Hamiltonian Cycles in Dirac Graphs*

Bill Cuckler
University of Delaware
Department of Mathematical Sciences
Newark, DE 19716
cuckler@math.udel.edu

Jeff Kahn
Rutgers University
Department of Mathematics
Piscataway, NJ 08854
jkahn@math.rutgers.edu

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Abstract

We prove that for any n-vertex Dirac graph (graph with minimum degree at least $n/2$) $G = (V, E)$, the number, $\Psi(G)$, of Hamiltonian cycles in G is at least

$$\exp_2[2h(G) - n \log e - o(n)],$$

where $h(G) = \max \sum_e \mathbf{x}_e \log(1/\mathbf{x}_e)$, the maximum over $\mathbf{x} : E \to \Re^+$ satisfying $\sum_{e \ni v} \mathbf{x}_e = 1$ for each $v \in V$, and $\log = \log_2$. (A second paper will show that this bound is tight up to the $o(n)$.)

We also show that for any (Dirac) G of minimum degree at least d, $h(G) \geq (n/2) \log d$, so that $\Psi(G) > (d/(e + o(1))^n$. In particular, this says that for *any* Dirac G we have $\Psi(G) > n!/(2 + o(1))^n$, confirming a conjecture of G. Sárközy, Selkow, and Szemerédi which was the original motivation for this work.

1 Introduction

A graph is said to be *Dirac* if its minimum degree is at least $n/2$; this is in honor of the seminal 1952 result of Dirac [8] proving that any such graph has a Hamiltonian cycle. In this paper we are interested in lower bounds for the *number* of Hamiltonian cycles in a Dirac graph. Write $\Psi(G)$ for the number of Hamiltonian cycles in an (arbitrary) graph G, and $\Psi(n)$ for the minimum of $\Psi(G)$ over n-vertex Dirac graphs G. (Throughout this discussion n is the default for $|V(G)|$.)

The question of estimating $\Psi(n)$ was raised by Bondy in [3, p.79], and also, according to [19], at several conferences. (See also Bollobás [2, p.1260].) Much earlier Nash-Williams [16] had proved that any Dirac graph has at least $\frac{5}{224}n$ edge-disjoint Hamiltonian cycles, so in particular, $\Psi(n) \geq \frac{5}{224}n$. (See [2] for more on the number of disjoint Hamiltonian cycles.) Sárkőzy et al. [19] proved, using the regularity lemma [20], that $\Psi(n) \geq c^n n!$ for some (very) small positive constant c, and conjectured that c can be improved to $1/2 - o(1)$; this is our first result:

Theorem 1.1 *For any n-vertex Dirac graph G,*

$$\Psi(G) \geq n!/(2 + o(1))^n. \tag{1}$$

Of course it's easy to give examples where this bound (apart from the factor $(1 + o(1))^n$) is attained; but the theorem is best possible in a stronger sense: Brégman's Theorem ([4], formerly the Minc Conjecture) on permanents of $\{0, 1\}$-matrices implies that for *any* $(n/2)$-regular (n-vertex) G one has $\Psi(G) \leq ((n/2)!)^2$ $(= O(\sqrt{n}2^{-n}n!)$.

We will actually prove something considerably more general than Theorem 1.1, for statement of which we need a few definitions. For an edge weighting $\mathbf{x} : E \to \Re^+$, set $h(\mathbf{x}) = \sum_e \mathbf{x}_e \log(1/\mathbf{x}_e)$. We will call this the *entropy* of $\mathbf{x}$ (but note it is not really entropy since $\sum \mathbf{x}_e$ will not usually be 1). Call an edge weighting $\mathbf{x}$ *proper* if $\sum_{e \ni v} \mathbf{x}_e = 1$ for each $v \in V$. (Such an $\mathbf{x}$ is also called a "perfect fractional matching.") Finally, let $h(G)$ (the "entropy" of G) be the maximum of $h(\mathbf{x})$ over proper edge weightings $\mathbf{x}$. (In the absence of proper weightings we may set $h(G) = 0$, but this won't be an issue here.) Our main result is

