 and by noting that $L^l(z_1^l/l!) = 1$ for all l, we obtain

$$f_{j+1} = f_1(z)H_j(z) + H_j^0(z), \tag{2.4.3}$$

where $z(\in M) \approx 0$ and $H_j^0(z)$ is real analytic for $j = 1, 2$. On the other hand, (2.4.2) tells us that

$$(2.4.3)' \qquad f_1 = -\overline{H_2} - \overline{H_1} f_2.$$

So, by combining (2.4.3) with (2.4.3)', we easily deduce the real analyticity of f at 0.

§**2.4.2**: We now study the case (BBB). Applying L to Equation (2.4.2), we then see that $\overline{f_2}(z) = -\frac{LH_2(z)}{LH_1(z)}$ for $z(\approx 0) \in M$.

Claim 4: $LH_1(z)$ cannot vanish on any open subset of M near 0.

Proof of Claim 4: Denote by $g(z, \omega)$ the complexification of $LH_1(z)$, i.e, $g(z, \overline{z}) = LH_1(z)$. By Lemma 2.1, Theorem 1 and the algebraicity of M, we know that $g(z, \omega)$ is also algebraic. Since Condition (BBB) indicates that $g(z, \omega) \not\equiv 0$, we therefore have a polynomial $P(\xi) = \sum c_j(z, \omega)\xi^j$ with $c_0(z, \omega) \not\equiv 0$ so that $P(g(z, \omega)) \equiv 0$. Now if $g(z, \overline{z})$ vanishes on an open subset of M, it then follows easily that $c_0(z, \overline{z}) \equiv 0$ on that same set. Notice that any open subset on M is a set of uniqueness. We see that $c_0(z, \omega) \equiv 0$ and thus obtain a contradiction. ■

We now prove Proposition $1'$ in case (BBB). By Claim 4, we can find two transversal complex lines, denoted by C_1 and C_2, which pass through 0 so that $L(H_1)$ is not identically zero on $C_j \cap M$ $(j = 1, 2)$ and $C_j \cap M$ divides $C_j \cap \Omega$ into two parts: the one inside $D \cap \Omega$ and the one in $D^c \cap \Omega$. Here $\Omega \subset \mathbf{C}^2$ is a small neighborhood of 0 (see §3.1 of [HK1] for more details on this matter). Translating C_1 or C_2, we then obtain an algebraic coordinates system in $\mathbf{C}^2$ near 0. In terms of the continuity of LH_1, it is easy to see that when C_ϵ is parallel and close to C_1 or C_2, then LH_1 is also not identical zero on $C_\epsilon \cap M$.

Now, we will extend $f_2(z)$ holomorphically to each of such C_ϵ $(\cap\Omega)$'s. We let ϕ be an algebraic embedding from Δ to $\mathbf{C}^2$ so that $\phi(\Delta^+) \subset \Omega \cap C_\epsilon$, $\phi(\Delta^-) \subset \Omega \cap D^c \cap C_\epsilon$, and $\phi(-1, 1) \subset M \cap C_\epsilon$. Here $\Delta^+ = \{\tau \in \Delta : \mathrm{Im}(\tau) > 0\}$ and $\Delta^- = \{\tau \in \Delta : \mathrm{Im}(\tau) < 0\}$. After shrinking Ω, we may assume that $LH_j \circ \phi(\tau)$ has a holomorphic extension to Δ^+ (by Theorem 1 and Lemma 2.1, the extension is also algebraic). Let $a^-(\tau) = \overline{\frac{LH_2\circ\phi(\overline{\tau})}{LH_1\circ\phi(\overline{\tau})}}$. Then $a^-(\tau)$ is meromorphic and algebraic on Δ^-. Define $a(\tau)$ by $f\circ\phi(\tau)$ if $\tau \in \Delta - \Delta^-$ and by $a^-(\tau)$ when $\tau \in \Delta^-$. We then see by the above argument, that $a(\tau)$ is meromorphic on $\Delta-$ E, where $E = \{\tau \in (-1, 1) : LH_1 \circ \phi(\tau) = 0\}$. We will show that $a(\tau)$ is actually holomorphic near $(-1, 1)$ and hence meromorphic on Δ. For this purpose, we let $\alpha(\tau) = \overline{LH_1 \circ \phi(\overline{\tau})}$.

Claim 5: E is a removable singular set of $\alpha(\tau)$.

Proof: As we noted above, $\alpha(\tau)$ is algebraic and holomorphic on Δ^-. Thus there is a polynomial $P(\xi) = \sum c_j(\tau)\xi^j$ with $c_0 \not\equiv 0$ so that $P(\alpha(\tau)) \equiv 0$. Obviously, it holds that $c_0(E) = 0$. Therefore, it follows that E can only be a finite set for $c_0(\tau)$ is a nonzero polynomial in τ.

On the other hand, from the equation $\sum c_j(\tau)(\alpha(\tau))^j = 0$, we easily see that for any $\tau_0 \in E$, it holds that $|\alpha(\tau) - \alpha(\tau_0)| \geq \text{const}|\tau - \tau_0|^n$ for some big n. Thus every point in E can at most be a pole of $a(\tau)$. However, since $a(\tau)$ is bounded on Δ^+, we can thus conclude the claim. ∎

Now we can verify the analyticity of f_2 at 0 by making use of the Hartogs theorem and by noting that we can make C_ϵ close enough to C_j so that the extension $a(\tau)$ of $f_2 \circ \phi(\tau)$ is holomorphic on a fixed small open subset of $(-1, 1)$. (We can also obtain the analyticity of f_2 at 0 in the following way: First, we prove by using algebraicity of f that f_3 is meromorphic. Then we prove the meromorphic property of f_1 and f_2. Finally, we see the holomorphic property of f by using the result of [Ch]. A much more general argument in this aspect will appear in a future work.)

Whenever we know the analyticity of f_2 at 0, returning to (2.4.2) and (2.4.1), we then can conclude the holomorphic extension of f at 0.

The proof of Proposition $1'$ is complete. ∎

We conclude this chapter by presenting a discussion of Corollary 1.

Corollary 1: Let M_1 and M_2 be two strongly pseudoconvex real algebraic hypersurfaces in (possibly different) complex spaces of dimension at least two. Then every C^∞ smooth CR mapping from M_1 to M_2 is real analytic on M_1.

§ **2.4.3: Proof of Corollary 1**: The proof follows from Theorem 1, the fact that M_1 is a uniqueness set of holomorphic functions defined on the pseudoconvex side of M_1, and the following known result (see [BR3] or [BJT]).

Lemma 2.16: Let $u(s)$ be a C^∞ function defined on the unit ball $\mathbf{B}$ in $\mathbf{R}^n$. Suppose that there exists a polynomial

$$h(s,z) = a_J(s)z^J + a_{J-1}(s)z^{J-1} + \cdots + a_0(s)$$

with coefficients real analytic in $\mathbf{B}$ and $a_J(s) \not\equiv 0$ so that $h(s,u(s)) \equiv 0$. Then u is real analytic in $\mathbf{B}$.

Proof of Lemma 2.16: This lemma follows from the arguments in [BK3]. However, we present the following details for completeness.

Let $v(s) = a_J(s)u(s)$ and $\widetilde{h}(s,z) = z^J + a_{J-1}(s)z^{J-1} + a_{J-2}(s)a_J(s)z^{J-2} + \cdots + a_J^{J-1}(s)a_0(s)$. Then by the given hypothesis, we see that $\widetilde{h}(s,v(s)) \equiv 0$. From Theorem 6.8.20, it thus follows that $v(s)$ is real analytic on $\mathbf{B}$. Now, to conclude the real analyticity of $u(s)$, we first notice that for certain open dense subset $S' \subset \partial\mathbf{B}$, it holds that $a_J(\omega t) \not\equiv 0$ for any given $\omega \in S'$, where $t \in (-1,1)$. Since $u(s)$ is smooth, we therefore see that $u(\omega t) = v(\omega t)/a_J(\omega t)$ is real analytic in $t \in (-1,1)$ for any given $\omega \in S'$. By Theorem 6.8.18, we thus get that the Taylor series of $u(s)$ at the origin is actually a convergent series. Noting the fact that this power series must coincide with $u(s)$ when restricted in each line $t\omega$ with $\omega \in S'$ and noticing that S' is dense in $\partial\mathbf{B}$, we thus see that the Taylor series of $u(s)$ at the origin converges to $u(s)$ everywhere; for u is continuous in $\mathbf{B}$. This completes the proof of Lemma 2.16 ∎

Chapter 3: Kobayashi extremal mappings and holomorphic self-mappings

The purpose of the present chapter is to give the proofs of the results which we described in §1.2. In §3.1 and §3.2, we will prove Theorem 4 and Theorem 5 (Theorem 6), respectively. The discussions of Theorem 7 and Theorem 3 can be found in §3.3. In the last section of this chapter (§3.4), we study further applications of previously obtained results, including the proof of Theorem 8.

§3.1 A preservation principle of extremal mappings—Proof of Theorem 4:

In this section, we give the proof of Theorem 4 and some of its immediate corollaries.

Theorem 4: Let $D \subset\subset \mathbf{C}^n$ be either a pseudoconvex domain with a Stein neighborhood basis or a pseudoconvex domain with C^∞ boundary. Suppose that $p \in \partial D$ is a strongly pseudoconvex point of ∂D with at least C^3 smoothness. Then for every open neighborhood U of p, there is a positive number ϵ such that for each extremal mapping ϕ of D, when $\|\phi(0) - p\| < \epsilon$ and $\|(\phi'(0))_N\| < \epsilon\|(\phi'(0))_T\|$, then ϕ is the complex geodesic of D and $\phi(\Delta) \subset U$.

Our idea is to make use of the C^k-version of the reflection principle to get the uniform Hölder continuity of the differentials of a sort of 'normalized' complex geodesics on strongly convex domains. We then apply it with the Fornaess embedding theorem and the Graham estimates of the Kobayashi metric to obtain our results.

We first fix some notation. In what follows, we fix the symbol $\langle , \rangle$ for the standard Hermitian inner product in $\mathbf{C}^n$ and the symbol $\|\cdot\|$ for the corresponding euclidean norm. For two domains D_1 and D_2, Hol(D_1,D_2) stands for the set of all holomorphic mappings from D_1 to D_2. When $f \in \mathrm{Hol}(D_1, D_2)$ with $D_1 = D_2$, we denote by f^m the m^{th}-iterate of f defined inductively by $f^1 = f$, ⋯, $f^m = f \circ f^{m-1}$.

For a bounded domain D in $\mathbf{C}^n$, denote by K_D the Kobayashi distance and by κ_D the Kobayashi metric of D (see [Kr1] for the definitions). For any $z \in D$, we use $\delta(z)$ to stand for the euclidean distance of z to ∂D. We recall that $\phi \in \mathrm{Hol}(\Delta, D)$ is said to be a complex geodesic (respectively, an extremal mapping) of D if $K_D(\phi(\tau_1), \phi(\tau_2)) = K_\Delta(\tau_1, \tau_2)$ for every pair $\tau_1, \tau_2 \in \Delta$ [Ve] (respectively, $\kappa_D(\phi(0), \phi'(0)) = \kappa_\Delta(0, 1) = 1$).

In what follows, we will also use the notation C^{k-} to denote the function space $\cap_{\alpha<1} C^{k-1+\alpha}$ in case k is an integer, and the space C^k otherwise.

Lemma 3.1: Let $D_1 \subset D_2$ be two bounded domains in $\mathbf{C}^n$. If ϕ is a complex geodesic (respectively, an extremal mapping) of D_2 such that $\phi(\Delta) \subset D_1$, then ϕ is also a complex geodesic (respectively, an extremal mapping) of D_1.

Proof: This follows immediately from the monotonicity properties of the Kobayashi metric and the Kobayashi distance [Kr1]. ■

Let D be a C^1-smoothly bounded domain in $\mathbf{C}^n$. Then, for every $p \in \partial D$, we may define the outward unit normal vector of ∂D at p, denoted by $\nu(p)$. When $D \subset\subset \mathbf{C}^n$ is a bounded C^k-strongly convex domain ($k > 2$), Lempert in [Lm1] [Lem 2] showed that a holomorphic mapping ϕ from Δ to D is an extremal mapping (or complex geodesic) of D if and only if it is proper and there exists a (unique) $C^{(k-1)-}$- smooth function $P_\phi : \partial\Delta \to \mathbf{R}^+$ so that the vector function $\xi P_\phi(\xi)\overline{\nu(\phi(\xi))}$, initially defined on $\partial\Delta$, can be extended to a holomorphic vector function $\widetilde{\phi}$ on Δ (which is called the dual mapping of ϕ) with $\langle \phi', \widetilde{\phi} \rangle \equiv 1$. The following lemma is an obvious consequence of this characterization:

Lemma 3.2: Let $D_1 \subset D_2$ be two bounded C^3-strongly convex domains in $\mathbf{C}^n$. Suppose that $\partial D_1 \cap \partial D_2$ is a piece of hypersurface. If $\phi \in \mathrm{Hol}(\Delta, D_1)$ is a complex geodesic of D_1 so that $\phi(\partial\Delta) \subset \partial D_2$, then ϕ is also a complex geodesic of D_2. ■

In the next two lemmas, we assume $D \subset\subset \mathbf{C}^n$ to be a C^3-strongly convex domain.

For each a >0, let $\mathcal{F}_a$ denote the set of all complex geodesics ϕ of D which satisfy $\delta(\phi(0)) \geq a$. From §7 of [Lm1], we see that there exist two positive constants C_0 and C_0', depending only on D and a, so that for every $\phi \in \mathcal{F}_a$, it holds that:

(3.1.a) $\|\phi(\tau_1) - \phi(\tau_2)\| < \mathrm{C}_0\|\tau_1 - \tau_2\|^{1/2}$, $\|\widetilde{\phi}(\tau_1) - \widetilde{\phi}(\tau_2)\| < \mathrm{C}_0|\tau_1 - \tau_2|^{1/2}$ for any $\tau_1, \tau_2 \in \Delta$;

(3. 1.b) $\mathrm{C}_0' < P_\phi < \mathrm{C}_0$.

Starting with these properties, we now prove the following:

Lemma 3.3: There exist two positive constants R_1 and R_2, depending only on D and a, so that for every $\phi \in \mathcal{F}_a$, it holds that $\mathrm{R}_1 < \|\widetilde{\phi}\| < \mathrm{R}_2$.

Proof of Lemma 3.3: We note that $\widetilde{\phi} = \xi \mathrm{P}_\phi(\xi)\overline{\nu(\phi(\xi))}$ for $\xi \in \partial\Delta$. Thus by applying the maximal principle to $\|\widetilde{\phi}\|$, we see that R_2 can be chosen to be C_0 in (3.1.b). To obtain the other inequality, we suppose not and seek a contradiction. Then there exist a sequence $\{\phi_k\} \subset \mathcal{F}_a$ and a sequence $\{\tau_k\} \subset \Delta$ which approaches some $\xi_0 \in \overline{\Delta}$ so that $\widetilde{\phi}_k(\tau_k) \to 0$. By (3.1.a) and the Arzela-Ascoli theorem we can assume, without loss of generality, that $\{\phi_k\}$ converges uniformly to some $\phi \in \mathcal{F}_a$ and $\{\widetilde{\phi}_k\}$ converges uniformly to some $\phi^* \in \mathrm{Hol}(\Delta, \mathbf{C}^n)$. Hence $\mathrm{P}_{\phi_k} = (\xi^{-1}\widetilde{\phi}_k \nu(\phi_k(\xi)))$ converges uniformly to some positive continuous function P^* defined on $\partial\Delta$. Since now $\phi^*(\xi) = \xi \mathrm{P}^*(\xi)\overline{\nu(\phi(\xi))}$ and $\widetilde{\phi}(\xi) = \xi \mathrm{P}_\phi(\xi)\overline{\nu(\phi(\xi))}$ we see that

$$\phi^* = \frac{\mathrm{P}^*}{\mathrm{P}_\phi}\widetilde{\phi}$$

on $\partial\Delta$. From the fact that $\widetilde{\phi} \neq 0$, it follows easily that $\frac{P^*}{P_\phi}$ is the boundary value of some holomorphic function defined on Δ. This implies that $\mathrm{P}^* = \mathrm{C}P_\phi$ and thus that ϕ^* and ϕ differ by a positive constant. That is a contradiction, for $\phi^*(\xi_0) = \lim_{k\to\infty} \widetilde{\phi}_k(\tau_k) = 0$ but $\widetilde{\phi}(\xi_0) \neq 0$. ∎

Lemma 3.4: There exists a positive constant C_1, depending only on D and a, such that for every $\phi \in \mathcal{F}_a$ and for any $\tau_1,\ \tau_2 \in \Delta$, it holds that $\|\phi'(\tau_1) - \phi'(\tau_2)\| < \mathrm{C}_1|\tau_1 - \tau_2|^{1/2}$.

Proof of Lemma 3.4: The argument is based on a careful examination of what is called the C^k version of the Schwarz reflection principle.

Let $\mathrm{S} = \{(p, \mathrm{T}_p^{(1,0)}\partial D) : p \in \partial D\}$ and let $\mathbf{CP}_{n-1}$ be the complex projective space of hyperplanes in $\mathbf{C}^n$. Then by a result of Webster (see [Ab1]), $\mathrm{S} \subset \mathbf{C}^n \times \mathbf{CP}_{n-1}$ is a compact totally real submanifold of maximal dimension . Let $\mathrm{B}(\mathrm{R}_1, \mathrm{R}_2) = \{z \in \mathbf{C}^n : \mathrm{R}_1 < \|z\| < \mathrm{R}_2\}$, and denote by $\pi : \mathbf{C}^n \times \mathrm{B}(\mathrm{R}_1, \mathrm{R}_2) \to \mathbf{C}^n \times \mathbf{CP}_{n-1}$ the natural projection, where R_1, R_2 are as in Lemma 2.3.

We first find two open coverings $\{\mathrm{E}_i\}_{i=1}^m$ and $\{\widetilde{\mathrm{E}}_i\}_{i=1}^m$ of S such that the following assertions hold for each i:

(3.1.c) $\mathrm{E}_i \subset\subset \widetilde{E}_i \subset\subset \mathbf{C}^n \times \mathbf{CP}_{n-1}$

(3.1.d): There exists a C^2-diffeomorphism $\Psi_i : \overline{\widetilde{E}_i} \to \overline{V_i} \subset\subset \mathbf{C}^{2n-1}$ so that $\Psi_i(\widetilde{E}_i \cap S) \subset \mathbf{R^{2n-1}} \subset \mathbf{C^{2n-1}}$ and $\mathrm{D}^\alpha(\overline{\partial}\Psi_i) = 0$ on $\widetilde{E}_i \cap \mathrm{S}$ for all multi-indices α with $\|\alpha\| \leq 1$;

(3.1.e) Let $O_i = \{z \in \mathbf{C}^n :$ There exists $w \in B(R_1, R_2)$ so that $\pi(z, w) \in \widetilde{E}_i\}$. Then $\max_{z \in \mathrm{O}_i}$ $\mathrm{dist}(z, \partial D) << 1$.

For every $\phi \in \mathcal{F}_a$, from Lemma 3.3, we can define a holomorphic mapping $\hat{\phi} \in \mathrm{Hol}(\Delta, \mathbf{C}^n \times B(R_1, R_2))$ by letting $\hat{\phi}(\tau) \equiv (\phi(\tau), \widetilde{\phi}(\tau))$. Let

$$b_0 \equiv \min_i \{\mathrm{dist}\left(\partial(\pi^{-1}(\widetilde{E}_i), \partial(\pi^{-1}(E_i))\right)$$

and let $U_i \equiv (\pi \circ \hat{\phi})^{-1}(E_i)$. We notice that the boundaries defined here are taken in the space $\mathbf{C}^n \times \mathrm{B}(R_1, R_2)$ and the 'dist' is inherited from the standard one. From (3.1.c), we see that $\mathrm{b}_0 > 0$. Furthermore the following properties are easy to verify:

(3.1.f) Let $\widetilde{U}_i = \{\tau \in \overline{\Delta} : \mathrm{dist}(\tau, U_i) < \left(\frac{b_0}{\mathrm{C}_0}\right)^2\}$, where C_0 is chosen as in Lemma 3.3. Then $\pi(\hat{\phi}(\widetilde{U}_i)) \subset \widetilde{E}_i$ (whenever $\mathrm{U}_i \neq \emptyset$).

(3.1.g): There exists a constant $b_1 > 0$, independent of the choice of ϕ, so that for any $\tau_1, \tau_2 \in \overline{\Delta}$, if $1 - |\tau_1| < b_1$ and $|\tau_1 - \tau_2| < b_1$, we may find some U_i, defined as above, which contains τ_1 and τ_2.

(3.1.h): There exists a constant $b_2 > 0$, independent of the choice of ϕ, so that for every $\tau \in \Delta$ with $|\tau| < b_2$, we have $\tau \notin \cup_i \widetilde{U_i}$.

In fact, (3.1.f) follows easily from (3.1.a) and the definition of b_0, (3.1.h) follows from (3.1.a) and (3.1.e), while (3.1.g) is a simple application of the *Lebesgue* *number* lemma and (3.1.a).

We let $\widetilde{U}_i^* = \{\tau \in \mathbf{C}^1 : \overline{\tau}^{-1} \in \overline{\widetilde{U}_i} \cap \overline{\Delta}\}$, $V_i^* = \{z \in \mathbf{C}^{2n-1} : \overline{z} \in V_i\}$, and $\widetilde{\Omega}_i = \widetilde{U}_i \cup \widetilde{U}_i^*$. Define $g_i : \widetilde{\Omega}_i \to V_i \cup V_i^*$ by $\Psi_i \circ \pi \circ \hat{\phi}(\tau)$ when $\tau \in \widetilde{U}_i$, and by $\overline{\Psi_i \circ \pi \circ \hat{\phi}(\overline{\tau}^{-1})}$ when $\tau \in \widetilde{U}_i^*$. Consider f_i, defined by $\frac{\partial g_i}{\overline{\partial} \tau}$. By the argument on Page 438 of [Lm1], we can conclude, from (3.1.a), (3.1.d), and the Hardy-Littlewood theorem, that f_i is uniformly bounded and uniformly Hölder -$\frac{1}{2}$ continuous on $\widetilde{\Omega}_i$ with respect to $\mathcal{F}_a$ (i.e, there is a constant C, independent of the choice of ϕ, so that for every $\phi \in \mathcal{F}_a$ and $\tau_1,\ \tau_2 \in \Delta$, the corresponding f_i satisfies $\|f_i(\tau_1) - f_i(\tau_2)\| < \mathrm{C}|\tau_1 - \tau_2|^{1/2}$)

Let

$$\psi_i(\tau) = \frac{1}{2\pi\sqrt{-1}} \int_{\widetilde{\Omega}_i} \frac{f_i(\xi)}{\xi - \tau} \mathrm{d}\xi \wedge \overline{\mathrm{d}\xi}.$$

We then have the following facts:

(3.1.i): $\frac{\partial \psi_i}{\overline{\partial} \tau} = f_i$;

(3.1.j): ψ_i is uniformly bounded on $\widetilde{\Omega}_i$ with respect $\mathcal{F}_a$ (by (3.1.h) and the uniform boundedness of f_i);

(3.1.k): $\frac{\partial \psi_i}{\partial \tau}$ is uniformly Hölder-$\frac{1}{2}$ continuous on Ω_i (by (3.1.a) and Proposition 2.6.40 of [Ab1]). Here $U_i^* = \{\tau \in \mathbf{C}^1 : \overline{\tau}^{-1} \in U_i\}$ and $\Omega_i = U_i \cup U_i^*$.

Since $\psi_i - g_i$ is holomorphic and uniformly bounded on $\widetilde{\Omega}_i$, it follows, from (3.1.f) and the Cauchy estimates, that $(\psi_i - g_i)'$ is uniformly bounded on U_i. Hence, by (3.1.k), $\frac{\partial g_i}{\partial \tau}$ is uniformly Hölder-$\frac{1}{2}$ continuous on U_i. So by (3.1.a), (3.1.d), (3.1.g), and the Cauchy estimates, we can now find a constant C_1, depending only on D and a, so that for every $\phi \in \mathcal{F}_a$ and for any $\tau_1,\ \tau_2 \in \Delta$, we have

$$\|\phi'(\tau_1) - \phi'(\tau_2)\| < \mathrm{C}_1|\tau_1 - \tau_2|^{\frac{1}{2}}.$$

This completes the proof. ■

Remark: Let $\mathcal{F}$ be the set of all complex geodesics ϕ satisfying

$$\delta(\phi(0)) = \max_{\tau}\{\delta(\phi(\tau))\}.$$

By making use of the uniform Hölder-1/4 continuity of $\mathcal{F}$ [CHL], the above argument can furthermore be modified to prove the following:

Proposition 3.1: Let $D \subset\subset \mathbf{C}^n$ be a C^k-strongly convex domain ($k > 2$). If $k = \omega$, then there exists an open neighborhood U of $\overline{\Delta}$ so that all elements in $\mathcal{F}$ can be extended holomorphically to U; if $k < \omega$, then for any $j \leq k-2$ there exists a constant C_j so that for every $\phi \in \mathcal{F}$ and $\tau_1, \tau_2 \in \Delta$, it holds that $\|\phi^{(j)}(\tau_1) - \phi^{(j)}(\tau_2)\| < \mathrm{C}_j|\tau_1 - \tau_2|^{1/4}$.

Another key lemma which we need is the following version of the Fornæss embedding theorem:

Lemma 3.5: Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let $p \in \partial D$ be a strongly pseudoconvex point with at least C^2-smoothness. Suppose that either D has a Stein neighborhood basis or D has a C^∞ boundary. Then there exist a neighborhood U of p, a bounded C^2-strongly convex domain Ω in $\mathbf{C}^n$, and a holomorphic mapping Φ from D to Ω such that

(a): Φ can be extended holomorphically to U with $\Phi^{-1}(\Phi(U \cap \overline{D})) = U \cap \overline{D}$;

(b): $\Phi(U \cap D) \subset \Omega$, $\Phi(U \cap \Omega^c) \subset \Omega^c$, and $\Phi(U \cap \partial\Omega) = \Phi(U) \cap \partial\Omega$.

Proof of Lemma 3.5: When D has a Stein neighborhood basis, the lemma is Proposition 1 of [Fn]. So it suffices for us to prove the lemma in case D is a smooth pseudoconvex domain with p being a strongly pseudoconvex point. The argument in this situation is also a slight modification of that in [Fn]. In fact, the only difference is that we now have to make use of Kohn's global regularity result for the $\overline{\partial}$-equations [Ko2] on smooth pseudoconvex domains to construct a nice bounded supporting function appearing in line $1-5$ of page 533 of [Fn] (this is the only place we need the global boundary smoothness of D). For the convenience of the reader, we present the following details:

First, let $\{w_1(z), \cdots, w_n(z)\}$ be a local coordinates system on a neighborhood U of p so that $w(p) = 0$ and $U \cap \partial D$ is defined by $\rho(w) = Rew_1 + \sum_{j=1}^n \|w_j\|^2 + o(\|w\|^2)$. Let $V \subset\subset U$ be a very small neighborhood of p (or $w = 0$). Choose χ to be a positive cut-off function with $\mathrm{Supp}\chi \subset\subset V$ and $\chi(0) = \chi(w(p)) = 1$. For a positive number ϵ, define $D_\epsilon = \{z \in \mathbf{C}^n : \text{either } z \in D \text{ or } z \in V \text{ with } \rho(w(z)) < \epsilon\chi(w)\}$. By the above discussions and (3.4.2.2) of Theorem 3.4.2 in [Kr1], it is easy to check that when ϵ is small enough then $D_\epsilon (\supset D)$ is also a smoothly bounded pseudoconvex domain. Now when $\|w\| < \lambda_0$ with $\lambda_0 << 1$, we may assume that $\chi(w) < 2$ and $\rho(w) > \mathrm{Re}w_1 + 1/2\sum_{j=1}^{n}|w_j|^2$. Thus for $w \in D_\epsilon \cap \{\|w\| < \lambda_0\}$, we have that

$$\mathrm{Re}w_1 < \rho(w) - 1/2\sum_{j=1}^{n}|w_j|^2 < \epsilon\chi(w) - 1/2|w|^2 < 2\epsilon - 1/2\|w\|^2,$$

where $\|w\|^2 = \sum_{j=1}^{n}\|w_j\|^2$. Hence, for $\lambda << 1$, if we let $\epsilon = \frac{1}{4}\lambda^2$, then the following claim holds (see also Lemma 5.2.8 of [Kr1]):

Claim: Let ϵ, λ, and λ_0 as above. Then, when $1 >> \lambda_0 > \lambda$, $\lambda < \|w\| < \lambda_0$, and when $w \in D_\epsilon$, it holds that $\mathrm{Re}(w_1) < 0$.

Now, define a cut-off function $\xi(t) : \mathbf{R^1} \to [0, 1]$ with $\xi(t) = 1$ for $|t| < \lambda'$ and 0 for $|t| > \lambda'_0$. Here $\lambda < \lambda' < \lambda'_0 < \lambda_0$. It thus follows that $\omega = \overline{\partial_z}(\xi(\|w\|)\mathrm{log}w_1$ is a well-defined C^∞ $(0, 1)$-form on $\overline{D_\epsilon}$; for in case $\overline{\partial}_z(\xi\|w\|) \neq 0$, $\mathrm{Re}w_1 < 0$ and thus $\log(w_1)$ is well defined (see page 186-187 of [Kr1] for more details on this matter). Furthermore, it is easy to verify that $\overline{\partial}\omega \equiv 0$. Therefore, by a theorem of Kohn [Ko2], there is a $g \in C^\infty(\overline{D_\epsilon})$ so that $\overline{\partial_z}g = \omega$.

Define $f(z)$ with $f(z) = \exp(g + \xi(\|w\|)\mathrm{log}w_1)$ for $w \in D_\epsilon \cap \{\|w\| < \lambda'_0\}$ and $f(z) = \exp(g)$ for $w \in D_\epsilon \cap \{\|w\| \geq \lambda'_0\}$. By the way these objects were constructed, we can conclude that

(i): $f(z) \in \mathrm{Hol}(D_\epsilon) \cap C(\overline{D_\epsilon})$ (see also page 186 of [Kr1]);

(ii): for w close to 0, $f(w(z)) = w_1 f^*(w)$ with $f^*(0) \neq 0$.

We now shrink λ_0 and λ (thus also ϵ) so that

(iii): $\|f^*(w) - f^*(0)\| < 1/2\|f^*(0)\|$ for $\|w\| < \lambda_0$ and $w \in D_\epsilon$

(iv): $\mathrm{Re}w_1 < 0$ for $w \subset (\overline{D} - \{p\}) \cap \{w : \|w\| < \lambda_0\}$.

Therefore, we can also define the smooth $(0, 1)$-closed form $\omega^* = \overline{\partial}(\xi(\|w\|)\mathrm{log}f \cdot f^{-3}$ on the closure of D_ϵ. Consider the similar equation $\overline{\partial}_z g^* = \omega^*$. By Kohn's theorem, we obtain again a solution g^* which is continuous on $\overline{D_\epsilon}$ (actually smooth, but for our purposes here, all we need is the existence of a bounded solution). Now, define $\eta^*(z) = \exp(g^* f^3 + \xi(\|w\|)\mathrm{log}f)$ for $w \in D_\epsilon \cap \{\|w\| < \lambda'_0\}$ and $\eta^*(z) = \exp(g^* f^3)$ for $w \in D_\epsilon \cap \{w : \|w\| \geq \lambda'_0\}$. Then we similarly see that $\eta^*(z) \in \mathrm{Hol}(D_\epsilon) \cap C(\overline{D_\epsilon})$. Moreover, it holds, for $w \approx 0$ (or $z \approx p$), that $\eta_1(w) \equiv$

$\eta^*(z)f^{*-1}(w(p)) = w_1 + O(\|w\|^3)$. As was done in [Fn], we now change the coordinates $\{w_1,\cdots,w_n\}$ to the globally defined functions $\{\eta_1,\cdots,\eta_n\}$ on $D_{\overline{\epsilon}}$ which also serve as the local coordinates near p, where η_1 is as above and η_j is the linear term of the Taylor expansion of $w_j(z)$ at $z = p(j > 1)$. Notice that, for $z \in D$, it still holds that $\rho(w(\eta)) = \mathrm{Re}\eta_1 + \sum_{j=1}^{n} |\eta_j|^2 + o(\|\eta\|^2) < 0$. We therefore see that $\mathrm{Re}\eta_1(z) < 0$ for $z(\approx p) \in \overline{D} - p$. Since $\eta_1(z) \neq 0$ for $z \in \overline{D} - p$ (by the construction of η_1 and the property (iv)), and since η_1 is continuous on $\overline{D}$, we therefore conclude that there is a small positive ϵ_0 so that $|\eta_1(z) - \epsilon_0| > \epsilon_0$ for $z \in \overline{D} - p$. Also, notice that $\eta_1 \in \mathrm{Hol}(D_\epsilon) \cap C(\overline{D_\epsilon})$ and η_1 is holomorphic near p. Thus starting from such a supporting function, we can now obtain the Φ in our lemma by copying the argument of [Fn] from line 6 of page 533 to line 11 of page 536. ∎

We now are ready to prove Theorem 4.

Proof of Theorem 4: Seeking a contradiction, we suppose that there is a sequence of extremal mappings $\{\phi_k\}$ of D so that $\phi_k(0) \to p$, $\frac{\|(\phi_k'(0))_T\|}{\|(\phi_k'(0))_N\|} \to \infty$, but for each k, $\phi_k(\Delta) \cap V \neq \emptyset$ for some fixed neighborhood V of p.

Let Ω, U, Φ be as in Lemma 3.5 and let $\phi_k^* = \Phi \circ \phi_k$. It is then easy to see that $\phi_k^*(0) \to \Phi(p)(\stackrel{\Delta}{=} q)$ and $\frac{\|(\phi_k^{*\prime}(0))_T\|}{\|(\phi_k^{*\prime}(0))_N\|} \to \infty$. Construct another strongly convex domain Ω_0, which is contained in $\Phi(U)$, so that $\partial\Omega_0 \cap \partial\Phi(D) (\subset \partial\Omega \cap \partial\Phi(D))$ is a piece of strongly convex hypersurface, and find a sequence of complex geodesics $\{\psi_k\}$ of Ω with $\psi_k(0) = \phi_k^*(0)$ and $\psi_k'(0) = \lambda_k \phi_k^{*\prime}(0)(\lambda_k > 0)$ for each k. We claim that $\psi_k(\Delta) \to p$ as $k \to \infty$, thus $\psi_k(\Delta) \subset \Omega_0$ for $k >> 1$. In fact, let $\{\sigma_k\} \subset \mathrm{Aut}(\Delta)$ be such that $\psi_k^* \stackrel{\Delta}{=} \psi_k \circ \sigma_k \in \mathcal{F}$ and $\psi_k^*(\tau_k) = \psi_k(0)$ for some $\{\tau_k\} \subset (0,1)$. If $\psi_k(\Delta)$ does not reduce to q as $k \to \infty$, it then follows easily from the assumptions that, for infinitely many $k, \tau_k \to 1$ and $\{\psi_k^*\} \subset \mathcal{F}_a$ for some $a > 0$. By a normal family argument, we may assume, without loss of generality, that $\psi_k^* \to \psi \in \mathcal{F}_a$ (see Proposition 4 of [CHL]). Hence, from Lemma 3.4 and the above hypotheses, we obtain $\psi'(1) \in \mathrm{T}_q^{(1,0)}\partial\Omega$. This is a contradiction [Lm1].

By Lemma 3.1, we see that ψ_k is also a complex geodesic of both Ω_0 and $\Phi(D)$ when $k >> 1$. Hence, by making use of the monotonicity property of the

Kobayashi metric and this fact, we have for k>>1 that

$$\kappa_{\Phi(D)}(\phi_k^*(0), \phi_k^{*\prime}(0)) \leq \kappa_D(\phi_k(0), \phi_k'(0)) = 1$$

and

$$\kappa_{\Phi(D)}(\phi_k^*(0), \phi_k^{*\prime}(0)) = \kappa_{\Omega_0}(\phi_k^*(0), \phi_k^{*\prime}(0))$$

$$= \kappa_{\Phi^{-1}(\Omega_0)}(\phi_k(0), \phi_k'(0)) \geq \kappa_D(\phi_k(0), \phi_k'(0)) = 1.$$

Thus,

$$\kappa_{\Phi(D)}(\phi_k^*(0), \phi_k^{*\prime}(0)) = 1.$$

On the other hand, since $\kappa_{\Phi(D)}(\phi_k^*(0), \lambda_k \phi_k^{*\prime}(0)) = 1$ (because ψ_k is a complex geodesic of $\Phi(D)$), we obtain $\lambda_k = 1$. So we can conclude that ϕ_k^* is a complex geodesic of Ω when k>>1. By the uniqueness property of complex geodesics on strongly convex domains [Lm1], we therefore have $\phi_k^* = \psi_k$ for k>>1. However, from the above argument this implies that $\phi_k^*(\Delta) \to q$ as $k \to \infty$. That is a contradiction and hence completes the proof for the second assertion of our theorem.

To conclude the proof, we let ϵ be small enough so that we can choose V in the theorem to be $\Phi^{-1}(\Omega_0)$. Suppose that ϕ is an extremal mapping of D with $\phi(\Delta) \subset V$. Then by Lemma 3.1, it is also an extremal mapping for V, thus a complex geodesic of V for V is biholomorphic to the strongly convex domain Ω_0. By making use of Lemma 3.2, we see that $\Phi \circ \phi$ is a complex geodesic of Ω. Now, by the monotonicity property for the Kobayashi distance, we have, for any $\tau_1, \tau_2 \subset \Delta$, that

$$K_\Delta(\tau_1, \tau_2) = K_V(\phi(\tau_1), \phi(\tau_2)) = K_\Omega(\Phi \circ \phi(\tau_1), \Phi \circ \phi(\tau_2))$$

$$\leq K_{\Phi(D)}(\Phi \circ \phi(\tau_1), \Phi \circ \phi(\tau_2)) \leq K_D(\phi(\tau_1), \phi(\tau_2)) \leq K_\Delta(\tau_1, \tau_2).$$

Therefore ϕ is a complex geodesic of D. The proof of Theorem 4 is complete. ■

We conclude this section by presenting several corollaries of Theorem 4.

Corollary 3.1: Let D and p as in Theorem 4. Suppose that ϕ is a complex geodesic of D. If there is a sequence $\{\tau_k\} \subset \Delta$ converging to 1, such that $\phi(\tau_k) \to p$, then ϕ' is bounded near $1 \in \partial\Delta$. Thus ϕ admits a Lipschitz-1 continuous extension near 1.

Proof of Corollary 3.1: Let ϕ be as in the corollary. Then from the argument in Theorem 1 of [FR], we easily see that ϕ is continuous at 1.

By well-known estimates of the Kobayashi metric near a strongly pseudoconvex point (see [Ala], for example), we may find a neighborhood U of p and a constant C so that for every $z \in U \cap D$ and $X \in \mathrm{T}_z^{(1,0)}D$, it holds that $\kappa_D(z, X) \geq \mathrm{C}\|X\|_N/\delta(z)$. Meanwhile, we recall that ϕ is also an infinitesimal complex geodesic (see [Ab1]), i.e., $\kappa_D(\phi(\tau), \phi'(\tau)) = \kappa_\Delta(\tau, 1) = 1/(1 - |\tau|^2)$. Hence, from the fact that $\delta(\phi(\tau)) \approx 1 - |\tau|^2$ ([Ab1]), it follows easily that $\|(\phi'(\tau))_N\| < \mathrm{C}$ near 1. To finish the proof, it now suffices to show that $\|(\phi'(\tau))_T\|$ is bounded near 1. Suppose this is not the case. Then there exists a sequence $\{\tau_k\}$ converging to 1, so that $\|(\phi'(\tau_k))_T\|/\|(\phi'(\tau_k)_N\|$ goes to the infinity as $k \to \infty$. Let ϕ_k be a reparametrization of ϕ so that $\phi_k(0) = \phi(\tau_k)$ for each k. From Theorem 4, it then follows that $\phi_k(\Delta)(= \phi(\Delta)) \to p$. This is obviously a contradiction. ■

We recall that a subset E of a bounded domain D is called a holomorphic retract if there is a $h \in \mathrm{Hol}(D, D)$ with $h^2 = h$ so that $h(D) = E$. An obvious observation is that, for a holomorphic retract E, it holds that $K_E(z_1, z_2) = K_D(z_1, z_2)$ for any $z_1, z_2 \in E$. Combining this fact with Corollary 3.1 , we have the following

Corollary 3.2: Let D and p be as in Theorem 1. Suppose that E is a simply connected one dimensional holomorphic retract of D with $p \in \overline{E}$, and suppose that ϕ is a biholomorphic mapping from Δ to E with $\phi(\tau_k) \to p$ for some $\tau_k \to 1$. Then ϕ is Lipschitz-1 continuous near 1.

Remark : From the proof, we can actually see that Theorem 4, Corollary 3.1, and Corollary 3.2 hold for all bounded domains which possess the local embedding property in Lemma 3.5. In particular, we can replace D by the bounded domain of the form $D - K$, where D is as in Theorem 1 and K is a compact subset of D.

3.2. A non-degeneracy property of extremal mappings–Proof of Theorem 5:

The purpose of this section is to prove Theorem 5. The immediate application to the proof of Theorem 6 is also presented.

Theorem 5: Let D be a bounded domain in $\mathbf{C}^{n+1}$ and $p \in \partial D$ a C^3 $(\alpha > 0)$ strongly pseudoconvex point. Then there is a small neighborhood U of p and a constant C depending only on U so that for any extremal mapping $\phi \in \mathrm{Hol}(\Delta, D)$ of D with $\phi(\Delta) \subset U \cap D$, it holds that $\|(\phi'(\tau))_N\| \leq \mathrm{C}\eta(\phi)\|(\phi'(\tau))_T\|$. Here, as before, $\|\cdot\|$ stands for the euclidean norm in $\mathbf{C}^n$ and $\eta(\phi) = \max_{\xi \in \overline{\Delta}} \|\phi(\xi) - p\|$.

Our point of departure is the characterization of extremal mappings in terms of their Euler-Lagrange equations (see [Lm1] or [P]), which leads to the study of their corresponding meromorphic disks attached to a totally real submanifold. Since we are only interested in the extremes near a boundary point, the poles of the meromorphic disks can be easily controlled. Using the technique of the Riemann-Hilbert problem, we then obtain a family of non-linear (but compact) operators, whose fixed points are exactly the boundary values of our meromorphic disks. Finally, a careful analysis of those operators completes the proof of Theorem 5.

Before proceeding, we recall again that an extremal mapping ϕ of D is a holomorphic map from the unit disk Δ to D so that for any $\psi \in \mathrm{Hol}(\Delta, D)$ with $\psi(0) = \phi(0)$ and $\psi'(0) = \lambda\phi'(0)$ (where, as usual, λ denotes a real number), it holds that $|\lambda| \leq 1$. A holomorphic mapping from Δ to D is called a complex geodesic in the sense of Vesentini if it realizes the Kobayashi distance between any two points on its image (see [Ve]). For a bounded convex domain, extremal mappings coincide with complex geodesics by a result of Lempert ([Lm1]).

Proof of Theorem 5: We let $D \subset\subset \mathbf{C}^{n+1}$ and $p \in \partial D$ a C^3 strongly pseudoconvex point. We then need to show that for any extremal mapping ϕ of D, when $\phi(\Delta)$ is close enough to p, it holds that $\|(\phi'(0))_N\| = O(\eta(\phi))\|(\phi'(0))_T\|$. For this purpose, we start by constructing a C^3 strongly convex domain $\Omega \subset D$ with $\partial\Omega \cap \partial D$ being a piece of hypersurface near p. More precisely, here we should say that Ω is the biholomorphic image of a C^3 strongly convex domain. However, we will not make this distinction in what follows; for all objects involved in this section are biholomorphically invariant. Let us assume that $\phi(\Delta) \subset \Omega$. It then follows from the monotonicity of the Kobayashi metric that ϕ is also an extremal mapping of Ω (thus a complex geodesic of Ω). Now we recall a result of Lempert [Lm1], which asserts that ϕ is proper and has a C^{2-} ($\alpha \in (0,1)$) smooth extension up to $\partial\Delta$. Write $\nu(q)$ for the unit outward normal vector of Ω at q. The key fact (see [Lm1] or [P]) for our later discussion is that ϕ satisfies the Euler-Lagrange equation in the sense that there exists a C^{2-} positive function P on $\partial\Delta$ so that $\widetilde{\phi(\xi)} = P\xi\overline{\nu(\phi(\xi))}$, initially defined on $\partial\Delta$, can be holomorphically extended to Δ (this $\widetilde{\phi}$ is called the dual mapping of ϕ).

Since extremal maps are preserved under holomorphic changes of variables, we can assume, without loss of generality, that $p = 0$ and Ω is locally defined by an equation: $\rho(z) = \bar{z}_{n+1} + z_{n+1} + h(z, \bar{z})$ with $h(z, \bar{z}) = \sum_{j=1}^{n} |z_j|^2 + o(\|z\|^2)$. Moreover, a simple application of the implicit function theorem tells that we can make $h(z, \bar{z})$ depending only on $z' = (z_1, \cdots, z_n)$ and $y_{n+1} = \mathrm{Im} z_{n+1}$.

Write $\nu = (v_1, \ldots, v_{n+1})$ and define

$$W =$$

$$\left\{ w = (z, \omega) \in \mathbf{C}^{2n+1} : z \in \partial\Omega,\ z \approx 0,\ \omega = \left(\overline{v_1(z)}/\overline{v_{n+1}(z)}, \ldots, \overline{v_n(z) v_{n+1}(z)} \right) \right\}.$$

Then, by an easy calculation, it can be seen that W is defined near 0 by the equation: $w = (z', iy_{n+1}, \bar{z'}) + O(\|z\|^2)$. Thus it follows that W is totally real near 0 (this is called the Webster lemma). In fact, the real tangent space of W at 0 is spanned by $\{T_{1,r}, \ldots, T_{n,r}, T_{n+1}, T_{1,i}, \ldots, T_{n,i}\}$, where, for $j \leq n$,

$$T_{j,r} = (0, \ldots, 1, \cdots, 1, \ldots, 0),$$

$$T_{j,i} = (0, \ldots, \sqrt{-1}, \cdots, -\sqrt{-1}, \ldots, 0),$$

and

$$T_{n+1} = (0, \ldots, \sqrt{-1}, \ldots, 0).$$

Write

$$A_0 = \begin{pmatrix} T_{1,r} \\ T_{2,r} \\ \vdots \\ T_{n+1} \\ \vdots \\ T_{n-1,i} \\ T_{n,i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \sqrt{-1} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \sqrt{-1} & \dots & -\sqrt{-1} & 0 \\ \sqrt{-1} & 0 & \dots & 0 & -\sqrt{-1} \end{pmatrix}$$

and let $W^* = WA_0^{-1} = \{wA_0^{-1} : w \in W\}$. Then we have that $\mathrm{T}_0 W^* = \mathbf{R}^{2n+1} \subset \mathbf{C}^{2n+1}$. From the implicit function theorem, W^* can thus be defined by an equation: $Y = H(X)$ with $X + iY \in \mathbf{C}^{2n+1}$ and $H(0) = \mathrm{d}H(0) = 0$.

We now return to the extremal mapping ϕ (of D and Ω). Assume that $\phi(\Delta)$ is close enough to 0 so that $\Phi(\xi) = (\phi(\xi), \phi^*(\xi))$, defined by

$$\left(\phi(\xi), \overline{v_1(\phi(\xi))}/\overline{v_{n+1}(\phi(\xi))}, \dots, \overline{v_n(\phi(\xi))}/\overline{v_{n+1}(\phi(\xi))}\right),$$

stays on W for $\xi \in \partial\Delta$. Write $\Phi^*(\xi) = \Phi(\xi)A_0^{-1}$. Then we have that $\Phi^*(\partial\Delta) \subset W^*$.

Lemma 3.6: There exists a $\sigma \in \mathrm{Aut}(\Delta)$ so that $\Phi^* \circ \sigma$ has a holomorphic extension to $\Delta \setminus \{0\}$. Furthermore, $0 \in \Delta$ is a simple pole of $\Phi^* \circ \sigma$.

Proof of Lemma 3.6: Write $\tilde{\phi}$, the dual mapping of ϕ, as $(\widetilde{\phi_1}, \dots, \widetilde{\phi_{n+1}})$. We then see that $\widetilde{\phi_{n+1}}(\xi) = \xi P(\xi)\overline{v_{n+1}(\phi(\xi))}$ for $\xi \in \partial D$ and some positive function P. Since $v_{n+1}(\phi(\xi)) \approx 1$, we can conclude that the winding number of $\widetilde{\phi_{n+1}}$ is 1. So it just has a simple zero on Δ, say a. Take $\sigma \in \mathrm{Aut}(\Delta)$ with $\sigma(0) = a$. Then $\widetilde{\phi_{n+1}} \circ \sigma$ has a simple zero at $0 \in \Delta$. Thus $\Phi \circ \sigma$ can be extended to Δ as

$$\left(\phi \circ \sigma, \frac{\widetilde{\phi_1} \circ \sigma}{\widetilde{\phi_{n+1}} \circ \sigma}, \dots, \frac{\widetilde{\phi_n} \circ \sigma}{\widetilde{\phi_{n+1}} \circ \sigma}\right),$$

which is obviously meromorphic on Δ with a simple pole at 0. Since Φ^* differs from Φ only by a linear transformation, we see that the proof of Lemma 3.6 is complete. ■

For simplicity, let us still write $\Phi^*(\xi) = X(\xi) + iY(\xi)$ ($\xi \in \partial\Delta$)for $\Phi^* \circ \sigma$ in what follows. Note that $\Phi^*(\partial\Delta) \subset M^*$. It follows that $Y(\xi) = H(X(\xi))$ ($\xi \in \partial\Delta$). Let $\xi = e^{i\theta}$ and take the derivative with respect to θ. We then see that $\frac{dY}{d\theta} = \frac{dX}{d\theta}\frac{\partial H}{\partial X}$, where $\frac{\partial H}{\partial X}$ is the Jacobian of H. So,

$$\frac{d\Phi^*}{d\theta} = \frac{dX}{d\theta} + i\frac{dY}{d\theta} = \frac{dX}{d\theta}(I_{2n+1} + i\frac{\partial H}{\partial X}),$$

or

$$\text{(3.2.1)} \qquad \mathrm{Im}(\frac{dX}{d\theta} + i\frac{dY}{d\theta})(I_{2n+1} + i\frac{\partial H}{\partial X})^{-1} = \mathrm{Im}\frac{dX}{d\theta} = 0.$$

Here I_{2n+1} denotes the identical $(2n+1)\times(2n+1)$ matrix and $\|g\| = \max_{\xi\in\partial\Delta}|g(\xi)|$ for each function g in the Banach space $L^\infty(\partial\Delta)$. An easy fact is that $\|X(e^{i\theta})\| << 1$ when $\eta(\phi) \approx 0$.

Consider the Riemann-Hilbert problem

$$\text{(3.2.2)} \qquad \mathrm{Im}\left(Q(X,\xi)(I_{2n+1} + i\frac{\partial H}{\partial X})^{-1}\right) = 0, \quad \xi \in \partial\Delta,$$

with $Q(X,\xi)$ holomorphic on $\xi \in \Delta$, L^2 integrable on $\partial\Delta$, and $\mathrm{Re}(Q(X,0)) = I_{2n+1}$.

Lemma 3.7: When $\|X\| << 1$, then (3.2.2) has a unique solution Q. Moreover, $Q^{-1}(X,\xi)$ exists and $\|Q(X,e^{i\theta}) - I_{2n+1}\|_2$, $\|Q^{-1}(X,e^{i\theta}) - I_{2n+1}\|_2 = O(\|X\|)$. Here, we write $\|\circ\|_2$ for the L^2 norm of the Hilbert space $L^2(\partial\Delta)$.

Proof of Lemma 3.7: Write $(I_{2n+1} + i\frac{\partial H(X)}{\partial X})^{-1} = e_1 + ie_2$ and $Q(X,\xi) = q_1(X,\xi) + iq_2(X,\xi)$. Then we see that $q_1(X,0) = I_{2n+1}$, $\|e_2(X,e^{i\theta})\|_2 = O(\|X\|)$, and (3.2.2) is equivalent to

$$\text{(3.2.3)} \qquad q_1e_2 + q_2e_1 = 0.$$

Since $q_1 = -\mathcal{H}(q_2) + I_{2n+1}$, where $\mathcal{H}$ is the standard Hilbert transform on $\partial\Delta$, (3.2.3) can therefore be written as

$$-\mathcal{H}(q_2)e_2e_1^{-1} + q_2 = -e_2e_1^{-1}.$$

So, when $\|X\| << 1$, it follows that $q_2 = (-\mathcal{H}(\circ) \times (-e_2 e_1^{-1}) + I_{2n+1})^{-1}(-e_2 e_1^{-1})$ and

$$\|q_2\|_2 \leq \frac{1}{(1 - \|e_2 e_1^{-1}\|_2)} \|e_2 e_1^{-1}\|_2 = O(\|X\|).$$

Thus Q is uniquely determined and $\|Q(X,\xi) - I_{2n+1}\|_2 \leq \|q_2\|_2 + \|\mathcal{H}(q_2)\|_2 = 2\|q_2\|_2 = O(\|X\|)$.

We now consider the following equation with respect to Q^*:

$$\mathrm{Im}\left((I_{2n+1} + i\frac{\partial H(X)}{\partial X})Q^*(X,\xi)\right) = 0, \qquad \text{with } \mathrm{Re}(Q^*(X,0)) = I_{2n+1}.$$

Similarly, we can obtain a unique solution with $\|Q^*(X,\xi) - I_{2n+1}\|_2 = O(\|X\|)$. Since the holomorphic matrix $Q \times Q^*$ has real values on $\partial\Delta$, it thus follows from the Schwarz reflection principle that $Q(X,\xi) \times Q^*(X,\xi) = C(X)$, some real constant matrix. Here, we remark that, to apply the Schwarz reflection principle, we need obtain $Q(X,\xi)Q^*(X,\xi) \in L^{\ell}(\partial\Delta)$ for some $\ell > 1$. But this can be easily seen by solving the equation (3.2.2) in the space $L^{\ell}(\partial\Delta)$ with $\ell >> 1$. We now notice that $\|Q(X,0) - I_{2n+1}\| \leq \frac{1}{2\pi}\int_{\partial\Delta} \|\frac{Q(X,\xi)-I_{2n+1}}{\xi} d\xi\| = O(\|X\|)$ and $\|Q^*(X,0) - I_{2n+1}\| = O(\|X\|)$ as $\|X\| \to 0$ (by the Hölder inequality). We see, especially, that $C(X) = Q(X,0)Q^*(X,0) = I_{2n+1} + O(\|X\|)$ as $\|X\| \to 0$. Hence $C(X)$ is invertible in case $\|X\| << 1$. This completes the proof of Lemma 3.7; for $Q^{-1}(X,\xi) = C^{-1}(X)Q^*(X,\xi)$. ∎

Now, by making use of Lemma 3.7, (3.2.1) becomes

$$\mathrm{Im}\left(\frac{dX}{d\theta} + i\frac{dY}{d\theta}\right) Q^{-1}(X,\xi) = 0, \quad \text{for } \xi \in \partial\Delta,$$

i.e,

$$\mathrm{Re}\left(\xi\frac{d\Phi^*}{d\xi}Q^{-1}(X,\xi)\right) = 0.$$

Note that $\xi\frac{d\Phi^*}{d\xi}Q^{-1}(X,\xi)$ is holomorphic on $\Delta \setminus \{0\}$ and has at most a simple pole at 0. We can conclude that

$$\xi\frac{d\Phi^*}{d\xi}Q^{-1}(X,\xi) = \frac{\alpha}{\xi} - \bar{\alpha}\xi + i\beta,$$

where α is a constant complex vector and β is a constant real vector (with respect to ξ, but depending only on X). In fact, since $\Phi(\xi) = \Phi^* \times A_0 = (\phi, \frac{\phi^{**}}{\xi})$ with $\phi^{**} = \xi\phi^*$ holomorphic on Δ by Lemma 3.6, it follows that:

$$\alpha = \lim_{\xi\to 0} \xi^2 \frac{d\Phi^*}{d\xi} Q^{-1}(X,\xi) = (0, -\phi^{**}(0))A_0^{-1}Q^{-1}(X,0). \tag{3.2.4}$$

Write $R(X,\xi) = Q(X,\xi)(I_{2n+1} + i\frac{\partial H(X)}{\partial X})^{-1}$ for $\xi \in \partial\Delta$ (we note that R is real). By Lemma 3.7, it then holds that $\|R - I_{2n+1}\|_2 = O(\|X\|)$. Therefore, the Hölder inequality implies that $\int_0^{2\pi} R(X,\xi)d\theta = 2\pi I_{2n+1} + O(\|X\|)$ is invertible when $\|X\| << 1$. On the other hand, we have

$$\frac{dX}{d\theta} = \frac{d\Phi^*}{d\theta}(I_{2n+1} + i\frac{\partial H(X)}{\partial X})^{-1} = i\xi\frac{d\Phi^*}{d\xi}(I_{2n+1} + i\frac{\partial H(X)}{\partial X})^{-1}$$

$$= i(\frac{\alpha}{\xi} - \bar{\alpha}\xi + i\beta)Q(X,\xi)(I_{2n+1} + i\frac{\partial H(X)}{\partial X})^{-1} = i(\frac{\alpha}{\xi} - \bar{\alpha}\xi + i\beta)R(X,\xi).$$

Integrating both sides with respect to θ, we obtain

$$0 = \int_0^{2\pi} (\frac{\alpha}{\xi} - \bar{\alpha}\xi + i\beta)R(X,\xi)d\theta.$$

Thus,

$$\beta = \sqrt{-1}\left(\int_0^{2\pi} (\frac{\alpha}{\xi} - \bar{\alpha}\xi)R(X,\xi)d\theta\right)\left(\int_0^{2\pi} R(X,\xi)d\theta\right)^{-1}. \tag{3.2.5}$$

Here, as usual, we identify $\xi \in \partial\Delta$ with $e^{i\theta}$. In particular, we easily see that $\alpha,\ \beta = O(\|X\|)$; for by the Cauchy formula and the Hölder inequality, it holds that $\phi^{**}(0) = O(\|X\|)$ (since $\|\phi^* * (\xi)\|$ with $\xi \in \partial\Delta$ is of quantity $O(\|X\|)$).

Consider now the following differential equation with parameters $\gamma \in \mathbf{C}^n$ and $X_0 \in \mathbf{R}^{2n+1}$:

(3.2.6)

$$\frac{dX(\xi,\gamma,X_0)}{d\theta} = i(\frac{\alpha(X,\gamma)}{\xi} - \overline{\alpha(X,\gamma)}\xi + i\beta(X,\gamma))R(X,\xi), \quad \text{with } X(1) = X_0,$$

or

$$X(\xi,\gamma,X_0) = i\int_0^{\theta} (\frac{\alpha(X,\gamma)}{\xi} - \overline{\alpha(X,\gamma)}\xi + i\beta(X,\gamma))R(X,\xi)d\theta + X_0, \tag{3.2.6$'$}$$

where $\xi = e^{i\theta}$,

$$\alpha(X,\gamma) = (0,\gamma)A_0^{-1}Q^{-1}(X,0), \tag{3.2.7}$$

and $\beta(X,\gamma)$ is given by (3.2.5).

Lemma 3.8: For any extremal mapping ϕ with $\phi(\Delta) \approx 0$, there correspond an automorphism σ of Δ, a $\gamma \approx 0$, and an $X_0 \approx 0$ so that the previously defined X is a solution of (3.2.6). Conversely, for any $\gamma,\ X_0 \approx 0$, (3.2.6) can be uniquely solved, and each of its solutions gives an extremal mapping ϕ of Ω with $\phi(\Delta) \approx 0$ and the last component of its dual mapping having a simple pole at 0. Moreover, the solutions of (3.2.6) are uniformly Hölder-$\frac{1}{2}$ continuous with respect to the parameters α and γ. In fact, denoting by $\| \circ \|_{\frac{1}{2}}$ the Hölder-$\frac{1}{2}$ norm in the Banach space $C^{\frac{1}{2}}(\partial\Delta)$, defined by

$$\|g\|_{\frac{1}{2}} = \|g\| + \sup_{\xi_1,\xi_2} \frac{|g(\xi_1) - g(\xi_2)|}{|\xi_1 - \xi_2|^{\frac{1}{2}}}, \quad \text{with } g \in C^{\frac{1}{2}}(\partial\Delta),$$

then for each solution X of (1.5), we have $\|X\|_{\frac{1}{2}} = O(\|X\|)$.

Proof of Lemma 3.8: The first part of the lemma follows from the above arguments.

We now present the proof of the last part of the lemma. To this aim, let $X(\xi,\gamma,X_0)$ be a solution of (3.2.6) with $\|X\| << 1$ and let $\Phi^*(\xi,\gamma,X_0) = X(\xi,\gamma,X_0) + iH(X(\xi,\gamma,X_0))$. Then we know from (3.2.6) that

$$\frac{d\Phi^*(\xi,\gamma,X_0)}{d\xi} = \left(\frac{\alpha(X,\gamma)}{\xi^2} - \overline{\alpha(X,\gamma)} + i\beta(X,\gamma)\frac{1}{\xi} \right) Q(X,\xi).$$

So (3.2.4) still holds. Notice that Φ^* must have a meromorphic extension to Δ (with at most a simple pole at the origin). Using the Cauchy formula and the Hölder inequality, we know from (3.2.4) that α and thus β (by (3.2.5)) are of $O(\|X\|)$. Now we note that $\|R\|_2 = O(1)$ and

$$\|X(e^{i\theta_1},\gamma,X_0) - X(e^{i\theta_2},\gamma,X_0)\| \leq \| \int_{\theta_1}^{\theta_2} (\frac{\alpha(\gamma)}{\xi} - \overline{\alpha(\gamma)}\xi + i\beta(\gamma))R(X,\xi)d\theta\|$$

$$\leq C(\|\alpha\| + \|\beta\|)\int_{\theta_1}^{\theta_2} \|R(X,\xi)\| d\theta \leq C(\|\alpha\| + \|\beta\|)\|R\|_2 \|\theta_1 - \theta_2\|^{1/2}$$

$$= O(\|X\|)\|\theta_1 - \theta_2\|^{1/2}.$$

It therefore follows that

$$\sup_{\xi_1,\xi_2} \frac{\|X(\xi_1,\gamma,X_0) - X(\xi_2,\gamma,X_0)\|}{|\xi_1 - \xi_2|^{1/2}} = O(\|X\|).$$

Thus, the Hölder -$\frac{1}{2}$ norm of X,

$$\|X\|_{\frac{1}{2}} = \|X\| + \sup_{\xi_1,\xi_2} \frac{\|X(\xi_1,\gamma,X_0) - X(\xi_2,\gamma,X_0)\|}{|\xi_1 - \xi_2|^{1/2}},$$

is bounded by $C\|X\|$ with some constant C independent of γ and X_0.

It remains to prove the existence of the solutions of (3.2.6)and study their behavior. For this purpose, we first notice that, by making use of the just obtained result and by solving (3.2.2) in the Hölder-$\frac{1}{2}$ space $C^{\frac{1}{2}}$, we see that the holomorphic matrix $Q(X,\xi)$ is also uniformly Hölder-$\frac{1}{2}$ continuous up to the boundary. Moreover it can be similarly seen that $\|Q - I_{2n+1}\|_{\frac{1}{2}}$ and thus $\|R(X,\xi) - I_{2n+1}\|_{\frac{1}{2}} = O(\|X\|)$. Now consider the operator

$$F: \qquad C^{\frac{1}{2}}(\partial\Delta) \times \mathbf{C}^n \times \mathbf{R}^{2n+1} \to C^{\frac{1}{2}}(\partial\Delta);$$

$$F(X,\gamma,X_0) = i\int_0^{\theta} (\frac{\alpha(X,\gamma)}{\xi} - \overline{\alpha(X,\gamma)}\xi + i\beta(X,\gamma))R(X,\xi)d\theta + X_0.$$

From the above discussions, it follows that in case γ, and $X_0 \approx 0$, we then have $\mathrm{d}_X F(0) \approx 0$. Hence, by the implicit function theorem in the Banach space, (3.2.6) and thus $(3.2.6)'$ can be uniquely solved for small γ and X_0. Now, for each solution $X(\xi,\gamma,X_0)$, let $\Psi^*(\xi) = X(\xi,\gamma,X_0) + iH(X(\xi,\gamma,X_0))$. Then

$$\frac{d\Psi^*}{d\xi} = \left(\frac{\alpha}{\xi^2} - \bar{\alpha} + \frac{i\beta}{\xi}\right) Q(X,\xi).$$

Denote by $(\psi,\psi^*) = \Psi^* A_0$, where ψ maps $\partial\Delta$ to $\mathbf{C}^{n+1}$. It follows easily that

$$(3.2.7)' \qquad \psi'_{\xi} = (\frac{\alpha(X,\gamma)}{\xi^2} - \overline{\alpha(X,\gamma)} + i\beta(X,\gamma)\frac{1}{\xi})Q(X,\xi)B.$$

Here we write B for the $(2n+1) \times (n+1)$ matrix, formed by the first (n+1) columns of A_0. Noting that $\alpha(X,\gamma)Q(X,0)B = (0,\gamma)A_0^{-1}B = 0$, we see that $p = 0$ can be at most a simple pole of ψ'. Since ψ is well-defined on $\partial\Delta$, we can conclude that ψ has a holomorphic extension to Δ. Meanwhile, it can be verified that $\psi(\partial\Delta) \subset \partial D$ and ψ satisfies the Euler-Lagrange equation. We thus conclude that ψ is an extremal map of Ω (and of D, in fact)([Lm1], Theorem 4 of the last section) with the property described in the lemma. The proof of Lemma 3.8 is complete. ■

We now are in a position to finish the proof of Theorem 5. For the sake of brevity, we retain the above notation and assume that σ in Lemma 3.6 is the identity.

Let ϕ be an extremal map of D with $\phi(\Delta)$ close to 0. First, we notice that both sides in $(3.2.7)'$, with ψ being replaced by ϕ, are holomorphic on $\Delta \setminus \{0\}$. We therefore have

$$\phi'(\xi) = \left(\frac{\alpha(\gamma)}{\xi^2} - \overline{\alpha(\gamma)} + i\beta(\gamma)\frac{1}{\xi}\right) Q(X,\xi)B,$$

for $\xi \in \Delta \setminus \{0\}$. Writing $Q_1(X,\xi) = Q(X,\xi) - Q(X,0)$ and $Q_2(X,\xi) = Q_1(X,\xi) - Q'_\xi(X,0)\xi$, we then obtain

$$\phi'_\xi = \left(\frac{\alpha Q_2(X,\xi)}{\xi^2} - \overline{\alpha(\gamma)}Q(X,\xi) + i\beta Q_1(X,\xi)\frac{1}{\xi}\right) B; \tag{3.2.8}$$

for ϕ is holomorphic on Δ.

Lemma 3.9: We have the estimates $\frac{Q_1(X,\xi)}{\xi} = O(\|X\|)$ and $\frac{Q_2(X,\xi)}{\xi^2} = O(\|X\|)$ as $\|X\| \to 0$.

Proof of Lemma 3.9: From the definition, we see that $\frac{Q_1(X,\xi)}{\xi}$ and $\frac{Q_2(X,\xi)}{\xi^2}$ are holomorphic on Δ. So, by the maximal principle, we have only to show that they converge uniformly to the 0-matrix at the rate of $\|X\|$, when $\xi \in \partial\Delta$ and $\|X\| \to 0$. But this follows obviously from the facts that $Q(X,\xi) = I_{2n+1} + O(\|X\|)$ and $Q'_\xi(X,0) = O(\|X\|)$ (by the Cauchy formula and Hölder inequality). ■

Note that $\alpha = (0,\gamma)A_0^{-1}Q(X,0) = (0,\gamma)A_0^{-1} + O(\|\gamma\|\|X\|)$ and $\beta = O(\|\gamma\|)$ by (3.2.5). It can be verified that (3.2.8) may be written as

$$\frac{1}{\|\gamma\|}\phi'(\xi) = -\overline{\frac{(0,\gamma)}{\|\gamma\|}A_0^{-1}} \times B + o(\|X\|),$$

as $\|X\| \to 0$. Now a direct computation shows that

$$A_0^{-1} = 1/2\begin{pmatrix} 1 & 0 & \dots & 0 & -i \\ 0 & 1 & \dots & -i & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -2i & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & i & 0 \\ 1 & 0 & \dots & 0 & i \end{pmatrix}.$$

So, writing $\overline{\frac{\gamma}{\|\gamma\|}} = (a_1, \dots, a_n)$, we then have

$$\frac{1}{\|\gamma\|}\phi'(\xi) = -\overline{\frac{(0,\gamma)}{\|\gamma\|}A_0^{-1}} \times B + O(\|X\|) = -(a_n, a_{n-1}, \dots, a_1, 0) + O(\|X\|),$$

when $\|X\| \to 0$. Hence, we obtain

$$\frac{|\phi'_{n+1}(\xi)|}{\|\phi'(\xi)\|} = O(\|X\|),$$

and

$$\frac{|(\phi'_1(\xi), \dots, \phi'_n(\xi))|}{\|\phi'(\xi)\|} = 1 + O(\|X\|).$$

Since

$$\frac{\|(\phi'(\xi))_N\|}{\|(\phi'(\xi))\|} = \frac{|\phi'_{n+1}(\xi)|}{\|\phi'(\xi)\|} + O(\|X\|)$$

and

$$\frac{\|(\phi'(\xi))_T\|}{\|\phi'(\xi)\|} = \frac{|(\phi'_1(\xi), \dots, \phi'_n(\xi))|}{\|\phi'(\xi)\|} + O(\|X\|) = 1 + O(\|X\|),$$

we finally conclude that

$$\|(\phi'(\xi))_N\| = O(\|X\|)\|(\phi'(\xi))_T\|, \quad \text{for } \xi \in \overline{\Delta},$$

as $\|X\| \to 0$. This completes the proof of Theorem 5; for $\|X\| \approx \eta(\phi)$. ■

We conclude this section by proving Theorem 7.

Theorem 7: Let D be a bounded C^3 strongly convex domain in $\mathbf{C}^n$ with $\alpha > 0$. For any given $p \in \partial D$ and complex vector $v \in \mathrm{T}^{(1,0)}\mathbf{C}^n$, but not in $\mathrm{T}_p^{(1,0)}\partial D$, there exists an extremal mapping ϕ so that $\phi(1) = p$ and $\phi'(1) = \lambda v$ for some real number λ (this ϕ then must be uniquely determined up to an automorphism of Δ according to Lempert [Lm1]).

Proof of Theorem 6: Let $D \subset\subset \mathbf{C}^n$ be a C^3 strongly convex domain and $p \in \partial D$. For any complex vector v, which is not contained in $T_p^{(1,0)}\partial D$, we then need to find an extremal mapping of D so that $\phi(1) = p$ and $\phi'(1)$ is different from v by a complex number. To this end, we choose a sequence $\{z_j\} \subset D$ converging to p and choose a sequence of normalized extremal mappings $\{\phi_j\} \subset \mathcal{F}(D)$ so that for each j, it holds that $\phi_j(\tau_j) = z_j$ with some $\tau_j \in (0, 1)$ and $\phi_j'(\tau_j) = \lambda_j v$ with $\lambda_j \in \mathbf{C}$. Since v is independent of j and is not contained in the complex tangent space of ∂D at p, it follows from Corollary 3, that $\inf_j \phi_j(\Delta) > 0$. In light of Lemma 3.4, we therefore see that there is a subsequence of $\{\phi_j\}$, which converges to an extremal mapping ϕ in the topology of $C^1(\overline{\Delta})$. Noting that $\tau_j \to 1$, we can thus conclude that $\phi'(1) = \lambda v$ for some $\lambda \in \mathbf{C}$. The proof is complete. ■

§ 3.3 Regularity of holomorphic retracts and iterates of holomorphic mappings–Proof of Theorem 3:

In this section, we will focus on the proof of Theorem 3. We will make decisive use of Theorems 4 and 5. The key step, as mentioned in §1.2, is to prove the following fixed point theorem:

Theorem 3.1: Let $D \subset\subset \mathbf{C}^n$ be a contractible strongly pseudoconvex domain with C^3 boundary and let M be a holomorphic retract of D. Suppose that $f \in \mathrm{Hol}(M, M)$ is elliptic, i.e, no subsequence of $\{f^k\}$ diverges to the boundary of M. Then f has a fixed point in M.

The main idea of the proof of this theorem is to obtain certain regularity results concerning holomorphic retracts so that the Lefschetz fixed point theorem can be applied. The argument will be carried out through several propositions, which are of interest in their own right.

Let $M \subset D$ be a holomorphic retract.We first recall that the Kobayashi metric and the Kobayashi distance of M are the same as those inherited from D. Another useful result regarding holomorphic retracts is a theorem of Rossi (see [Ab1] for example), which states that all holomorphic retracts of D are closed complex sub-manifolds of D.

We now start with Proposition 3.2, which will play a crucial role in the whole discussion.

Proposition 3.2: Let $D \subset\subset \mathbf{C}^n$ be either a smooth pseudoconvex domain or a taut domain with a Stein neighborhood basis. Let $p \in \partial D$ be a strongly pseudoconvex point with at least C^3 smoothness. Suppose that $M \subset D$ is a holomorphic retract with complex dimension greater than 1 and suppose that $p \in \partial M$. Then for any neighborhood U of p, there is a C^{2-} complex geodesic ϕ of D with $\phi(\Delta) \subset U \cap M$ and $\phi(1) = p$.

Proof of Proposition 3.2: Choose a sequence $\{z_j\} \subset M$ converging to p and define

$$t_j = \sup_{v \in T^{(1,0)}_{z_j} M} \frac{\|v_N\|}{\|v_T\|}.$$

Then we first claim that $\inf_j(t_j) > 0$, i.e, M intersects ∂D transversally at p. If that is not the case, we may just assume that $t_j \to 0$. Then we let

$$M_j$$

$$= \cup\{\phi(\Delta) : \ \phi \text{ is extremal with respect to D}, \phi(0) = z_j, \text{ and } \phi'(0) \in T^{(1,0)}_{z_j} M\}.$$

We first note that M_j is a non-empty set by the tautness of D. In light of the preservation principle (Theorem 4), we see that, for every C^3 strongly convex domain $\Omega \subset D$ with $\partial\Omega \cap \partial D$ being a piece of hypersurface near p, when $j >> 1$, it holds that $t_j << 1$ and each ϕ in the definition of M_j stays in Ω and thus $M_j \subset \Omega$. Therefore each ϕ, described in the definition of M_j, is also an extremal mapping of Ω. Now, we notice the tautness of M and the uniqueness property of extremal mappings in Ω. We see, by the fact that each extremal map of M is also extremal with respect to D, that M_j is also a subset of M. We now need to use Lempert's spherical representation $\Psi_j(z_j, \circ)$ of Ω with the base point z_j, i.e, we define the map $\Psi_j(z_j, \circ)$ from the closed unit ball $\overline{\mathbf{B}}_n$ to $\bar{\Omega}$ by $\Psi_j(z_j, 0) = z_j$ and $\Psi_j(z_j, b) = \psi_{z_j}(b, \|b\|)$ for each $b \in \mathbf{B}^n$. Here $\psi_{z_j}(b, \xi)$ stands for the unique extremal mapping of Ω with $\psi_{z_j}(b, 0) = z_j$ and $\psi'_{z_j}(b, 0) = \lambda b$ for some positive λ. Writing $E = \{v \in T^{(1,0)}_{z_j} M : \ \|v\| \leq 1\}$, we then get $M_j = \Psi_j(E)$. Notice that Ψ_j is a homeomorphism (in fact, it is a C^{2-} diffeomorphism on $\mathbf{B}_n \setminus \{0\}$, as showed in [Lm2]) and notice that E is a closed submanifold of $\overline{\mathbf{B}}_n$ with real dimension equal to 2 $\dim_{\mathbf{C}} M$. We therefore see that M_j is a closed open subset of M. From the connectedness of M (since all domains in this thesis are automatically assumed to be connected), it hence follows that $M = M_j$. That is a contradiction; for Ω can be made arbitrarily small.

So, there is an $\epsilon_0 > 0$ such that $t_j > \epsilon_0$ for every $j >> 1$. Pick two independent unit vectors v_1 and v_2 in the complex tangent space of M at z_j. By the above claim and by passing to a simple linear combination, we may assume that $(v_1)_N = 0$ and $\|(v_2)_N\| > \epsilon_0 \|(v_2)_T\|$. Let $v(t) = \frac{v_1 + tv_2}{\|v_1 + tv_2\|}$. Then it is easy to see that

$$\frac{\|(v(t))_N\|}{\|(v(t))_T\|} = \frac{\|t(v_2)_N\|}{\|(v_1)_T + t(v_2)_T\|},$$

can be made to be any number between 0 and ϵ_0 by just varying t.

To finish the proof of the proposition, we let U be a small neighborhood of p and construct a C^3 strongly convex domain $\Omega \subset D \cap U$ with $\partial\Omega \cap \partial D$ being a piece of hypersurfaces near p. Again, by making use of Theorem 4, for $j >> 1$ and some $\epsilon << 1$, we can find a complex geodesic ϕ_j of D with $\phi(0) = z_j$, $\phi'(0) \in T^{(1,0)}_{z_j} M$, $\phi_j(\Delta) \subset \Omega$, and $\|(\phi'_j(0))_T\| = \epsilon \|(\phi'_j(0))_N\|$. As argued in Theorem 6, since ϵ is independent of j, after a normalization, Theorem 5 indicates that a subsequence of $\{\phi_j\}$ will converge to a complex geodesic ϕ of D (and also Ω) in the topology of C^1. Noting that $\phi_j(\Delta) \subset M$ for each j, we thus conclude that $\phi(\Delta) \subset M \cap \Omega$ and $\phi(1) = p$. Finally, the regularity of ϕ follows from the reflection principle [Lm1]. ∎

We now turn to the regularity result for holomorphic retracts.

Proposition 3.3: Let $D \subset\subset \mathbf{C}^n$ be either a smooth pseudoconvex domain or a taut domain with a Stein neighborhood basis. Suppose that $p \in \partial D$ is a strongly pseudoconvex point with C^k smoothness ($k > 2$) and suppose that M is a holomorphic retract of D with complex dimension greater than 1. If $p \in \overline{M}$, then $\overline{M}$ is a complex submanifold with a $C^{(k-1)-}$ smooth boundary near p.

Proof of Proposition 3.3: As before, we first construct a small C^k strongly convex domain Ω with $\partial D \cap \partial\Omega$ being an open subset of $\partial\Omega$ near p. By Proposition 3.2, we have a complex geodesic ϕ of D, M, and Ω, staying close to p , and with $\phi(1) = p$. Let $z = \phi(0)$ and $v_0 = \frac{\phi'(0)}{\|\phi'(0)\|}$. By Theorem 5, it holds that $\|(\phi'(0))_N\| << \|(\phi'(0))_T\|$. Hence, from Theorem 4, it follows that all extremal mappings of D starting from z and with the initial velocity close to v_0 should also stay in Ω. To be more precise, by shrinking ϕ if necessary, there exists a small $\epsilon > 0$ so that, for each extremal mapping ψ with $\psi(0) = z$ and $\|\frac{\psi'(0)}{\|\psi'(0)\|} - v_0\| < \epsilon$, then $\psi(\Delta) \subset \Omega$. Write $E^* = \{v \in \overline{\mathbf{B}_n} : v \in T_z^{(1,0)}M, \ \|\frac{v}{\|v\|} - v_0\| < \epsilon|\}$ and still denote by $\Psi(z, \circ)$ the spherical representation of Ω with the base point z. Since E^* is a submanifold of $\overline{\mathbf{B}}_n$ with smooth boundary near v_0, hence, by a theorem of Lempert, $M^* = \Psi(z, E^*)$ is a submanifold with $C^{(k-1)-}$ boundary near p, whose real dimension is obviously 2 $\dim_{\mathbf{C}} M$. As we have argued before, by noting the fact that all extremal mappings of M are also extremal with respect to D and the uniqueness of extremal mappings in strongly convex domains, we can conclude that $M^* \subset \overline{M}$. Now, to complete the proof of the proposition, we need only show that for some small neighborhood U^* of p, it holds that $U^* \cap \overline{M} = U^* \cap M^*$.

For this purpose, we proceed by seeking a contradiction if there is no such U^*. Then we can find a sequence $\{z_j\} \subset M \setminus M^*$, which converges to p. Choose U_0, a small neighborhood of p, with $U_0 \cap M^*$ being a simply connected submanifold with smooth boundary, and choose a sequence $\{w_j\} \subset M^*$, converging to p.

From an estimate of the Kobayashi distance $K_D(\cdot, \cdot)$ of D (see [Ab1], for example), we know that

(3.3.1)

$$K_D(z_j, w_j) \leq -\frac{1}{2}\log\delta(z_j) - \frac{1}{2}\log\delta(w_j) + \frac{1}{2}\log(|z_j - w_j| + \delta(z_j) + \delta(w_j)) + C,$$

with C independent of j. On the other hand, since M is connected, there is a curve $\gamma(t)$ on M, connecting z_j to w_j, so that

$$K_D(z_j, w_j) = K_M(z_j, w_j) \geq \int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt - 1.$$

Here $\kappa_D(z, v)$ denotes the Kobayashi metric of D at z and in the direction v. We remark that such a curve must intersect the boundary $U_0 \cap M^*$ if we choose U_0 small enough. Let t_0 be such that $\gamma(t_0) \in \partial U_0 \cap M^*$ but $\gamma(t) \notin U_0 \cap M^*$ for $t < t_0$. Then we see that

$$\int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt$$

$$= \int_0^{t_0} \kappa_D(\gamma(t), \gamma'(t))dt + \int_{t_0}^1 \kappa_D(\gamma(t), \gamma'(t))dt \geq K(z_j) + K(w_j), \tag{3.3.2}$$

where $K(z) = \inf_{w \in \partial U_0 \cap M^*} K_D(z, w)$. Now, from the strong pseudoconvexity of D at p, it follows that (see [Ab1], for example) $K(z) \geq -\frac{1}{2} \log \delta(z) + C$. Thus, combining (3.3.1) with (3.3.2), we arrive at

$$\log(|z_j - w_j| + \delta(z_j) + \delta(w_j)) \geq C.$$

Since C is independent of j and $|z_j - w_j| + \delta(z_j) + \delta(w_j) \to 0$, we obtain a contradiction. Therefore the proof of Proposition 3.3 is complete. ■

Proposition 3.4: Let $D \subset\subset \mathbf{C}^n$ be a C^k strongly pseudoconvex domain with $k \geq 3$. Suppose that M is a holomorphic retract of D with complex dimension greater than 1. Then the following holds:

(1) Every automorphism of M has $C^{(k-1)-}$ smooth extension up to $\overline{M}$.

(2) Let $\{f_j\}_j,\ f \subset \mathrm{Aut}(M)$ with $\{f_j\}$ converging to f uniformly on compacta. Then it follows that $f_j \to f$ in the topology of $C^{k-1-}(\overline{M})$.

Proof of Proposition 3.4: First of all, Proposition 3.3 tells that M is a complex submanifold with $C^{(k-1)-}$ boundary. Thus, it makes sense to talk about the regularity (less than $C^{(k-1)-}$) extension up to the boundary for its automorphisms.

Choose $p \in \partial M$. By using Proposition 3.2, we can find a sequence of complex geodesics $\{\phi_j\}$ of M with $\phi_j(\Delta)$ shrinking to p as $j \to \infty$ and with $\phi_j(1) = p$.

Let f be an automorphism of M. Then we claim that the diameter of $f \circ \phi_j(\Delta)$ goes to 0 as $j \to \infty$. If that is not the case, then since $\{f \circ \phi_j\}$ are also complex geodesics, we may assume, without loss of generality, that $f \circ \phi_j \in \mathcal{F}_D$ for each j. Thus the $f \circ \phi_j$'s can be easily shown to be uniformly Hölder-$\frac{1}{4}$ continuous on $\overline{\Delta}$ (see [CHL], for example). Hence, by passing to a subsequence, we may assume that $f \circ \phi_j$ converges uniformly to certain complex geodesic ϕ of D. This implies that there is a sequence $\{z_j\} \to p$ with $f(z_j) \to z \in M$, and this contradicts the properness of f.

The rest of the argument for (1) is now similar to that in [Lm1]. For simplicity, we assume that $\phi_j(\overline{\Delta})$ converges to $q \in \partial D$. As we did before, construct two small C^k strongly convex domains Ω_1 and Ω_2 near p and q, respectively. Choose $j >> 1$ so that ϕ_j and $f \circ \phi_j$ are, respectively, complex geodesics of Ω_1 and Ω_2. Denote by Ψ_1 the spherical representation of Ω_1 based at $z_j = \phi_j(0)$, and by Ψ_2 the spherical representation of Ω_2 based at $z_j^* = f \circ \phi_j(0)$. Then

$$f(z) = \Psi_2\left(z_j^*, \frac{df(\phi_j(0))\Psi_1^{-1}(z_j, z)}{\|df(\phi_j(0))\Psi_1^{-1}(z_j, z)\|}\|\Psi_1^{-1}(z_j, z)\|\right)$$

for $z \approx p$. Since Ψ_1 and Ψ_2 give the local coordinate charts of M at p and q, respectively, we see that f has the same regularity at p as M does at p and q. Because p is arbitrary, we have obtained the proof for (1).

To prove (2), we still pick an arbitrary boundary point p of M, and write $q = f(p)$. Define similarly Ω_1, Ω_2, ϕ, Ψ_1 and Ψ_2. Using the fact that f_j converges uniformly to f on a small neighborhood of $z_0 = \phi(0)$, we know, by Theorem 4, that $f_j \circ \phi$ is also a complex geodesic of Ω_2 for $j >> 1$. Denote by $\Psi_2(z_j, \circ)$ the spherical representation of Ω_2 at $z_j = f_j(z_0)$ when $j >> 1$. Then we see that

$$f_j(z) = \Psi_2\left(z_j, \frac{df_j(z_0)\Psi_1^{-1}(\phi(0), z)}{\|df_j(z_0)\Psi_1^{-1}(\phi(0), z)\|}\|\Psi_1^{-1}(\phi(0), z)\|\right)$$

for z near $p \in \overline{M}$. Thus we can conclude that f_j converges to f in the topology of $C^{k-1-}(p)$; for the matrix sequence $\{df_j(z_0)\}$ converges to $df(z_0)$ and $\Psi_2(z_j, \circ)$ converges to $\Psi_2(z^*, \cdot)$ ($z^* = \lim_j z_j$) in C^{k-1-}-topology near p by the fact that $\Psi(z, w)$ depends $C^{(k-1)-}$ on the base point z when $w \approx \partial \mathbf{B}_n$. Let p vary, we then complete the proof of Proposition 3.4. ∎

Remark: In case M has the top dimension (i.e, $M = D$), result (2) of Proposition 3.4 can also be obtained by using the asymptotic expansion of the Bergman kernel function (see [GK]). However, we do not know whether there is a similar Bergman kernel functions argument if M is a holomorphic retract of lower dimension.

Now with all these Propositions at our disposal, the proof of Theorem 3.1 can be easily achieved by using an idea in [GK].

Proof of Theorem 3.1: Since a holomorphic retract of M is also a holomorphic retract of D, by results of Bedford [Be1] and Abate [Ab1] we may simply assume that $f \in$Aut(M) and $\dim_{\mathbf{C}} M > 0$. In case M is a Riemann surface, then the theorem follows easily from the Riemann mapping theorem and the classical Denjoy-Wolff theorem. So we assume that $\dim_{\mathbf{C}} M \geq 2$. Let ρ be a C^3 defining function of D. Then, when restricted to $\overline{M}$, it also gives a C^{2-} defining function of $\overline{M}$ by using the fact that M intersects ∂D transversally (see the Claim in the proof of Proposition 3.2). Let H be the closed subgroup of Aut(M) generated by f. Then by the Cartan theorem and the given condition, H is a compact Lie group. It thus possesses a regular Haar measure μ. Define $\rho_f = \int_H \rho \circ g d\mu(g)$. By (2) of Proposition 3.4 and a lemma in [Hu1], it follows that ρ_f is also C^{2-} up to $\overline{M}$ and moreover it is easy to check that ρ_f serves a new defining function of M (an easy application of Hopf's lemma). We now let $M_\epsilon = \{z \in M : \ \rho_f \leq -\epsilon\}$, for $\epsilon << 1$. Then Morse theory tells that M_ϵ has the same topology type as $\overline{M}$ does; for ρ_f has no critical values between $-\epsilon$ and 0 (including the end points). Since $f(M_\epsilon) \subset M_\epsilon$, we conclude, by using the hypothesis and the Lefschetz fixed point theorem, that f has a fixed point on M_ϵ, which is obviously an interior point of M. ∎

We now are ready to complete the proof of Theorem 3.

Proof of Theorem 3: We keep the previous notation and consider the sequence $\{f^k\}$. First, by making use of results of Bedford [Be1] and Abate [Ab1], we see that either $\{f^k\}$ diverges to the boundary or there is a holomorphic retract M of

D so that $f|_M$ is an elliptic element of $\mathrm{Aut}(M)$. In the latter case, Theorem 3.1 tells that f has an interior fixed point.

So it only remains to explain why the sequence $\{f^k\}$ converges on compacta to a boundary point in case it diverges to the boundary. This part has actually been argued in [Ma] and [Ab3]. However, for completeness, we include here a proof which is slightly different but much simpler. First, the strong pseudoconvexity of D indicates that there is no non-trivial complex sub-variety in ∂D. Hence, if a subsequence of $\{f^k\}$ converges on compacta, the limit has to be a boundary point. Pick $z_0 \in D$, and choose, by induction, a subsequence $\{m_1 < m_2 < \cdots, m_j, \cdots\}$ so that $K_D(z_0, f^j(z_0)) \geq K_D(z_0, f^{m_1}(z_0))$ for each $j \geq 1$, …, $K_D(z_0, f^j(z_0) \geq K_D(z_0, f^{m_l}(z_0))$ for every $j > m_{l-1}$. By passing to a subsequence, we assume that $\{f^{m_j}\}$ converges on compacta to $p \in \partial D$. We will complete the proof by showing that $\{f^k\}$ converges on compacta to p. In fact, if that is not the case, there would be a subsequence f^{k_i}, which goes to $q(\in \partial D) \neq p$. Since $f^{m_j+k_i}(z_0) = f^{m_j}(f^{k_i}(z_0)) \to p$ as $j \to \infty$), for each fixed k_i, we therefore are able to find a subsequence $\{m_{j_i}\}$ of $\{m_j\}$ so that $f^{m_{j_i}+k_i}(z_0) \to p$ as $i \to \infty$. Noting the length decreasing property for the Kobayashi distance and the way we chose $\{m_j\}$, we have

$$(3.3.3) \quad K_D(f^{k_i}(z_0), f^{k_i+m_{j_i}}(z_0)) \leq K_D(z_0, f^{m_{i_j}}(z_0)) \leq K_D(z_0, f^{k_i+m_{j_i}}(z_0)).$$

On the other hand, by making use of the fact that $f^{m_{j_i}+k_i}(z_0) \to p$ and $f^{k_i}(z_0) \to q(\neq p)$, it follows from the estimates for the Kobayashi distance that

$$\begin{aligned} &K_D(f^{k_i}(z_0), f^{k_i+m_{j_i}}(z_0)) - K_D(z_0, f^{k_i+m_{j_i}}(z_0)) \\ \geq & -\frac{1}{2}\log\delta(f^{k_i}(z_0)) - \frac{1}{2}\log\delta(f^{k_i+m_{j_i}}(z_0)) + \frac{1}{2}\log\delta(f^{k_i+m_{j_i}}(z_0)) + C \\ \geq & -\frac{1}{2}\log\delta(f^{k_i}(z_0)) + C \to +\infty, \ (\text{as } i \to \infty), \end{aligned}$$

where C is a constant independent of i. This contradicts (3.3.3) and thus finishes the proof of Theorem 3. ∎

Remark: The boundary point in Theorem 3 is the so-called Wolff point of f, which is a fixed point of f when we understand the value of f there as the non-tangential boundary limit. Meanwhile, it is worth mentioning that the assumption of the triviality of the topology of D can be weakened to the condition where the Lefschetz fixed point theorem can be applied.

Theorem 6: Let $D \subset \mathbf{C}^n$ be a bounded strongly pseudoconvex domain with C^k smooth boundary and let f be a holomorphic self-mapping of D. Then the following holds:
(1): Every holomorphic retract of complex dimension greater than 1 of D is actually a closed complex sub-manifold with $C^{(k-1)-}$ smooth boundary.
(2): Suppose that $\{f^k\}$ is a precompact family, but does not converge to a single point. Then there exists a unique holomorphic retract E, depending only on f, such that (a) in case the dimension of E is greater than 1, $f|_E$ is an automorphism of E and admits a $C^{(k-1)-}$ smooth extension up to the boundary of E (b) for each point $z_0 \in D$, the limit points of $\{f^k(z_0)\}$ stay in E.

Proof of Theorem 6: This follows directly from Proposition 3.3., the result of Bedford and Abate, and Proposition 3.4. In fact, E in the second part of the theorem is exactly the set $\{z \in D : \ z \text{ is a limit point of } \{f^k(z_0)\} \text{ for some } z_0 \in D\}$.

§ **3.4: Further applications**

We will now present two more applications of the results in previous sections. The first application is the proof of Theorem 8 which can be viewed as a boundary version of the classical Cartan uniqueness theorem, while the second one is concerned with the compactness of composition operators on simply connected strongly pseudoconvex domains.

Theorem 8: Let $D \subset\subset \mathbf{C}^n$ be either a simply connected smooth pseudoconvex domain or a simply connected taut domain with Stein neighborhood basis. Let $p \in \partial D$ be a strongly pseudoconvex point with at least C^3 smoothness. Suppose that $f \in \mathrm{Hol}(D, D)$ is a holomorphic self mapping of D, $f(z) \neq \mathrm{id}$, so that $f(z) = z + o(\|z - p\|^k)$ as $z \to p$. Then the following hold:
(1) $k \leq 2$

(2) If $k = 1$, then either f fixes a holomorphic retract with positive dimension or $f^m \to p$. In case D is not biholomorphic to the ball, then f cannot be an automorphism.
(3) If $k = 2$, then f can not be an automorphism of D and the sequence $\{f^m\}$ converges to p on compacta.

Corollary 3.3: Let D and p be as in Theorem 8. Suppose that f$\in$ Hol(D, D) is such that $f(z_0) = z_0$ for some $z_0 \in D$ and $f(z) = z + o((z-p)^2)$ as $z \to p$. Then $f \equiv \mathrm{id}$.

Remarks: Theorem 6 can be viewed as a boundary version of the classical Cartan theorem. The case (1) is the local version of the Burns-Krantz theorem (see [BK] and [H2]). For the disk in $\mathbf{C}^1$, as noted in [Lm3], the exponent in Corollary 3 can be reduced to just 1. However, the following examples show that the situation in the higher dimensional case is different and our result is actually quite sharp:

Example (a): Let $\sigma(z_1, z_2) = (\frac{(1-2i)z_1-1}{z_1-1-2i}, \frac{-2iz_2}{z_1-1-2i})$ for $(z_1, z_2) \in \mathbf{B}_2$. Then $\sigma \in \mathrm{Aut}(\mathbf{B}_2)$ with $\sigma(p) = p$ and $\sigma'(p) = \mathrm{id}$, where $p = (1, 0)$. But $\sigma \neq \mathrm{id}$.

(b): Let D be a bounded strongly pseudoconvex domain defined by $D = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + h(|z_2|) < 1\}$ for some smoothly increasing function $h(\cdot)$ with $h(0) = 0$. Denote by p the boundary point $(1, 0)$. Define $f(z_1, z_2) = (z_1, z_1 z_2)$. Then f fixes the holomorphic retract of D : $\{(z_1, 0) : \|z_1\| < 1\}$ and $f(z) = z + o(\|z - p\|)$ as $z \to p$. But f$\neq$id.

(c): Let $B_2 = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ be the unit 2-ball and let $p = (1, 0)$. For every $a > 0$, define a holomorphic mapping f_a from $\mathbf{B}_2$ to $\mathbf{C}^2$ by

$$f_a(z_1, z_2) = \left(\frac{z_1 + a(1-z_1)^2}{1 + a(1-z_1)^2} \, , \frac{z_2}{1 + a(1-z_1)^2} \right).$$

Then it is easy to check that f_a is a self-mapping of $\mathbf{B}_2$ and $f_a(z) = z + O(\|z-p\|^3)$ as $z \to p$. By Theorem 8, $\{f^k\}$ converges compactly to p.

We procceed with the proof of Theorem 8 by way of several lemmas. We first start with the following

Lemma 3.10: Let D and p be as in Theorem 8. Suppose that σ is a biholomorphism of D such that $\sigma(z) = z + o(\|z-p\|^k)$ as $z \to p$. Then $\sigma \equiv \mathrm{id}$ if either $k = 2$ or $k = 1$ and D is not biholomorphic to the ball.

Remark: We notice that Lemma 3.10 is sharp even in the one complex variable case:

Example : Let $\sigma(\tau) = \frac{1+(2i-1)\tau}{2i+1-\tau}$ for $\tau \in \Delta$. Then $\sigma(1) = \sigma'(1) = 1$ and $\sigma \in \mathrm{Aut}(\Delta)$. However, by a direct computation, it can be seen that an automorphism of Δ which has contact of order 2 with the identity at some boundary point must be the identity. In fact, let $\sigma = e^{i\theta}\frac{z-a}{1-\bar{a}z}$ with $\theta \in \mathbf{R}$ and $a \in \Delta$. Then $\sigma'' \neq 0$ when $a \neq 0$.

Proof of Lemma 3.10: Let Ω, Ω_0, Φ, and U be as in the proof of Theorem 4, and let ϕ be a complex geodesic of Ω with $\phi(1) = \Phi(p)$ (see [Lm1]). As argued before, we see that when $\phi'(1)$ is close enough to the tangential direction, then $\phi(\Delta) \subset \Omega_0$ and $\phi_0 \overset{\Delta}{=} \Phi^{-1}|_{\Omega_0} \circ \phi$ is a complex geodesic of D. Hence, $\sigma \circ \phi_0$ is also a complex geodesic of D. Now when $\sigma \circ \phi_0(\Delta)$ is close enough to p (we can do this by shrinking $\phi(\Delta)$ and by the continuity of σ at p), it follows from Lemma 3.1 that $\sigma \circ \phi_0$ is also a complex geodesic of $\Phi^{-1}(\Omega_0)$. So $\Phi \circ \sigma \circ \phi_0$ is a complex geodesic of Ω_0 and therefore a complex geodesic of Ω (by Lemma 3.2). We note that ϕ and $\Phi \circ \sigma \circ \phi_0$ coincide at 1 up to the first order. Thus, by the uniqueness property of complex geodesics on strongly convex domains, we can find a biholomorphism α of Δ so that $\alpha(1) = 1, \alpha'(1) = 1$, and $\Phi \circ \sigma \circ \phi_0 = \phi \circ \alpha$. If $k = 2$ or $k = 1$ and α is elliptic (i.e, the sequence $\{\alpha^n\}$ is a pre-compact family), we have that $\alpha(\tau) \equiv \tau$ and hence that σ fixes $\phi_0(\Delta)$. If α is non-elliptic, then by noting the fact that $\sigma(\phi_0(\Delta)) \subset \phi_0(\Delta)$, we have $\Phi \circ \sigma^m \circ \Phi^{-1} \circ \phi_0 = \phi \circ \alpha^m \to \Phi(p)$. Thus p is a boundary accumulation point of the automorphism sequence $\{\sigma^m\}_{m=1}^{\infty}$ of D. By the Wong-Rosay theorem [Kr1], this implies that D is biholomorphic to the ball. So when D is not biholomorphic to the ball, by making use of the uniqueness theorem for holomorphic functions and the fact that the union of all such $\phi_0(\Delta)'$s occupies an open subset of D, we see that $\sigma \equiv \mathrm{id}$. ∎

Lemma 3.11: Let $D \subset\subset \mathbf{C}^n$ $(n > 1)$ be a pseudoconvex domain and $p \in \partial D$ a C^2-strongly pseudoconvex point. Assume furthermore that either D has a Stein neighborhood basis or D has a C^∞ boundary. If $f \in \mathrm{Hol}(D, D)$ is such that $f(z) = z + o(\|z-p\|)$ as $z \to p$, then for any neighborhood V of p, there exists a point $z \in V \cap D$ such that $f^k(z) \in V$ for $k = 1, 2, 3, \cdots$.

Proof of Lemma 3.11: Let D, p, f be as in the lemma, and let $\overrightarrow{n}$ be the inward normal vector of D at p. Denote by L the inward $\frac{\pi}{4}$-cone at p, i.e,

$$L = \{z \in D : \text{ the angle between } \overrightarrow{pz} \text{ and} \overrightarrow{n} \text{ is less than } \frac{\pi}{4}\}.$$

We then define the big and small horospheres for any $z_0 \in D$ and R >0 as follows (we note that the definition is somewhat different from that in [Ab2], but is more suitable for our purpose here):

$$E(z_0, R) \stackrel{\Delta}{=} \{z \in D : \limsup_{w(\in L) \to p} (K_D(z,w) - K_D(z_0,w)) < 1/2 \log R\},$$

$$F(z_0, R) \stackrel{\Delta}{=} \{z \in D : \liminf_{w(\in L) \to p} (K_D(z,w) - K_D(z_0,w)) < 1/2 \log R\}.$$

Claim 1: Let $R > 0$ and $z_0 \in \overrightarrow{n}$ close to p. Then it holds that $z \in E_D(z_0, R)$ for $z \in \overrightarrow{n}$ close enough to p.

Proof of claim 1: Let $z_0 \in \overrightarrow{n}$ be close to p and let z be in the segment $\overrightarrow{z_0 p}$. Denote by $B(z)$ the ball with center z and radius $\delta(z)(= \|z-p\|)$. We then see that $B(z) \subset D$ when $z \sim p$. By the estimate that $K_D(z_0, w) \geq \mathrm{C} - \frac{1}{2}\log\delta(w)$ for $w(\in L) \sim p$ (see Claim 2 for more discussion on this matter) and the monotonicity properties of the Kobayashi distance we then have for $w(\in L) \sim p$ that:

$$\begin{aligned} K_D(z,w) - K_D(z_0,w) &\leq K_{B(z)}(z,w) - K_D(z_0,w) \\ &\leq \frac{1}{2}\log(\frac{1+\|z-w\|/\delta(z)}{1-\|z-w\|/\delta(z)}) - \mathrm{C} + \frac{1}{2}\log\delta(w) \end{aligned}$$

$$\leq \frac{1}{2}\log\delta(z) + C + \frac{1}{2}\log\frac{\delta(w)}{\delta(z) - |z-w|},$$

where, as before, C denotes a constant which may be different in different context.

We now use the special property of L which makes $\frac{\delta(w)}{\delta(z)-\|z-w\|} \to 1$ as $w(\in L) \to p$. We therefore obtain that $K_D(z,w) - K_D(z_0,w) \leq \frac{1}{2}\log\delta(z) + \mathrm{C}$. So for any $R > 0$, when $z(\in \overrightarrow{n})$ is close enough to p, we have $z \in E(z_0, R)$.

Claim 2: Let z_0 be as in Claim 1. For every small neighborhood V of p, there exists an $R > 0$ such that $F(z_0, R) \subset V$.

Proof of Claim 2: Let Ω_0, Ω, U and Φ be as in the proof of Theorem 4. Without loss of generality, we assume that $\Phi^{-1}(\Omega_0) \subset V$ and $\mathrm{d}\Phi(p) = \mathrm{id}$. Let $z_0^* = \Phi(z_0)$, $w^* = \Phi(w)$, and $\overrightarrow{n^*}$ the inward normal vector of Ω at $q(=\Phi(p))$. Then $\Phi(\overrightarrow{n})$ is tangent to $\overrightarrow{n^*}$ at q.

By noting the fact that the Kobayashi distance of Ω between any two points can be realized by a complex geodesic (because Ω is a bounded strongly convex domain), we then have for any $z^* \notin \Omega_0$ and $w^* \sim q$, that $K_\Omega(z^*, w^*) \geq \inf_{u \in \partial\Omega_0 - \partial\Omega} K_\Omega(u, w^*)$. Let $B^* \supset \Omega$ be a ball, which is tangent to Ω at q and has $\overrightarrow{n}$ as part of its diameter. Then, from the direct computation of the Kobayashi distance for B^*, we obtain $K_\Omega(u,w^*) \geq K_{B^*}(u,w^*) \geq -1/2\log\delta^*(w^*) + \mathrm{C}$. Here C is a constant independent of the choice of $u \in \partial\Omega_0 - \partial\Omega$, $\delta^*(w^*)$ denotes the distance from w^* to $\partial\Omega$, and $w^*(\in \Phi(L)) \sim q$. So from the monotonicity property of the Kobayashi distance, it follows that

$$\begin{aligned} K_D(z,w) - K_D(z_0,w) &\geq K_\Omega(z^*,w^*) - K_{\Omega_0}(z^*,w^*) \\ &\geq -\frac{1}{2}\log\delta^*(w^*) + \mathrm{C} - \mathrm{C}' + \frac{1}{2}\log\delta^*(w^*) \geq \mathrm{C} - \mathrm{C}'. \end{aligned}$$

Thus, if we choose $\frac{1}{2}\log R = \mathrm{C} - \mathrm{C}'$, then $z \notin F(z_0, R)$ when $z \notin \Phi^{-1}(\Omega_0)$. This completes the argument for Claim 2.

Claim 3: $f(E(z_0,R)) \subset F(z_0,R)$.

Proof of Claim 3: Let $z_k = p + \frac{\overrightarrow{n}}{k}(\in \overrightarrow{n})$. Then for $k >> 1$, $f(z_k)$ is in L and converges to p as $k \to \infty$. For any $z_0 \in D$, $R > 0$, and $z \in E(z_0, R)$, we have the following:

$$\liminf_{w(\in L)\to p}(K_D(f(z),w)-K_D(z_0,w))\le\liminf_{k\to\infty}(K_D(f(z),f(z_k))-K_D(z_0,f(z_0)))$$

$$\le\liminf_{k\to\infty}(K_D(z,z_k)-K_D(z_0,f(z_0)))$$

$$\le\liminf_{k\to\infty}(K_D(z,z_k)-K_D(z_k,z_0))+\limsup(K_D(z_k,z_0)-K_D(z_0,f(z_k))$$

$$\le\frac{1}{2}\log R+\limsup_{k\to\infty}(K_D(z_k,z_0)-K_D(z_0,f(z_k)).$$

So to complete the proof of Claim 3, we have only to show that

$$\limsup(K_D(z_0,z_k)-K_D(z_0,f(z_k)))\le 0.$$

In fact, let $\gamma_k:[0,1]\to D$ be the segment joining z_k and $f(z_k)$. Obviously, when $k>>1$, then γ_k stays in D. Denote by $B(a,r)$ the ball of center a and radius r. We then have, for every $X\in \mathrm{T}^{(1,0)}_{\gamma_k(t)}D$, that

$$\kappa_D(\gamma_k(t),X)\le\kappa_{B(\gamma_k(t),1/(2k))}(\gamma_k,X)\le \mathrm{C}\|X\|k.$$

Here C is a constant which is independent of k and t. Hence

$$K_D(z_0,z_k)-K_D(z_0,f(z_k))\le K_D(z_k,f(z_k))\le\int_{\gamma_k}\kappa_D(\gamma_k(t),\gamma_k'(t))\mathrm{d}t$$

$$\le \mathrm{C}k\|f(z_k)-z_k\|\le o(1),$$

as $k\to\infty$. This completes the argument for Claim 3.

Now for any given V, a small neighborhood of p, by Claim 1 and Claim 2 we can find a point z_0 and $R>0$, so that $V\supset F(z_0,R)\supset E(z_0,R)\neq\emptyset$. From Claim 3, it follows easily that $f^k(E(z_0,R))\subset F(z_0,R)$ for each k, since for any k, f^k also satisfies the condition in Lemma 3.11. Hence, every element in $E(z_0,R)$ does the job. ■

Lemma 3.12: Let D, p be as in Theorem 8, and let M be a holomorphic retract of D with complex dimension greater than 1. Suppose that $p\in\partial M$ and $f\in\mathrm{Aut}(M)$ is an elliptic element such that $f=z+o(z-p)$ as $z(\in M)\to p$. Then $f(z)\equiv z$.

Proof of Lemma 3.12: By Proposition 3.2, we can find a complex geodesic ϕ of M with $\phi(1) = p$ and $\phi(\Delta)$ close enough to p. By the hypothesis, it then follows that $\mathrm{Diam}(f \circ \phi(\Delta)) << 1$. Since ϕ and $f \circ \phi$ are actually two complex geodesics of a C^3 strongly convex domain (see the proof of Proposition 3.2) with $|\phi(\xi) - f(\phi(\xi))| = o(|\xi - 1|)$ and since f is elliptic, it thus follows that $\phi = f \circ \phi$. (This can be seen by an argument similar to that for Lemma 3.10 and the classical Wolff-Denjoy theorem). So f fixes $\phi(\Delta)$. Now, noting that all such $\phi(\Delta)$'s fill in an open subset of M, we have the proof of Lemma 3.12. ■

Proof of Theorem 8: Let D, p, and f be as in the theorem. Then Case (1) is the local version of the Burns-Krantz theorem (see [BK] and [H2]). Now if $\{f^l\}$ does not converge compactly to p, then by the iteration theory of holomorphic mappings (see [Be1] or [Ab1]) and Lemma 2.11, we have the following possibilities:

(i) $\{f^l\}$ converges compactly to some $z_0 \in D$;

(ii) Some subsequence of $\{f^l\}$ converges to a non-trivial holomorphic retract h of D so that $f \in \mathrm{Aut}(h(D))$ $(\dim(h(D) > 0)$;

(iii) f is an automorphism of D.

In view of Lemma 3.11, (i) cannot happen, while by Lemma 3.10 (iii) can occur only when $k = 1$ and D is biholomorphic to the ball. Hence, all we actually have to study is the case (ii).

Notice that $h(D)$ is either a simply connected hyperbolic Riemann Surface (this follows from the simple connectivity of D) or a holomorphic retract of dimension greater than 1.

In case $h(D)$ has dimension 1, since $\{f^n\}$ is a precompact family, we may conclude that f fixes some point on $h(D)$ ([Ab1]). From Lemma 3.11, it follows easily that $p \in \overline{h(D)}$. Hence we may choose a biholomorphism ϕ from Δ to $h(D)$ and a sequence $\{\tau_k\}$, converging to 1, so that $\phi(\tau_k) \to p$ as $k \to \infty$. By Corollary 3.2, we see that $\phi(1) = p$ and ϕ is Lipschitz-1 continuous near 1. Since $\phi^{-1} \circ f \circ \phi(\in \mathrm{Aut}(\Delta))$ fixes two points on $\overline{\Delta}$; one is in Δ and the another one is on $\partial\Delta$, we can easily conclude that f fixes h(D).

When $\dim(h(D)) > 1$, by applying Lemma 3.12 and noting that $f|_{h(D)}$ is elliptic, it also follows that $f(z) = z$ for $z \in h(D)$.

So, if the k in the lemma is 2, we then let $\lambda(\tau) = (\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}) \circ \phi(\tau)$, which is the sum of the eigenvalues of the Jacobian of f at $\phi(\tau)$. We claim that $\mathrm{Re}\lambda(\tau) \equiv 2$ under these assumptions. In fact, by using the Cauchy estimates, the Lipschitz-1 continuity of ϕ at 1, and the fact that $\delta(\phi(\tau)) \simeq \mathrm{C}(1 - |\tau|)$, we have the following estimate for $\tau \in (0, 1)$:

$$\|\frac{\partial f_j}{\partial z_j} \circ \phi(\tau) - 1\| \leq \mathrm{C}\frac{1}{\delta(\phi(\tau))} \sup_{\|z-\phi(\tau)\| \leq \delta(\phi(\tau))} \|f_j(z) - z_j\|$$

$$= \frac{1}{|1-\tau|} o((\delta(\phi(\tau)) + \|\phi(\tau) - p\|)^2) = o(|1-\tau|)$$

as $\tau \to 1$. On the other hand, since $\mathrm{Re}(\lambda(\tau))$ is harmonic and is never greater than 2, it follows from the Hopf lemma that $\mathrm{Re}(\lambda(\tau)) \equiv 2$. However, note that $|\lambda(\tau)| \leq 2$. From the Cartan-Carathéodory-Kaup-Wu theorem, we therefore conclude that it implies that $f(z) \equiv z$ on D. This contradicts our assumption and thus completes the proof. ∎

We end this section by proving:

Proposition 3.5: Let $D \subset\subset \mathbf{C}^n$ be a C^{3+} simply connected strongly pseudoconvex domain and let ϕ be a holomorphic self mapping of D. Denote by $H^r(D)$ the standard Hardy space (see [Kr1]) of D with $r > 1$. Suppose that the composition operator C_ϕ, defined by $C_\phi(g) = g \circ \phi$ for each $g \in H^r(D)$, is a compact self-operator of $H^r(D)$. Then $\{\phi^k\}$ converges uniformly on compacta to a fixed point $z_0 \in D$.

Remark: When D reduces to the ball or a strongly convex domain, Proposition 3.5 follows from the work of MacCluer or Mercer, respectively. The argument we will present for the general situation is based on the regularity result in Proposition 3.3 and the extension theorem for certain Hardy spaces obtained by Cumenge in 1983 [Cu].

Proof of Proposition 3.5: Under the given hypothesis, we first claim that ϕ must be an elliptic element. In fact, if that is not the case, then $\phi^k \to p \in \partial D$ and the angular derivative of ϕ at p is a positive number (see [Ab1]). Thus it follows from a standard argument (see [Me], for example), that C_ϕ cannot be a compact operator on $H^r(D)$.

Now suppose that there is a non-trivial holomorphic retract M of D with $\phi|_M \in \mathrm{Aut}(M)$. Notice that M is a closed complex submanifold of D with C^2 boundary and intersects ∂D transversally (Proposition 3.2, Proposition 3.3 and Corollary 3.3). Let $H^r(M, \mu_{k-1}) = \mathrm{Hol}(M) \cap L^r(\mu_{k-1})$ (where k is the codimension of M in D and the notation μ_{k-1} is explained on Page 59 of [Cu]). Then Theorem 0.1 of [Cu] tells us that there exists a bounded linear extension operator $E : H^r(M, \mu_{k-1}) \to H^r(D)$ and moreover the restriction operator $\pi : H^r(D) \to H^r(M, \mu_{k-1})$ is also bounded (see the argument of Corollary 4.1 in [Cu]). Since $C_\phi|_{H^r(M,\mu_{k-1})}$ is an isomorphism of $H^r(M, \mu_{k-1})$ to itself (see Proposition 3.4), we can easily conclude that C_ϕ is not compact; for C_ϕ cannot map the closed unit ball in $E(H^r(M, \mu_{k-1}))$ to a compact subset of $H^r(D)$.

Applying results in [Be1] and [Ab1], we can thus conclude that $\{\phi^k\}$ converges uniformly on compacta to some point $z \in D$. ■

Chapter 4: Local hull of holomorphy of a surface in $\mathbf{C}^2$

§4.1**: Regularity of local hull of holomorphy–Proof of Theorem 9**:

We retain all the notation which we have set up in § 1.3. In the present section, we prove Theorem 9 by using Bishop disks.

Theorem 9: Let $M \subset \mathbf{C}^2$ be a real surface of class C^a, where $a = \infty$ or ω. Suppose that p is a degenerate elliptic point of M of degree $2m$ and suppose that M can be flattened to any order at p. Then the local hull of holomorphy of M at p is a C^a Levi flat hypersurface with C^a boundary M near p.

Our idea of the proof can be described as follows: We first blow up the point p many times. This process makes the manifold M stated in Theorem 9 into a twisted totally real cylinder in $\mathbf{C}^2$ with regular boundary. We then use a suitable infinite dimensional implicit function theorem to obtain a real parametrized family of analytic discs that are attached to this cylinder. In case $a = \infty$, to verify the smoothness of $\widetilde{M}$ near p, we prove a unique determination of the holomorphic hull in terms of the locally attached analytic disks. When $a = \omega$, we verify the real analyticity of our foliation in the normal direction at p. We then finally obtain the full statement of real analyticity of $\widetilde{M}$ near p by using the uniqueness of analytic functions.

For clarity, we divide our discussion into two subsections. We first study the C^a dependence of analytic discs on a real parameter. Then we investigate the local hull of M near the exceptional point.

§ 4.1.1 C^a dependence on a real parameter

Let $M \subset \mathbf{C}^2$ be an embedded, real, two dimensional manifold of class C^a. Let p be an isolated complex tangent point of M. Moreover, we assume that p is a (possibly degenerate) elliptic point of degree $2m$, which can be flattened to any order (as defined in §1.3).That is, for any ℓ, there is a holomorphic change of coordinates, which maps p to 0 and such that the manifold M is given, in the new coordinates (z, w), by

$$w = h(z) = p_0(z) + h^*(z), \tag{4.1.1}$$

where p_0 is a positive polynomial of degree 2m, $h^*(z) = O(|z|^{2m})$, and $\text{Im}h^*(z) = o(|z|^\ell)$. Here we notice that $2m$ is invariantly associated to p (i.e, independent of the coordinates chosen).

In what follows, an analytic disc is a continuous function ϕ from the closed unit disc $\overline{\Delta}$ in $\mathbf{C}$ to $\mathbf{C}^2$ that is holomorphic on the interior Δ. We say that an analytic disc ψ is attached to M if $\psi(\partial\Delta) \subset M$.

Next we set up the equation that will describe our analytic discs. Set $I_\epsilon = (-\epsilon, \epsilon) \subset \mathbf{R}$, with $\epsilon > 0$ and $\epsilon << 1$. Let S^1 denote the unit circle in $\mathbf{C}$. We consider a function $\Phi : I_\epsilon \times S^1 \to \mathbf{C}^2$. The function Φ acts on variables (r, ξ) with $r \approx 0$ and $\xi \in S^1$. We would like to arrange for $\Phi(r, \ \cdot \ )$ to have a holomorphic extension to Δ for each fixed r and also that $\Phi(r, \xi) \in M$ when $\xi \in S^1$. We will write $\Phi(r, \xi) = (\phi_1(r, \xi), \phi_2(r, \xi))$.

For $r \in I_\epsilon$, we let D_r denote the domain

$$D_r \equiv \{z \in \mathbf{C}^1 : p_0(z) + \frac{1}{r^{2m}} p_1(rz) < 1\}.$$

Here we write $p_1(z)$ for $\text{Re}h^*(z)$. We notice that D_0 is a bounded star-like (with respect to the origin) domain with real analytic boundary. In fact, the starlike property follows from the homogeneity of p_0. So, to see the claim, we need only to show that $\frac{\partial p_0(z)}{\partial z} \neq 0$ when z is away from 0. For this purpose, we let $p_0(z) = \sum_{i+j=2m} a_{ij} z^i \overline{z}^j$ ($\overline{a_{ij}} = a_{ji}$. Then

$$z\frac{\partial p_0}{\partial z} = \sum_{i+j=2m} i a_{ij} z^i \overline{z}^j = \sum_{i+j=2m} (2m - j) a_{ij} z^i \overline{z}^j$$

$$= 2m \sum_{i+j=2m} a_{ij} z^i \overline{z}^j - \sum_{i+j=2m} j a_{ij} z^i \overline{z}^j$$

$$= 2m \sum_{i+j=2m} a_{ij} z^i \overline{z}^j - \overline{\sum_{i+j=2m} i a_{ij} z^i \overline{z}^j},$$

from which we see that $\mathrm{Re}\left(z\frac{\partial p_0}{\partial z}\right) = m\sum_{i+j=2m} a_{ij} z^i \overline{z}^j > 0$ when $z \neq 0$.

Now, for each $|r| \ll 1$, as a small perturbation of D_0, D_r is also simply connected. Let $\sigma_r(\xi) = \sigma(r, \xi)$ be a conformal mapping of Δ to D_r. Assume in advance that $\sigma(r, 0) = 0$ and $\sigma'(r, 0) > 0$. These last two conditions can always be arranged and make our choice of $\sigma(r, \xi)$ unique.

Lemma 4.1: In case $a = \infty$, $\sigma(r, \xi)$ is smooth in $(r, \xi) \in I_\epsilon \times S^1$. If $a = \omega$, we have

$$\sigma(r, \xi) = \sum_{i \geq 0} \sigma_i(\xi) r^i \qquad \xi \in \overline{\Delta},\ r \in I_\epsilon.$$

Here $\|\sigma_i\|_n \lesssim R^i$ for some $R >> 1$, depending only on n. Moreover each σ_j is holomorphic on Δ and $\sigma(r, \xi)$, as a function of two variables r and ξ, is real analytic. Here and in what follows, we use $\| \ \|_n$ to stand for the standard norm in the Banach space $\mathrm{E}_n = C^{n,1/2}(S^1)$ (which, in some context, will be used for the space of real functions with smoothness $n + 1/2$) and R for a large constant which may be different in different contexts (depending only on n, an apriori given non-negative number).

The proof is based on the implicit function theorem in Banach spaces. Since the technique also plays an important role in the later discussion, we present the following details on this matter.

We first recall some definitions. Let E and F be two Banach spaces (over $\mathcal{K} = \mathbf{R}$ or $\mathbf{C}$) and let $\mathcal{O} \subset \mathrm{E}$ be an open subset. Suppose that $T : \mathcal{O} \to \mathrm{F}$ is a continuous map. We say that T is of class C^1 if for every $x_0 \in \mathcal{O}$, $T(x + x_0) = T(x_0) + A(x_0)(x - x_0) + o(\|x - x_0\|)$ where $A(x_0) \in \mathcal{L}(\mathrm{E}, \mathrm{F})$ for each x_0 and depends continuously on x_0. We write $T'(x_0)$ for $A(x_0)$. The map T is C^2 if T' is of class C^1. Inductively, we can speak of the C^k smoothness of T for every k. The map T is said to be smooth or of class C^∞ if T is C^k for each k. Usually, we identify $T^{(k)}$ with a symmetric k-multiple linear mapping from $\overbrace{\mathrm{E} \times \cdots \times \mathrm{E}}^{k \text{ times}} \to \mathrm{F}$ (see [Die] and [Dei]). For $u \in \mathrm{E}$, we write $T^{(k)}(u^k) = T^{(k)}(\overbrace{u, \cdots, u}^{k \text{ times}})$. We use $L_k(E, F)$ to denote the spaces of k-multiple linear mappings between E and F, and $S_k(E, F)$ for the space of symmetric k-multiple linear mappings.

Let T as above. We say T is analytic if, for any $x_0 \in \mathcal{O}$, there exist a small $r > 0$ and a sequence of symmetric k-multiple linear mappings A_k with $\sum_{k\geq 0} \|A_k\| r^k < \infty$ such that $T(x_0 + x) = T(x_0) + \sum_{k\geq 0} A_k(x^k)$ for $\|x\| \ll 1$. Here $\|A_k\| = \sup\{\|A_k(y_1, \cdots, y_k)\| : \ \|y_j\| \leq 1 \ j = 1, \cdots, k\}$.

Claim 0: (a): Let E and F be two Banach spaces. Let $A_k \in L_k(E, F)$ be such that $\limsup_{k\to\infty} \|A_k\|^{1/k} = R$ $(R < \infty)$. Then the map defined by $T(x) = \sum_{k=0}^{\infty} A_k(x^k)$ is smooth and analytic on $B(1/R) = \{x \in E : \|x\| < 1/R\}$. Moreover, $T^{(p)}(0) = p!(SA_p)$. Here, for $L \in L_k(E, F)$, we define SL to be the symmetrization of L. That is, $SL(x_1, \cdots, x_k) = 1/k! \sum_{\sigma_1, \cdots, \sigma_k} L(x_{\sigma_1}, \cdots, x_{\sigma_k})$, where the sum is taken over all permutations of $\{1, \cdots, k\}$.

(b): Let $\mathcal{K} = \mathbf{C}$ and $\mathcal{O} \subset E = \mathcal{K}^n$ be an opens subset. Then any C^1 mapping from $\mathcal{O}$ to F is analytic.

Proof of Claim 0: The proof is similar to the finite dimensional case.

(a): Under the assumptions in (a), we first notice that $\sum A_k(x^k)$ converges uniformly on $B(\epsilon 1/R)$ for each $\epsilon < 1$. Thus T is continous on $B(1/R)$. Consider the map from E to F which sends $x \in E$ to $A_k(x^k)$. It is easy to verify that $A_k(x^k)' = k(SA_k)(\cdot, x^{k-1})$, i.e., for each $h \in E$, $(A_k(x^k))'h = k(SA_k)(h, x^{k-1})$. Obviously,

$$\|A_k(x^k)'\| = \sup_{\|s\|\leq 1} \|A_k(x^k)'s\| = \sup_{\|s\|, \|x\|\leq 1} \|k(SA_k)(s, x^{k-1})\| \leq k\|A_k\|\|x\|^{k-1}.$$

Therefore, it follows that $\limsup \|A_k(x^k)'\| \leq R\|x\|$, and thus $\sum A_k(x^k)'$ converges uniformly on each $B(\epsilon/R)$. By [Die], we see that T is of class C^1 on $B(1/R)$ and $T'(X) = \sum_{k\geq 1} A_k(x^k)'$. Meanwhile, one can see that $T'(0) = A_1(x)' = SA_1$. Define $B_k \in L_k(E, L(E, F))$ by

$$B_k(h_1, \cdots, h_k) = k(SA_{k+1})(\cdot, h_1, \cdots, h_k).$$

Then

$$\|B_k\| = \sup_{\|h_j\|\leq 1} \|B_k(h_1, \cdots, h_k)\| =$$

$$= \sup_{\|s\|\leq 1, \|h_j\|\leq 1} k\|(SA_{k+1})(s, h_1, \cdots, h_k)\| \leq k\|A_k\|$$

and hence $\limsup \|B_k\|^{1/k} \leq R$. Now, applying the previous argumnet to the series $T'(x) = \sum B_k(x^k)$, we obtain T is of class C^2, $T''(x) = \sum B_k(x^k)'$ for $x \in B(1/R))$, and $B_1' = T''(0)$. Since

$$B_k(x^k)' = (k+1)\left(SA_{k+1}(., x^k)\right)' = (k+1)k(SA_{k+1})(\cdot, \cdot, x^{k-1}),$$

we have $T''(0) = 2!(SA_2)$. By induction, we then conclude that T is of class C^∞ and

$$T^{(j)}(x) = \sum_{j\leq k}(A_k(x^k))^{(j)} = \sum k \cdots (k-j+1)(SA_k)(\cdots, x^{k-j}),$$

and $T^{(j)}(0) = j!(SA_j)$.

Next we will show that T is analytic on $B(1/R)$. To this aim, we notice that

$$\|T^{(j)}(h)\| \lesssim \sum k \cdots (k-j+1)\|SA_k(\cdots, h^{k-j})\| \lesssim \sum k \cdots (k-j+1)\|A_k\|\|h\|^{k-j}.$$

From this, one derives that, when $h \in B(\epsilon/R)$, it holds that $\|T^{(j)}(h)\| \lesssim j!R_\epsilon^j$. Here the positive number R_ϵ depends only on ϵ.

We now fix $h_0 \in B(1/R)$ and use the Taylor formula of T (see [Die]) at h_0. We then have

$$T(h_0 + s) = \sum_0^N \frac{T^{(j)}(h_0)s^j}{j!} + \mathcal{R}_{\mathcal{N}}(h_0, s),$$

where $\mathcal{R}_{\mathcal{N}} = 1/(N-1)! \int_0^1 (1-t)^N T^{(N)}(h_0 + ts)dt \cdot s^N$. Hence, when $\|s\| << 1$, one see that $\|\mathcal{R}_{\mathcal{N}}\| \to 0$ and thus $T(h_0 + s) = \sum \frac{T^{(j)}(h_0)s^j}{j!}$. This completes the proof of (a).

Now we proceed to the proof of (b): Let T be a C^1 map from $\mathcal{O}$ to F and let $x \in \mathcal{O}$. For each $y^* \in F^*$, we consider the complex function $y^*T(x+z)$ in z, which is then holomorphic on its defining domain. Thus we have the following Cauchy formula:

$$\phi(z) = y^*T(x+z) = (\frac{1}{2\pi i})^n \int_\Gamma \frac{y^*T(x+\xi)}{\xi - z}d\xi,$$

where $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$, the principal boundary of some polydisk centered at the origin. Taking away y^* in the above formula, we have

$$T(x+z) = (\frac{1}{2\pi i})^n \int_\Gamma \frac{T(x+\xi)}{\xi - z}d\xi.$$

From this, we see that all partial derivatives of $T(x+z)$ exist and thus T is smooth on its defining domain (see pp 178 of [Die]). Meanwhile, we have the power series expansion: $T(x+z) = \sum a_\alpha z^\alpha$ with $a_\alpha = (\frac{1}{2\pi i})^\alpha \int_\Gamma \frac{T(x+\xi)}{\xi^{\alpha+1}} d\xi$, which is also equal to $\frac{1}{\alpha!}(T^{(\alpha)}(x))$. We observe that the norms of partial derivatives and thus those for the total derivatives of T satisfy the Cauchy estimates (see [Die]) at each fixed x. Hence we may conclude the analyticity of T at x. (See also a similar argument in pp. 150 of [Dei]). ∎.

Claim 1: Let $B_\epsilon^{n,1/2}(x_0(\xi))$ be the ϵ-ball of E_n (over $\mathcal{K} = \mathbf{R}$) centered at the $x_0(\xi)$ and let $f(\xi, r, x, y)$ be a C^∞ function in $(\xi, r, x, y) \in S^1 \times \mathrm{I}_\epsilon \times U$, where U is an open subset of $\mathbf{R}^2$ (or $\mathbf{C}^2$, in case E_n consists of complex valued functions) Suppose that

$$T_1,\ T_2, \in \mathcal{L}(C^{n+1/2}(S^1), C^{n+1/2}(S^1)),$$

and $(x_0(\xi), T_1(x_0)(\xi)) \in U$ for each $\xi \in S^1$, then the map $F : \mathrm{I}_\epsilon \times B_\epsilon^{n,1/2}(x_0) \to C^{n+1/2}(S^1)$ defined by

$$F(r, x) = T_2(f(\xi, r, x(\xi), (T_1 x)(\xi)))$$

is of class C^∞ when $\epsilon \ll 1$. Moreover, when $x_0 = 0$ and

$$f(\xi, r, x, y) = \sum_{ijk} a_{ijk}(\xi) r^i x^j y^k$$

with $\|a_{ijk}\|_n \lesssim R^{i+j+k}$. Then the F defined above is real analytic (after shrinking ϵ).

Proof: The smoothness part follows from an induction argument. First, since $F'_r = T_2(\frac{\partial f}{\partial r}(\xi, r, x, T_1(x))$ and $F'_x(\cdot) = T_2(\frac{\partial f}{\partial x}(\cdot) + \frac{\partial f}{\partial y} T_1(\cdot))$ are continous on (r, x), we see that F is C^1. Moreover,

$$F' = T_2(\frac{\partial f}{\partial r} e_1 + \frac{\partial f}{\partial x} e_2 + \frac{\partial f}{\partial y} T_1 \circ e_2).$$

Here $e_1(r, h) = r$ and $e_2(r, h) = h$ are two natural projections.

Suppose that F is of class C^p and

$$F^{(p)} = T_2 \left(\sum_{i+j+k \le p} a_{ijk} \frac{\partial^{i+j+k} f}{\partial r^i \partial x^j \partial y^k} I_{ijk}) \right),$$

where $I_{ijk}(\in L_p(\mathbf{R}^1 \times E_n, E_n))$ are p- multiple linear mappings. Then

$$\frac{\partial F^{(p)}}{\partial r} = T_2(\sum_{i+j+k\le p} a_{ijk} \frac{\partial^{i+j+k+1} f}{\partial r^{i+1} \partial x^j \partial y^k} I_{i+j+k}),$$

and

$$\frac{\partial F^{(p)}}{\partial x} h = T_2(\sum_{i+j+k\le p} a_{ijk} \frac{\partial^{i+j+k+1} f}{\partial r^{i} \partial x^{j+1} \partial y^k} h I_{i+j+k} +$$

$$+ \sum_{i+j+k\le p} a_{ijk} \frac{\partial^{i+j+k+1} f}{\partial r^{i} \partial x^{j} \partial y^{k+1}} T_1(h) I_{i+j+k}),$$

which are continuous on (r, x). Thus, T is of class C^{p+1} and

$$F^{(p+1)} = T_2(\sum_{i+j+k\le p} a_{ijk} \frac{\partial^{i+j+k+1} f}{\partial r^{i+1} \partial x^j \partial y^k} e_1 \otimes I_{ijk} +$$

$$+ \sum_{i+j+k\le p} a_{ijk} \frac{\partial^{i+j+k+1} f}{\partial r^{i} \partial x^{j+1} \partial y^k} e_2 \otimes I_{ijk} +$$

$$+ \sum_{i+j+k\le p} a_{ijk} \frac{\partial^{i+j+k+1} f}{\partial r^{i} \partial x^{j} \partial y^{k+1}} (T_1 \circ e_2) \otimes I_{ijk}).$$

Hence, by induction, we see the smoothness of T.

Now, assume the latter case, i.e,

$$f(\xi, r, x, y) = \sum a_{ijk}(\xi) r^i x^j y^k$$

with $\|a_{ijk}\|_n \lesssim R^{i+j+k}$. Let $C_0 \gg 1$ so that $\|T_j\| < C_0$ $(j = 1, 2)$ and $\|e_1 \cdot e_2\| \le C_0 \|e_1\| \|e_2\|$ for $e_1,\ e_2 \in C^{n,1/2}(S^1)$ (see [GT] for the existence of such a C_0). Consider the power series

$$F(r, e) = \sum T_2\left(a_{ijk}(\xi) r^i e^j (T_1(e))^k\right) = \sum_k \sum_i A_{i,k}((r, e)^{i+k}),$$

where

$$A_{i,k}\left((r_1, e_1), \cdots, (r_i, e_i), (r_{i+1}, e_{i+1}), \cdots, (r_{i+k}, e_{i+k})\right) =$$

$$= \sum_{l\le k} T_2\left(a_{i,l,k-l} r_1 \cdots r_i e_{i+1} \cdots e_{i+l} T_1(e_{i+l+1}) \cdots T_1(e_{i+k}\right).$$

Here we understand $A_{i,k} \in L_{k+i}(\mathbf{R} \times E_n, E_n)$. Then

$$\|A_{i,k}\| \lesssim \sum_{l \le k} C_0^{2k-l} \|a_{i,l,k-l}\| \lesssim k C_0^{2k} R^{i+k}.$$

Write $B_N = \sum_{i+k=N} A_{i,k}$. Then $\|B_N\| \lesssim N^2 C_0^{2N} R^{2N}$ and

$$T(r,e) = \sum B_N((r,e)^N)$$

. From (a) of Claim 0, we see the proof of Claim 1.

The following is the starting point for our later discussion:

Implicit Function Theorem: Let X, Y, Z be Banach spaces, $U \subset$X and $V \subset$Y be neighborhoods of x_0 and y_0 respectively. Let $F : U \times V \to$Z be of class C^∞. Suppose that $F(x_0, y_0) = 0$ and $F'_y(x_0, y_0)$ is an invertible bounded linear map from Y to Z. Then there exist balls $B_r(x_0) \subset U$, $B_{r'}(y_0) \subset V$, and a unique map $T : B_r(x_0) \to B_{r'}(y_0)$ of class C^∞ such that $T(x_0) = y_0$ and $F(x, T(x)) \equiv 0$ on $B_r(x_0)$. Moreover, in case X= $\mathcal{K}^n$ and F is analytic, then T is also analytic.

Proof: The proof in the C^∞ case can be found in many text books (see [Die], for example). In the real analytic case, there is a general argument in [Dei] (Theorem 15.3 of [Dei]). It seems to me that there is a gap in the proof there, but it works well in case X is of finite dimension. For completeness, we give the following discussion:

When $\mathcal{K} = \mathbf{C}$, the proof follows from the first part and (b) of Claim 0. When $\mathcal{K} = \mathbf{R}$. We complexify X, Y, and Z. For example, we define $X_c = \{x + iy : x, y \in X\}$ and define $\|x + iy\| = \|x\| + \|y\|$. Next, for a symmetric multiple linear map, say $A_k \in L_k(X, Y)$, we complexify it in a natural way: To define $A_k^*(x_1 + iy_1, \cdots, x_k + iy_k)$, one expands $A_k^*(x_1 + ty_1, \cdots, x_k + ty_k)$ as a polynomial in t and then replace t by i. An important observation is then that the complexified sequence also satisfies the Cauchy estimates if the original one does.

So, by the above argument we can extend F as analytic map near (x_0, y_0) which is from $X_c \times Y_c$ to Z_c. Now using the smooth version of the implicit function theorem, we have a smooth solution T_c which, when restricted to X, coincide with

T by the uniqueness. By (b) of Claim 0, we see that T_c and thus T is analytic near x_0. ∎.

After all these preparations, we now turn to the proof of Lemma 4.1.

Proof of Lemma 4.1: Let σ_0 be the conformal mapping from Δ to D_0 with $\sigma_0(0) = 0$ and $\sigma_0'(0) > 0$. By the Schwarz reflection principle, σ_0 is actually holomorphic on $\overline{\Delta}$. Thus $D_r^* = \sigma_0^{-1}(D_r)$ is also a domain with C^a boundary (when $|r| << 1$).

Let $p_0(\sigma_0(\xi)) = (1 - |\xi|^2)p_0^*(\xi)$ with $p_0^*(\xi) \neq 0$ and of class C^a for ξ close to S^1. Let

$$\rho(r, z) = 1 - |z|^2 + \rho^*(r, z) = 1 - |z|^2 + \frac{1}{r^{2m}} \frac{p_1(r\sigma_0(z))}{p_0^*(z)}.$$

Then $\rho(r, z)$ is a C^a defining function for D_r^*.

Now $\sigma^*(r, \xi) = \sigma_0^{-1}(\sigma(r, \xi))$ is a conformal map from Δ to D_r^*. Obviously it suffices for us to show that $\sigma^*(r, \xi)$ has all the properties stated in the lemma.

In fact, we will show that $\sigma^*(r, \xi)$ can be uniquely written as $\xi(1 + \sigma^{**}(r, \xi))$ with

$$\sigma^{**}(r, \xi) = \Xi(r, \xi) + \sqrt{-1}\mathcal{H}(\Xi(r, \xi)),$$

where $\mathcal{H}$ stands for the standard Hilbert transform and $\Xi(r, \xi)$ is real and of class C^∞ when $(r, \xi) \in \mathrm{I}_\epsilon \times S^1$. Moreover, when $a = \omega$, then $\Xi(r, \xi) = \sum_{j \geq 0} \Xi_j(\xi) r^j$ and $\|\Xi_j\|_n \lesssim R^j$ for $R \gg 1$. We observe that $\mathcal{H}$ is a bounded isomorphism of $E_n = C^{n,1/2}(S^1)$. (Here we will take E_n as the space of real functions).

To this end, we first note that $\rho(r, \sigma^*(r, \xi)) \equiv 0$ for $\xi \in S^1$ and

$$\rho(r, \sigma^*(r, \xi)) = \rho(r, \xi) + 2\mathrm{Re}\left(\frac{\partial \rho(r, \xi)}{\partial z} \xi \sigma^{**}(r, \xi)\right) + \rho^{**}(r, \xi, \sigma^*(r, \xi)).$$

Here $\|\rho(r, \xi)\|_n = O(r)$,

$$\frac{\partial \rho(r, \xi)}{\partial z} = -\overline{\xi} + O(r),$$

and $\|\rho^{**}(r, \xi, \sigma^*)\|_n = O(\|\sigma^{**}\|_n^2)$. Thus we obtain

$$(*) \qquad 2\mathrm{Re}\sigma^{**} = F(r, \xi, \sigma^{**})$$

with $\|F(r,\xi,\sigma^{**})\|_n = O(|r| + \|\Xi\|^2)$. Moreover,

$$F(r,\xi,z) = \rho(r,\xi) + \mathrm{Re}\left(\frac{\partial\rho(r,\xi)}{\partial\xi} + \bar{\xi}\right) z + \rho^{**}(r,\xi,z),$$

which is obviously of class C^∞. In case $a=\omega$, $F(r,\xi,z)$ is real analytic in $(r,\xi,z) \approx I_\epsilon \times S^1 \times \Delta_\epsilon$. Note that $(*)$ can be written as

$$(**) \qquad \Xi = \widetilde{F}(r,\Xi,\mathcal{H}\Xi) = G(r,\xi,\Xi,\mathcal{H}\Xi)$$

where $\widetilde{F}(r,\Xi) = 1/2F(r,\xi,\Xi+\sqrt{-1}\mathcal{H}\Xi)$ is also of class C^∞ by Claim 1 (when $(r,x,y)\approx(0,0)$). When $a=\omega$, we notice that $G(r,\xi,x,y)$ is real analytic in (r,ξ,x,y). So for each fixed $\xi_0 \in S^1$, there exist a s small neighborhood U of $0\in\mathbf{R}$ and a small arc C_{ξ_0} such that when $r,x,y\in U$ and $\xi\in C_{\xi_0}$, it holds that

$$G(r,\xi,x,y) = \sum a_{ijk}(\xi) r^i x^j y^k,$$

where $\|a_{ijk}(\xi)\|_n^* \lesssim R^{i+j+k}$ and the norm $\|\cdot\|_n^*$ is taken over C_{ξ_0}. Now, by using the covering lemma, the uniqueness of the power series, and by changing R, we can assmue that U is fixed and a_{ijk} is independent of the choice of ξ_0. Meanwhile, we also see that the global Cauchy estimates for a_{ijk} hold.

Now, since $\widetilde{F}(0,0)=0$ and $\widetilde{F}'_\Xi(0,0)=0$, by the above mentioned implicit function theorem, Equation $(**)$ has a unique solution $\Xi(r)$ which is also of class C^a with respect to the parameter r. Especially, When $a=\omega$, then $\Xi(r) = \sum_{j\geq 0} r^j \frac{D^j(\Xi)(1^j)}{j!}$ with $\|\frac{D^j(\Xi)(1^j)}{j!}\|_n \lesssim R^j$. (We note that $\Xi(0)=0$).

Returning to $\sigma^*(r,\xi)$, when $a=\omega$, we thus have $\sigma^*(r,\xi) = \sum_{j\geq 0}\sigma_j(\xi) r^j$ with $\sigma_j \in C^{n,1/2}(S^1)$ for every n (thus it is smooth). The holomorphic extendibility of σ_j^* to Δ follows from a normal family argument. To simplify the notation, we drop the superscript '*' in what follows.

Now, we show that $\sigma(\xi,r)$ is smooth in (ξ,r). In fact, we can find an η so that $(**)$ can be uniquely solved in $C^{1/2}(S^1)$ for $r\in I_\eta$. (The solution will be denoted by $\Xi_0(r)$). Moreover we may assume that $\|G'_x(r,\xi,\Xi_0(r),\mathcal{H}(\Xi_0(r))\|_0$ and $\|G'_y(r,\xi,\Xi_0(r),\mathcal{H}(\Xi_0(r))\|_0$ are very small when $r\in I_\eta$. We claim that $\Xi_0(r)$ is also a smooth map from I_η to E_n for any n. This then will imply th smoothness of $\sigma(r,\xi)$ in two variables. For this purpose, we let $r_0\in I_\eta$. We first notice that

$\Xi_0(r_0)$ is in $C^\infty(S^1)$, for by the Kellog theorem $\sigma(r_0, \xi)$ is smooth. Now, consider the derivative of $\Xi - \widetilde{F}$ with respect to Ξ at $\mathcal{P} = (r_0, \Xi_0(r_0))$, which is given by

$$\mathcal{L}(\mathcal{P})h = h - \frac{\partial G}{\partial x}h - \frac{\partial G}{\partial y}\mathcal{H}h,$$

where $h \in E_0$. By the assumption, $\mathcal{L}(\mathcal{P})$ is a bounded invertible linear self operator of E_0. Let $a(\xi) = 1 - \frac{\partial G}{\partial x}|_{\mathcal{P}}$ and $b(\xi) = -\frac{\partial G}{\partial y}|_{\mathcal{P}}$. Then $a(\xi)$ and $b(\xi)$ are smooth. Notice that $\mathcal{L}(\mathcal{P})$ is also a linear bounded one to one operator from E_n to E_n for each n. By the well-know theorem in Banach space theory, to verify that $\mathcal{L}$ is a bounded invertible self-operator on E_n, it suffices for us to show that $\mathcal{L}$ is onto. But this follows from the following claim and the fact that $\mathcal{L}$ is onto when acting on E_0.

Claim 2: $\mathcal{L}$ is an hypoelliptic operator in the sense that for each n, $\mathcal{L}(\mathcal{P})h \in E_n$ if only if $h \in E_n$.

Proof: We need only to prove that $\mathcal{L}h \in E_n$ implies $h \in E_n$. By the simple connection between the Cauchy singular integral and the Hilbert transform (pp 63 of [MP]), it suffices for us to show that the solution of the equation

$$a^*(\xi)h + \frac{b^*(\xi)}{i\pi}\int_{S^1}\frac{h(\tau)}{\tau - \xi}d\tau = g$$

stays in E_n when $g \in E_n$, where $\|a^*\|_0 \approx 1$ and $\|b^*\|_0 \approx 0$.

Let $h(z) = \frac{1}{2\pi i}\int_{S^1}\frac{h(\tau)}{\tau - z}d\tau$. By the Plemelj formula, $h = h^+ - h^-$ and

$$\frac{1}{\pi i}\int_{S^1}\frac{h(\tau)}{\tau - \xi}d\xi = h^+ + h^-.$$

Thus we obtain the Riemann-Hilbert equation:

$$(EQ) \qquad h^+ = \frac{a^* - b^*}{a^* + b^*}h^- + g^*$$

with $g^* = g/(a^* + b^*) \in E_n$. Since the index of $\frac{a^*-b^*}{a^*+b^*}$ is 0, we can well define

$$d(z) = \exp\left(\frac{1}{2\pi i}\int_{S^1}\frac{\log\frac{a^*-b^*}{a^*+b^*}}{\tau - z}d\tau\right).$$

By the Privalov theorem and the Plemelj formula, we see that $d^+ = \frac{a^*-b^*}{a^*+b^*}d^-$ and $d^\pm \in E_n$. Returning to Equation (EQ), we see that

$$h(z) = \frac{1}{2\pi i}\int_{S^1} \frac{g^*(\tau)}{d^+(\tau)}\frac{d\tau}{\tau - z} + C,$$

from which follows the fact that $h(t) \in E_n$. This completes the proof of Claim 2.

Now, we see that the derivative of $\Xi - \widetilde{F}(r, \Xi)$ with respect to Ξ at $(r_0, \Xi_0(r_0))$ is also bounded invertible when acting on the space E_n. Thus by the implicit function theorem, we conclude that $\Xi_0(r)$ depends smoothly on r in E_n near r_0 for each n. Therefore we know that $\sigma(r, \xi)$ is smooth in (r, ξ).

Finally, we assume that $a = \omega$ and prove the real analyticity of σ in r and ξ. Pick r_0 and ξ_0 with $|r_0| << 1$ and $\xi_0 \in S^1$. Let $q_0 = \sigma(r_0, \xi_0)$. Since D_r near p_0 is defined by a function $g(r, z)$ which is real analytic in $(r, z) \approx q_0$. Using the implicit function theorem, we may assume that $g(r, z) = y - g^*(r, x)$. Thus, there is a biholomorphic map which is real analytical in (r, z) and sends, for each fixed r, an (fixed) open interval in $\mathbf{R}^1$ to an arc of ∂D_r near q_0. Denote the inverse of this map by $\psi(r, z)$. Thus we can assume, for each r, that $\psi(r, U \cap D_r)$ into the upper half space and $\psi(r, U \cap \partial D_r)$ into $\mathbf{R}^1$. Here U is a fixed open neighborhood of p_0.

Write $\widetilde{\sigma}(r, \xi) = \psi(r, \sigma) = \sum \psi_j(\xi)(r - r_0)^j$. By the extendibility of ϕ_j, we see that ψ_j is holomorphic on $V \cap \Delta$, where V is a small neighborhood of ξ_0. By shrinking V, we may also assume that the supreme norm of each ψ_j over $U \cap \overline{\Delta}$, denoted by $\|\psi_j\|'$, satisfies the Cauchy estimates: $\|\psi_j\|' \lesssim R^j$. We notice that when $\xi \in S^1$ $(\approx \xi_0)$, $\widetilde{\sigma}$ takes the real value. Thus, we easily see that ψ_j also takes the real value for $\xi \in S^1$. Hence, by the Schwarz reflection principle, each ψ_j can be holomorphically extended to some fixed $\xi_0 \in V' \subset V$. Meanwhile, we also see that the extension (still denoted by ψ_j) does not increase the supreme norm by the way we construct the extension.

Now on V', suppose that $\psi_k(\xi) = \sum_j a_{kj}(\xi - \xi_0)^j$. Here, by the Cauchy formula and the above discussion, we have some large R so that $|a_{jk}| \lesssim R^{j+k}$. From this, it follows that $\psi(r, \xi) = \sum_{jk} a_{jk}(\xi - \xi_0)^j(r - r_0)^k$. This finally completes the proof of Lemma 4.1. ■

Now, after a suitable renormalization, the mapping

$$\Phi_0(r, \xi) = (r\sigma(r, \xi), r^{2m})$$

gives rise to all of the analytic discs attached to the model surface M_0 of M. Here the "model" surface is given by

$$M_0 = \{(z, w) : w = p_0(z) + p_1(z)\}.$$

Typically, the strategy for constructing discs attached to M is that (i) it is easy to attach discs to the model surface, (ii) the surface M osculates the model surface to high order at z_0, and (iii) we may then obtain discs attached to M itself from those attached to M_0 by a deformation process.

The ideas in the last paragraphs motivate us to consider a mapping

$$\Phi(r, \xi) = (\phi_1(r, \xi), \phi_2(r, \xi))$$

such that

$$\phi_1(r, \xi) = r\sigma(r, \xi)\left(1 + \mathcal{F}(r, \xi)\right), \tag{4.1.2}$$

where $\mathcal{F}(r, \xi) \approx 0$ when $r \approx 0$ and is holomorphic in ξ for each fixed r. Our plan is to construct such a function $\mathcal{F}$ later on.

Given (4.1.1) and (4.1.2), we find that

$$\phi_2(r, \xi) = p_0(\phi_1(r, \xi)) + p_1(\phi_1(r, \xi)) + \sqrt{-1}k(\phi_1(r, \xi)),$$

where $\xi \in S^1$ and $k(z) = \mathrm{Im}h^*(z)$. In particular, we see that

$$\frac{1}{r^{2m}}\phi_2(r, \xi) = p_0\left(\sigma(r, \xi)(1 + \mathcal{F}(r, \xi)\right) + \frac{1}{r^{2m}}p_1(\phi_1(r, \xi)) + \sqrt{-1}\frac{1}{r^{2m}}k(\phi_1(r, \xi)).$$

Thus

$$p_0(\sigma(r, \xi)(1+\mathcal{F}(r, \xi))) + \frac{1}{r^{2m}}p_1(r\sigma(r, \xi)(1+\mathcal{F})) = -\frac{1}{r^{2m}}\mathcal{H}(k(r\sigma(r, \xi)(1+\mathcal{F})) + \mathrm{C}$$

for some real constant C. We seek a function $\mathcal{F}$ such that $\mathrm{C} = 1$.

Let

$$\Omega(r,\mathcal{F}) = p_0(\sigma(r,\xi)(1+\mathcal{F})) + \frac{1}{r^{2m}} p_1(r\sigma(r,\xi)(1+\mathcal{F})).$$

Linearizing near $\mathcal{F}=0$, we find that

$$\Omega(r,\mathcal{F}) = \Omega(r,0) + \Omega'(r,\mathcal{F}) + \Omega^*(r,\mathcal{F}).$$

These terms are defined as follows:

$$\Omega(r,0) = 1;$$

$$\Omega'(r,\mathcal{F}) = \lim_{t\to 0} \frac{\Omega(r,t\mathcal{F}) - \Omega(r,0)}{t} = 2\mathrm{Re}\{\sigma \frac{\partial p_0(\sigma)}{\partial z} + \frac{\sigma}{r^{2m}} \frac{\partial p_1(\sigma)}{\partial z}\}\mathcal{F};$$

$$\|\Omega^*(r,\mathcal{F})\|_n \lesssim \|\mathcal{F}\|_n^2.$$

Let us write

$$c(r,\xi) = \sigma(r,\xi)\frac{\partial p_0}{\partial z}(\sigma) + \frac{\sigma}{r^{2m}} \frac{\partial p_1(\sigma)}{\partial z}.$$

We see that

$$\mathrm{Re}\{c(r,\xi)\mathcal{F}\} + \Omega^*(r,\mathcal{F}) = -\frac{1}{r^{2m}} \mathcal{H}\left(k(r\sigma(1+\mathcal{F}))\right). \tag{4.1.3}$$

Lemma 4.2: We have that $c(r,\xi) \neq 0$ and $\mathrm{Ind}_{S^1} c(r,\xi) = 0$ for $|r| \ll 1$. So, there exists a positive function $d(r,\xi) \in C^\infty(S^1 \times I_\epsilon)$ (when $a=\omega$, we furthermore have $d(r,\xi) = \sum_{j\geq 0} d_j(\xi) r^j$ with $\|d_j\|_n \lesssim R^j$) such that $d^*(r,\xi) = d(r,\xi)c(r,\xi)$ has a holomorphic extension to Δ for each fixed r. Meanwhile, $d^*(r,\xi) \in C^\infty(S^1 \times I_\epsilon)$ and and, when $a=\omega$, $d^*(r,\xi) = \sum_{j\geq 0} d_j^*(\xi) r^j$ with $\|d_j^*\|_n \lesssim R^j$. Furthermore, in this case c, d, and d^* are real analytic in (r,ξ).

Proof of Lemma 4.2: Let $p_0 = \sum_{i+j=2m} a_{ij} z^i z^j$. Then we see that

$$c(r,\xi) = \sigma(r,\xi)\frac{\partial p_0}{\partial z}(\sigma) + \frac{\sigma}{r^{2m}} \frac{\partial p(\sigma)}{\partial z}$$

$$= \sum_{i+j=2m} a_{ij} i \sigma^i \overline{\sigma^j} + O(|r|\|\sigma\|^{2m}).$$

In analogy with what we did before, it holds that

$$\mathrm{Re}\sum_{i+j=2m} a_{ij} i\sigma^i\overline{\sigma^j} = \mathrm{Re}\sum a_{ij}(2m-j)\sigma^i\overline{\sigma^j}$$

$$= 2m\sum a_{ij}\sigma^i\overline{\sigma^j} - \mathrm{Re}\sum a_{ij}\sigma^i\overline{\sigma^j}.$$

Therefore we have

$$\mathrm{Re}\sum a_{ij} i\sigma^i\overline{\sigma^j} = m\sum a_{ij}\sigma^i\overline{\sigma^j} \geq \mathrm{C}\|\sigma\|^{2m}.$$

Thus, when $|r| \ll 1$, we obtain

$$\mathrm{Re}c(r,\xi) \geq \mathrm{C}\|\sigma\|^{2m} > 0 \ (\xi \in S^1).$$

Now we let $d(r,\xi) = \mathrm{e}^{\mathrm{i}\mathcal{H}(\log \mathrm{c}(\mathrm{r},\xi))}$, and $d^*(r,\xi) = d\cdot c$. The d and d^* possess the properties mandated in the lemma. ■

Returning to Equation (4.1.3), we have

$$\mathrm{Re}\left(d^*(r,\xi)\mathcal{F}\right) = -d(r,\xi)\Omega^*(r,\mathcal{F}) - \frac{d(r,\mathcal{F})}{r^{2m}}\mathcal{H}\left(k(r\sigma(r,\xi)(1+\mathcal{F}))\right).$$

Let $\widetilde{\mathcal{F}} = d^*(r,\xi)\mathcal{F} \equiv U(r,\xi) + \sqrt{-1}\mathcal{H}(U(r,\xi))$. Then we obtain the equation

$$U(r,\xi) = -d(r,\xi)\Omega^*\left(r, \frac{U+\sqrt{-1}\mathcal{H}(U)}{d^*(r,\xi)}\right)$$

$$-d(r,\xi)\mathcal{H}\left(\frac{1}{r^{2m}}k\left(r\sigma(r,\xi)\left(1+\frac{U+\sqrt{-1}\mathcal{H}U}{d^*(r,\xi)}\right)\right)\right).$$

Let

$$\Lambda_1(r,U) = -d(r,\xi)\Omega^*\left(r, \frac{U+\sqrt{-1}\mathcal{H}U}{d^*(r,\xi)}\right),$$

and

$$\Lambda_2(r,U) = -d(r,\xi)\mathcal{H}\left(\frac{1}{r^{2m}}k\left(r\sigma(r,\xi)\left(1+\frac{U+\sqrt{-1}\mathcal{H}U}{d^*(r,\xi)}\right)\right)\right).$$

Then we need to solve the equation

$$U = \Lambda_1(r,U) + \Lambda_2(r,U) \tag{4.1.4}$$

for $U \approx 0$. We note here that for the model surface, i.e. $k \equiv 0$, the only solution for each $r \sim 0$ is $U \equiv 0$.

Now we are going to apply the implicit function theorem to (4.1.4) to obtain a solution U that is C^a in the variable r. To this end, we still write

$$B_\epsilon^{n,1/2} = \{\phi \in C^{n,1/2}(S^1) : \phi \text{ is real valued and } \|\phi\|_n < \epsilon\}.$$

The index n is the same as above.

Consider the operator

$$\Lambda : \mathrm{I}_\epsilon \times B_\epsilon^{n,1/2} \to C^{n,1/2}(S^1),$$

$$\Lambda((r,U)) = \Lambda_1(r,U) + \Lambda_2(r,U).$$

By the boundedness of the Hilbert transform acting on the Banach space $C^{n,1/2}(S^1)$, we easily see that Λ is a well-defined operator when $\epsilon << 1$. In fact, we have

Lemma 4.3: For $\epsilon \ll 1$, Λ is a smooth map from $I_\epsilon \times B_\epsilon^{n,1/2}$ to $C_\epsilon^{n,1/2}(S^1)$. Moreover, when $a = \omega$, Λ is real analytic in (r,U) for any $(r,U) \sim 0$.

Proof of Lemma 4.3: This follows from Claim 0, Claim 1, Lemma 4.1, Lemma 4.2, and the fact:

$$\Lambda(r,U) = F_1(r,\xi,U,\mathcal{H}U) + d(r,\xi)\mathcal{H}(F_2(r,\xi,U,\mathcal{H}U)).$$

Here

$$F_1(r,\xi,x,y) = -d(r,\xi)\Omega^*\left(r, \frac{x+\sqrt{-1}y)}{d^*(r,\xi)}\right),$$

$$F_2(r,\xi,x,y) = -\left(\frac{1}{r^{2m}}k\left(r\sigma(r,\xi)\left(1+\frac{x+\sqrt{-1}y}{d^*(r,\xi)}\right)\right)\right). \blacksquare$$

Now we notice that $\Lambda(0,0) = 0$ and $\Lambda'_U(0,0) = 0$. Thus, from the implicit function theorem, (4.1.4) can be uniquely solved. Moreover, if we denote by $U(r)$ the solution, then $U(r)$ depends C^a on the parameter r.

A useful observation is that $\|U\|_n \lesssim r^{\ell-2m}$. In fact, by (4.1.4),

$$\|U(r,\xi)\|_n \leq \|\Lambda_1(r,U)\|_n + \|\Lambda_2(r,U)\|_n \leq \epsilon\|U\|_n + O(r^{\ell-2m}),$$

that is,

$$\|U(r,\xi)\|_n \leq C \cdot \frac{1}{1-\epsilon} r^{\ell-2m}.$$

Therefore, $\|\mathcal{F}(r,\xi)\|_n \lesssim r^{\ell-2m}$. In particular, we see that $r\sigma(r,\xi)(1+\mathcal{F})$ is an embedding when $|r| \ll 1$. Moreover, by the reflection principle [Chik], we see that this embedding (thus $U(r)(\xi)$) is of class C^a for each fixed r. Similar to what we got in Lemma 4.1, we have

Claim 3: When $\|r\| << 1$, $U(r)(\xi)$ is smooth in (r,ξ).

Proof of Claim 3: The proof is similar to that for Lemma 4.1. First, by working in E_0, we have a solution for (4.1.4), denoted by $U_0(r)$ with $r \in I_\eta$. We can also assume that at each point $\mathcal{P} = (r, U(r))$, $\Lambda'_U(\mathcal{P})$ has a very small $\|\cdot\|_0$ norm. Similiar to the argument in Lemma 4.1, to prove the smoothness of U_0 in (r,ξ), we need only study the hypoellipticity of the following operator at a given $\mathcal{P}$:

$$\text{(!)} \qquad \mathcal{L}h = h - (F_1)'_x h - (F_1)'_y \mathcal{H}(h) - d(r,\xi)\mathcal{H}\left((F_2)'_x h + (F_2)'_y \mathcal{H}(h)\right).$$

As mentioned before we may replace $\mathcal{H}$ in the above equation by the Cauchy singular integral operator $\mathcal{S}$: $\mathcal{S}h(\xi) = \frac{1}{\pi i}\int_{S^1} \frac{h(\tau)d\tau}{\tau-\xi}$. We now apply to (!) the Poincaré-Bertrand theorem (see pp 59 of [MP]), which asserts that

$$\int_{S^1} \frac{dt}{t-\xi} \int_{S^1} \frac{a(t,\tau)d\tau}{\tau-t} = -(\pi)^2 a(\xi,\xi) + \int_{S^1} d\tau \int_{S^1} \frac{a(t,\tau)dt}{(t-\xi)(\tau-t)}.$$

By the fact that $U(r)$ is smooth in ξ for the fixed r, we can simplify the above equation to an equation of the following form:

$$\text{(@)} \qquad a^* h + b^* \mathcal{S}h + \mathcal{K}h = g.$$

Here, a^* and b^* are smooth with $\|a^*\|_0 \approx 1$ and $\|b^*\|_0 \approx 0$. The operator $\mathcal{K}$ is defined as

$$\mathcal{K}h = \int_{S^1} k(t,\xi)h(t)dt$$

for certain $k(t,\xi)$ smooth in (t,ξ). Since $\mathcal{K}h$ is always smooth, the hypoellipticity of (@) thus follows from Claim 2. ∎

Remark: One can also prove by using the reflection principle (see the argument in Lemma 4.1) that $U(r)(\xi)$ is real analytic in (r,ξ) when r is small but not equal to 0, in cas $a=\omega$.

Notice that, for $\xi \in \overline{\Delta}$,

$$\Phi(r,\xi) = (\phi_1, \phi_2) = \big(\mathcal{C}(r\sigma(1+\mathcal{F})), \mathcal{C}\big(h(r\sigma(r,\xi)(1+\mathcal{F}))\big)\big),$$

where $\mathcal{C}$ is the Cauchy integral operator. We have the following properties:

$$\phi_1(r,\xi) = r\sigma(r,\xi)(1+\mathcal{C}(\mathcal{F})) = r\sigma(r,\xi) + \psi_1(r,\xi) \ \text{ with } \psi_1(r,\xi) = O(r^{\ell-2m});$$

$$\mathcal{C}(h(r\sigma(r,\xi)(1+\mathcal{F}))) = \phi_2 = r^{2m} + \psi_2(r,\xi) \quad \text{with } \ \psi_2(r,\xi) = O(r^{\ell})$$

In particular, when $a=\omega$, we obtain

$$\psi_1(r,\xi) = \sum_{\ell-2m}^{\infty} \psi_{1,j}(\xi) r^j, \quad \text{and } \psi_2(r,\xi) = \sum_{j}^{\infty} \psi_{2,j}(\xi) r^j.$$

Here $\psi_{1,j}$ and $\psi_{2,j} \in C^\infty(S^1)$ and $\|\psi_{1,j}\|_n, \ \|\psi_{2,j}\|_n \lesssim R^j$ for some $R \gg 1$ (R depends only on n).

§**4.2.2 Local Hull of Holomorphy**:

We now study the hull of holomorphy of M near $0 \in M$. For $0 < u \ll 1$,let

$$\widetilde{M_0} = \{(z,w): \ \mathrm{Im} w = 0, \ \mathrm{Re}(w) \le p_0(z) + p_1(z)\}$$

be the hull of holomorphy of M_0 and let $E^* = \cup_{0 \le r \ll 1} \Psi(r, \overline{\Delta})$ Define $\Psi: \ \widetilde{M_0} \setminus \{0\} \to E^* \setminus \{0\}$ by $\Psi(z,u) = \Phi(u^{\frac{1}{2m}}, \xi(z,u))$, where $\xi(z,u)$ is determined by the equations

$$z = r\sigma(r,\xi), \quad r = u^{\frac{1}{2m}}.$$

Lemma 4.4: The function Ψ is of class C^∞ on $\widetilde{M_0} \setminus \{0\}$, C^ω on $\widetilde{M_0} \setminus \{M_0\}$ (in case $a = \omega$), and admits a $C^{\frac{\ell-2m}{2m}}$ extension near 0. Moreover, when $\ell > 4m$, $\mathrm{d}_0\Psi = \mathrm{id}$.

Proof of Lemma 4.4: Let $\psi_1(r,\xi) = r^{\ell-2m}\psi_1^*(r,\xi)$ and $\psi_2(r,\xi) = r^\ell \psi_2^*(r,\xi)$ with $\|\psi_j^*\|_n = O(1)$ $(j = 1,2)$. Then $\Psi(z,u) = (z,u) + (\widetilde{\phi_1}, \widetilde{\phi_2})$, where

$$\phi_1\widetilde{(z,u)} = \psi_1(u^{1/2m}, \xi(u,z)) = u^{\frac{\ell-2m}{2m}}\psi_1^*(u^{1/2m}, \xi(u,z)),$$

$$\phi_2\widetilde{(z,u)} = \psi_2(u^{1/2m}, \xi(u,z)) = u^{\frac{\ell}{2m}}\psi_2^*(u^{1/2m}), \xi(u,z)).$$

We note that

$$\xi(u,z) = [(\sigma(r,\ \cdot\ )]^{-1}\left(\frac{z}{u^{\frac{1}{2m}}}\right) = [\sigma(u^{1/2m},\ \cdot)]^{-1}(\frac{z}{u^{1/2m}}) \equiv \sigma^{-1}(u^{1/2m}, z/u^{1/2m}),$$

and $\sigma^{-1}(\,\cdot\,,\,\cdot\,)$ is jointly smooth in its variables, and depends real analytically on the parameter r when $a = \omega$ (by the implicit function theorem).

Thus we easily see that $\Psi(z,u)$ is smooth off $u = 0$ and, in case $a = \omega$, real analytic in z, u when $(z,u) \in \widetilde{M_0} \setminus M_0$. Observe that $\|z/u^{1/2m}\| = O(1)$ when $(z,u) \in \widetilde{M_0} \setminus \{0\}$. We have, for $\ell > 4m$

$$\frac{\partial^{i+j+s}\widetilde{\phi_k}(z,u)}{\partial z^i \partial \overline{z}^j \partial u^s} = O(u^{\frac{\ell-2m}{2m} - s - \frac{j}{2m}}) \qquad \text{when} \qquad k = 1,2.$$

Thus, when $s + \frac{j}{2m} < \frac{\ell-2m}{2m}$,

$$\|\frac{\partial^{i+j+s}\widetilde{\phi_k}(z,u)}{\partial z^i \partial \overline{z}^j \partial u^s}\| = o(1)$$

as $(z,u) \to 0$. We therefore see that $\Psi(z,u)$ admits a $C^{\frac{\ell-2m}{2m}}$ extension near 0 when $\frac{\ell-2m}{2m} > 1$, ie, $\ell > 4m$. In this case, we obviously have $\mathrm{d}_0\Psi(z,u) = \mathrm{id}$. ∎

Lemma 4.5: When $\ell > 8m$, then E^* is the local hull of holomorphy of M in the sense that for any small piece M' of M near 0, when $\epsilon \ll 1$, $\mathbf{B}_\epsilon \cap E^*$ is an open subset of the holomorphic hull of M' near 0. Here $\mathbf{B}_\epsilon$ is the open ϵ-ball in $\mathbf{C}^2$ centered at the origin.

Proof of Lemma 4.5: For any fixed small piece M' of M near 0, we first claim that $E^* \cap B_\epsilon$ is contained in its hull of holomorphy when $\epsilon \ll 1$. This follows from a continuity argument. In fact, let Ω be any pseudoconvex domain containing M'. We notice that when $r << 1$, then $\Psi(r, \xi) \in \Omega$ for $\xi \in \overline{\Delta}$. Thus by the pseudoconvexity of Ω and the well-known family of disks argument, it follows that $\Phi(r, \xi) \in \Omega$ for all r (whenever $\Phi(r, \xi) \in M'$ for any $\xi \in S^1$)[Kr1]. To prove the rest, it suffices for us to show that there exists a sequence of pseudoconvex domains $\{\Omega_j\}$ such that $\Omega_j \supset M' \cap B_\eta$ and $E^* \cap B_\epsilon$ an open subset of $\cap \Omega_j$. Here we take $\eta << 1$ but independent of ϵ.

For this purpose, we return to the mapping $\Psi(z, u)$: $\widetilde{M_0} \to E^*$ and notice that $\partial_{\overline{z}} \Psi \equiv 0$.

For each point $(z, u) \in \widetilde{M_0} \approx (0, 0)$, we let

$$\mathcal{J}_\alpha \Psi = D^\alpha \Psi = \frac{\partial^{|\alpha|} \Psi}{\partial^{\alpha_1} z \partial^{\alpha_2} \overline{z} \partial^{\alpha_3} u}$$

($\|\alpha\| = \|(\alpha_1, \alpha_2, \alpha_3)\| \le 3$) and

$$\mathcal{T}(\Psi) = \sum_{\|\alpha\| \le 3} \frac{(Z - Z_0)^\alpha}{\alpha!} \mathcal{J}_\alpha \Psi(Z_0), \quad \text{where } Z = (z, u).$$

Then, by the fact that $\Psi \in C^2(\widetilde{M_0})$ and the holomorphic property along z-direction, it follows that $\|D^\alpha \mathcal{T}(\Psi)(Z_1) - \mathcal{J}_\alpha \Psi(Z_0)\| = o(\|Z_1 - Z_0\|)^{3-|\alpha|}$. Thus, by making use of the Whitney extension theorem (see [Mal]), we can obtain a C^3-extension Ψ^* for Ψ to a neighborhood of $\widetilde{M_0}$ near $(0, 0)$ with $\|\partial_W \partial_{\overline{W}} \Psi^*(z, w)\| = o(|\mathrm{Im} w|)$ (where we write $W = (z, w)$). Obviously $\mathrm{d}\Psi^*|_0 = \mathrm{id}$. We now let $\Psi^{*-1} = (\hat{\psi}_1, \hat{\psi}_2)$ and $\rho(z, w) = \mathrm{Im} \hat{\psi}_2$. Then $\mathrm{d}\rho(z, w) \neq 0$ on E^*, and $\rho(z, w) = 0$ if and only if $(z, w) \in E^*$; also $|\partial \overline{\partial} \rho(z, w)| = o(\delta(z, w))$ where we use $\delta(z, w)$ to denote the distance from (z, w) to E^*.

Now, we set

$$\Omega_{\eta, \epsilon_0, \tau} = \{(z, w) \in \mathbf{C}^2 :$$

$$(z, w) \in D_{\tau, \eta}, \rho(z, w) < \epsilon_0 (1 - |z|^2 - |w|^2),\ \rho(z, w) > -\epsilon_0 (1 - |z|^2 - |w|^2)\},$$

where

$$D_{\tau, \eta} = \{(z, w) :\ u > p_0(z) + p_1(z) - \tau |z|^{4m} - \tau,\ |z|^2 + |w|^2 < \eta\}.$$

Obviously, $E^* \cap B_\epsilon$ is an open subset of $\cap_{\tau,\epsilon_0}\Omega_{\eta,\epsilon_0,\tau}$. By the positivity of $p_0(z)$, we notice that $D_{\tau,\eta}$ is pseudoconvex when $\tau \ll 1$. Meanwhile, we will always choose $\tau \ll \epsilon_0$. Thus, to verify that $\Omega_{\eta,\epsilon_0,\tau}$ is pseudoconvex, we need only to show that the hypersurfaces defined by $\rho^\pm = \rho(z,w) \mp \epsilon_0(1-|z|^2-|w|^2) = 0$ are strongly pseudoconvex towards $\Omega_{\eta,\epsilon_0,\tau}$. In fact, The Levi form of ρ^+ is the Levi form of $\rho(z,w)+\epsilon_0\mathrm{id}$. Since $\|\mathrm{Levi}\ (\rho(z,w))\| \lesssim |o(\delta(z,w))|$ and $\delta(z,w) \approx \epsilon_0$ when $\rho(z,w) = -\epsilon_0(1-|z|^2-|w|^2)$, we see that the Levi form of ρ^+ is positive when η is small and $\tau \ll \epsilon_0 \ll 1$.

Similarly, we can show that the Levi form of $-\rho(z,w)-\epsilon_0(1-|z|^2-|w|^2)$ is also positive near its zero set. This completes the proof. ∎

Now we are ready to prove the smooth part of Theorem 9.

Lemma 4.6: Let M and p be as given in Theorem 9. Then the local hull of holomorphy of M near p is C^∞ with a piece of M near p as part of the smooth boundary.

Proof of Lemma 4.6: First, by Lemma 4.3 and Lemma 4.4, we need only to show that $\widetilde{M}$ has a smooth extension at p. To this end, we note that $2m$ and the smoothness of $\widetilde{M}$ at p are local biholomorphic invariants of M. Thus from Lemma 4.4 (or, Lemma 4.7 in the following subsection), it follows that $\widetilde{M}$ is of class $C^{(\ell-2m)/2m}$ for each ℓ and thus $\widetilde{M}$ is a smooth manifold with C^∞ boundary. ∎

For the rest of this section, we assume that $a=\omega$ and proceed to prove the real analyticity of $\widetilde{M}$ near p.

For this purpose, we return to the function Ψ. From the fact that $d\Psi|_0 = \mathrm{id}$, we see that the tangent space $T_0\widetilde{M} = \mathbf{R^3}\ \left(= T_0(\widetilde{M_0})\right)$.

Now let us define the projection mapping $\pi : \widetilde{M} \to \mathbf{R^3}$ by $\pi(z,u+iV) = (z,u)$. Then we can conclude that π is a C^∞ diffeomorphism by using Lemma 4.6. Moreover, since $\pi(\partial\widetilde{M}) = \pi(M) = M_0$ and $\left(\pi(\widetilde{M}\setminus M)\right) \cap \left(\widetilde{M_0}\setminus M_0\right)$ is not empty, we see that $\pi(\widetilde{M}) = \widetilde{M_0}$. That is, $\widetilde{M}$ can be viewed as the graph of some function $V(z,u)$ over $\widetilde{M_0}$. That is, $\widetilde{M} = \left\{\left(z, u+iV(z,u)\right) : (z,u) \in \widetilde{M_0}\right\}$.

We remark that the graph of a function is a C^α smooth manifold if and only if the function itself is C^α smooth (here α is either ∞ or ω). Thus we have that $V(x,u) \in C^\infty(\widetilde{M_0})$ and is real analytic on $\widetilde{M_0} \setminus M_0$.

We are going to show next that

$$V(z,u) = \sum_{i,j,s} \frac{1}{i!j!s!} \left(\frac{\partial V^{i+j+s}(z,u)}{\partial z^i \partial \overline{z}^j \partial u^s} \right)_{(0,0)} z^i \overline{z}^j u^s$$

when $|z|, u \approx 0$. That is, $V(z,u)$ is real analytic near 0. This will complete the proof of Theorem 9.

To this end, we first note that the point $(z, u+iV(z,u)) \in \widetilde{M} \setminus M$ if and only if there is a unique pair $(r,\xi) \in I_\epsilon^+ \times \Delta$ such that $u \neq 0$ and

$$z = \phi_1(r,\xi), \quad u + iV = \phi_2(r,\xi);$$

that is,

$$(a) \qquad z = r\sigma(r,\xi)\big(1 + \mathcal{F}(r,\xi)\big)$$

$$(b) \qquad u = \mathrm{Re}\phi_2 = r^{2m} + \mathrm{Re}\psi_2(r,\xi)$$

$$(c). \qquad V = \mathrm{Im}\phi_2 = \mathrm{Im}\psi_2(r,\xi)$$

Here we note that $\psi_2(r,\xi)$ is holomorphic in $\xi \in \Delta$ and real analytic in the parameter r. Thus it follows that $\psi_2(r,\xi) = \sum_{i+j\geq \ell} \alpha_{ij} r^i \xi^j$ with $(r,\xi) \in I_\epsilon \times \Delta$ for $\epsilon << 1$. From (b), we obtain $u^{\frac{1}{2m}} = r + r^2 \ell_1^*(r,\xi)$ with $\ell_1^*(r,\xi)$ real analytic jointly in r, ξ when $(r,\xi) \in I_\epsilon \times \Delta$ ($\epsilon << 1$).

When $|r| << 1$ then the implicit function theorem tells us that

$$r = \widetilde{g}(\eta_1, \xi) = \eta_1 \cdot (1 + \widetilde{g}^*(\eta_1, \xi)),$$

where $\eta_1 = u^{\frac{1}{2m}}$ in case $1 >> u \geq 0$ and $\widetilde{g}^* = o(|\eta_1|)$ is jointly real analytic in (η_1, ξ). Thus by (a) we see that

$$z = u^{\frac{1}{2m}} (1 + \widetilde{g}^*(\eta_1, \xi) \sigma(\widetilde{g}(\eta_1, \xi), \xi)(1 + \mathcal{F}(r,\xi)).$$

Write $\eta_2 = z/u^{\frac{1}{2m}}$. We then have

$$\eta_2 = (1+\widetilde{g}^*(\eta_1,\xi)\sigma(\widetilde{g}(\eta_1,\xi),\xi)(1+\mathcal{F}(\widetilde{g}(\eta_1,\xi),\xi)) = \eta_2(\eta_1,\xi). \qquad (d)$$

Notice that when $\eta_1, \xi \approx 0$, we have (a) η_2 is real analytic in ξ and η_1; (b) $\eta_2(0,0) = 0$; (c) $d_\xi(1+\widetilde{g}^*(\eta_1,\xi)), d_\xi\mathcal{F}(\tilde{g}(\eta_1,\xi),\xi) \approx 0$; and (d) $d_\xi\sigma(\widetilde{g}(\eta_1,\xi),\xi) \neq 0$.

We see from the implicit function theorem that (d) can be solved as

$$\xi = f(\eta_1, \eta_2)$$

with f real analytic near $(0,0)$ and $f(0,0) = 0$. Now

$$r = \widetilde{g}(\eta_1, f(\eta_1,\eta_2)) = g(\eta_1,\eta_2),$$

which is also real analytic near 0.

Returning to (c), we see that

$$V(z,u) = \mathrm{Im}\phi_2(g(\eta_1,\eta_2), f(\eta_1,\eta_2)),$$

which is also analytic in η_1, η_2 when $\eta_1, \eta_2 \approx 0$. For η_1 real, write

$$\mathrm{Im}\phi_2(g(\eta_1,\eta_2), f(\eta_1,\eta_2)) = \sum_{i,j,s\geq 0} S_{ijs}\eta_1^i\eta_2^j\overline{\eta}_2^s$$

with $|S_{ijs}| \lesssim R^{i+j+s}$ for some $R >> 1$.

Now, when $|u| < \epsilon^{2m}$ and $|z|/u^{\frac{1}{2m}} < \epsilon$ with $0 < \epsilon << 1$, we have that

$$V(z,u) = \sum_{i,j,s} S_{ijs}u^{\frac{1}{2m}(i-j-s)}z^j\overline{z}^s.$$

However we note that $V(z,u)$ is C^∞ near 0. In particular,

$$\frac{\partial^{j+s}V(z,u)}{\partial z^j\partial\overline{z}^s}\Big|_{(0,u)}$$

is C^∞ in u, as long as $0 \leq u << 1$.

Meanwhile,

$$V^{(j,s)}(0,u) = \frac{\partial^{j+s}V(z,u)}{\partial z^j\partial\overline{z}^s}\Bigg|_{(0,u)} = \sum_{i=0}^{\infty} j!s!S_{ijs}u^{\frac{1}{2m}(i-j-s)}\ .$$

This obviously implies that $S_{ijs} = 0$ when $(1/2m)(i-j-s)$ is not a non-negative integer. Thus

$$V(z,u) = \sum_{i,j,s} S_{ijs} u^{\frac{1}{2m}(i-j-s)} z^j \overline{z}^s = \sum_{\tau,j,s\in \mathbf{Z}^+} S_{2m\tau+j+s,j,s} u^\tau z^j \overline{z}^s$$

when $0 < u < \epsilon^{2m}$, $|z| < \epsilon u^{\frac{1}{2m}}$.

On the other hand,

$$|S_{2m\tau+j+s,j,s}| \lesssim R^{2m\tau+j+s+j+s} \lesssim (R^{2m})^{\tau+j+s}.$$

Thus we conclude that

$$\widetilde{V}(z,u) = \sum_{\tau,j,s} S_{2\tau+j+s,j,s} u^\tau z^j \overline{z}^s$$

is real analytic when $u, |z| \approx 0$. Also $\widetilde{V}(z,u) \equiv V(z,u)$ when $0 < u < \epsilon^{2m}$ and $|z| < \epsilon u^{1/2m}$. Notice that $V(z,u)$ is real analytic on $\widetilde{M_0} \setminus M_0$ and it is C^∞ on $\widetilde{M_0}$. By the unique continuation property of real analytic functions, it follows that $\widetilde{V}(z,u) \equiv V(z,u)$ for all $z, |u| \approx 0$ and $(z,u) \in \widetilde{M_0}$.

At last this completes the proof of Theorem 9.

The proof of Theorem 10 obviously follows from Theorem 9 and the fact that any real analytic Levi-flat surface is locally biholomorphic to an open subset in $\mathbf{R}^3 \subset \mathbf{C}^3$ [Cat].

§4.2**: A concrete example–Proof of Proposition 2**:

When p is a non-degenerate elliptic point in Theorem 9, then a lemma of Kenig-Webster tells that it can be flattened to any order. However, the situation for a degenerate elliptic point is different. As an example, consider the analytic manifold

$$M_n = \{(z,w) : w = |z\overline{z}|^2 + |z\overline{z}|^{3+2n}(|z|z + \sqrt{-1})\}$$

in two dimensional complex space. It turns out that the manifold $\widetilde{M}$ (at least when n is big) in this case is only $C^{3/2+n}$ up to the point $0 \in M$—certainly not real analytic. Meanwhile, M_n cannot be flattened to order $\ell = 10 + 4n$.

To verify this last assertion, we first prove the following uniqueness of attached analytic disks:

Lemma 4.7: Let M be a smooth surface and $p \in M$ be a degenerate elliptic point of order $2m$. Suppose that M can be flattened to some order $\ell > 4m$. Then, for any non-constant holomorphic mapping $\phi : \Delta \to \mathbf{C}^2$ with $\phi(\partial\Delta) \subset M$ and $\|\phi\| \ll 1$, there exists a proper holomorphic mapping σ of Δ and $0 < r \ll 1$ such that $\phi(\xi) = \Psi(\sigma(\xi), r)$.

Proof of Lemma 4.7: The argument of this lemma is similar to that for Proposition 4.3 of [KW]. However, for completeness, we present the following details.

Without loss of generality, we let $p = 0$ and E^* be as constructed in the above section. Define $D = \{(z, w) : |z|^2 + |w|^2 < \epsilon_0, \ \rho(z, w) = q_0(z) + q_1(z) - u < 0\}$. Then D is pseudoconvex and has finite type boundary near 0. So the surface $\rho = 0$ does not contain any non-trivial analytic disks near 0. Now let $\|\phi\| \ll 1$. Then, by the maximum principle, we have $\phi(\Delta) \subset D$. Also, when $r \ll 1$, we have $E^* \subset D$. Note that E^* sweeps out D near 0 when we move E^* along the v-axis ($v = \mathrm{Im} w$). Thus, if $\phi(\Delta)$ is not contained in E^*, then there is a v_0 so that $E_{v_0} \equiv \{(z, u + \sqrt{-1}v_0) : (z, u) \in E^*\}$ will touch some $Z_0 = \phi(\xi_0)$; also near $\phi(\xi_0)$, $\phi(\Delta)$ stays on one side of E_{v_0}. We will claim that this contradicts the Levi-flatness of E_{v_0} near Z_0. In fact, as argued in Lemma 4.5, we can find a small pseudoconvex domain D^* with E_{v_0} as part of the boundary. Meanwhile, we can also choose D^* so that a small piece of $\phi(\Delta)$ near Z_0 is contained in D^*. But this is impossible by the classical results (see [Kr1], for example).

We note that (the interior of) E^* is a smooth Levi flat surface. So $\phi(\Delta)$ is contained in a unique analytic disk of E^*, say in $\Psi(r_0, \Delta)$. Let $\sigma = \Psi^{-1}(\phi(\xi), r_0)$. Using the fact that $\phi(\partial\Delta) \subset M$, one easily sees that σ is a proper self-mapping of Δ. ∎

An immediate corollary of Lemma 4.7 is that the regularity of E^* near 0 is a local biholomorphic invariant. In particular, we see that if M at p can be flattened to some order ℓ, then the image of the local disks attached to M near p fills in a manifold with $C^{\frac{\ell-2m}{2m}}$ boundary at p, where $2m$ is the degeneracy degree of $p \in M$.

For our M_n here, we first consider the model surface $M_0 = \{(z, w) : w = |z|^4\}$. Near the point $(0, 0) \in M_0$, we may attach analytic discs to M_0 in this way: $\{(z, u) :\quad \text{for each fixed } u, |z|^4 < u\}$. Here the parameter is the real variable $u \geq 0$: for each value of u, the associated disc is attached to M_0. Now we may map M_0 to M_n, and $\widetilde{M_0}$ to what will turn out to be E_n^*, by way of the mapping

$$\Phi(z, w) = \big(\phi_1(z, w), \phi_2(z, w)\big)$$

with

$$\phi_1(z, w) = z \quad \phi_2(z, w) = u + u^{3/2+n}(u^{1/2}z + \sqrt{-1}).$$

One can see that this Φ takes each analytic disc in $\widetilde{M_0}$ to an analytic disc attached to M_1. Thus $\Phi(\widetilde{M_0})$ becomes E_n^* (by Lemma 4.5, at least when E^* has C^2 smoothness near 0 it is the local hull of holomorphy of M_n near 0).

Lemma 4.8: E_n^* is of class $C^{n+3/2}$ at 0.

Proof of Lemma 4.8: Let $z = x + \sqrt{-1}y$ and $w^* = u^* + v^*$. Then $E_n^* = \{(z, w^*) :\; u^* = u^{3+n}x + u,\; v^* = u^{3+n}y + u^{3/2+n}\}$. Applying the implicit function to $u^* = u^{3+n}x + u$, we obtain $u = u^* + h^*(x, u^*)$ with h^* analytic in $(u^*,\ x)$ and $h^*(0, u^*) = 0$. Thus E^* is exactly the graph of the function

$$v^*(z, u^*) = (u^* + h^*(x, u^*))^{3+n}y + (u^* + h^*(x, u^*))^{3/2+n}.$$

This function is obviously only $C^{3/2+n}$ at 0. Thus E^* has only $C^{3/2+n}$ smoothness at 0. ∎

Now by Lemma 4.2 and the regularity result obtained in the last section, we see that M_n cannot be flattened to order $\ell = 10 + 4n$ (we remark that E_n^* is the local hull of M_n at least when $n > 3$.) Thus we also conclude that the local hull of M_n has only smoothness $C^{3/2+n}$ at 0).

The argument for Proposition 2 is now complete. ∎

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