Theorem 1.2 *For any n-vertex Dirac graph G,*

$$\log \Psi(G) \geq 2h(G) - n \log e - o(n).$$

To get Theorem 1.1 from this, we need a lower bound on h for Dirac graphs. This is our second, though easier, main point:

Theorem 1.3 *If $\delta(G)$ (the minimum degree of G) is at least $d \geq n/2$, then*

$$h(G) \geq (n/2) \log d. \tag{2}$$

This is easy (and sharp) when the graph is d-regular ($\mathbf{x}_e = 1/d$ $\forall e$ attains the bound and is easily seen to maximize h), but does not seem obvious in general.

Of course Theorems 1.2 and 1.3 give a natural extension of Theorem 1.1:

Corollary 1.4 *For $d \geq n/2$, any n-vertex G of minimum degree at least d satisfies $\Psi(G) \geq (d/(e+o(1)))^n$.*

In particular for regular G this again matches the Brégman upper bound ($\Psi(G) \leq (d!)^{n/d}$ for d-regular G).

In fact Theorem 1.2 is *always* sharp (again up to the error factor). Write $\Phi(G)$ for the number of perfect matchings of a (general) graph G.

Theorem 1.5 *For any n-vertex Dirac graph G,*

$$\log \Psi(G) = 2h(G) - n \log e - o(n),$$

(note $o(n)$ is not necessarily positive), and, if n is even,

$$\log \Phi(G) = h(G) - (n/2) \log e - o(n),$$

so that $\log \Psi(G) = 2 \log \Phi(G) - o(n)$.

Of course $\log \Psi(G) < 2 \log \Phi(G)$ (i.e. $\Psi(G) < \Phi^2(G)$) is trivial for any graph G with an even number of vertices. Thus the lower bound on $\log \Phi(G)$ given by Theorem 1.5 is contained in that on $\log \Psi(G)$, which is Theorem 1.2. The upper bounds will be proved in a separate paper [7] as part of a more general Brégman-like result. (The upper bound on Ψ follows from that for Φ; this is trivial when n is even, and turns out to be only slightly less so when n is odd.)

Note that in Theorem 1.5 we have, quite surprisingly, a simple parameter that essentially captures the behavior of Ψ (and also Φ in case n is even) for an arbitrary Dirac graph. This has algorithmic implications: since $h(G)$, the maximum of a concave function subject to linear constraints, can be estimated efficiently, we have an efficient algorithm for estimating both Ψ and Φ for Dirac graphs to within subexponential factors.

This is reminiscent of a beautiful result of Linial et al. [12] on approximating permanents of nonnegative matrices. They show in particular that one can approximate such a permanent to within a factor e^n (actually meaning to within $e^{n/2}$) in *deterministic* polynomial time. (One can do much better with randomization [10].) Of course this includes estimating Φ for bipartite graphs.

In fact bipartite analogues of the preceding results are also true; precisely, for a balanced bipartite G on $2n$ vertices, we have

$$\log \Psi(G) = 2h(G) - 2n \log e - o(n), \tag{3}$$

if $\delta(G) > n/2$, and

$$\log \Phi(G) = h(G) - n \log e - o(n) \tag{4}$$

if $\delta(G) \geq n/2$, as well as a bipartite version of Theorem 1.3 (Theorem 3.1). The latter is thought to be of some independent interest and is proved in Section 3, but apart from this we will not pursue the bipartite assertions here. Proofs of the lower bounds in (3) and (4) are similar to the main arguments of the present paper, though there are some additional graph theoretic complications. The upper bounds in (3) and (4) will again follow from [7].

In the rest of this introduction we will try to say roughly what's involved in the proof of Theorem 1.2. Our basic approach—analysis of a self-avoiding walk on G—is similar to that in [6], which in turn was inspired by [11]. In [6], which proves a result analogous to Theorem 1.1 for regular tournaments, the walk is just the natural one: the next vertex is chosen uniformly from the as yet unvisited outneighbors of the current vertex. The present procedure will also reduce to this in case G is regular (except we should replace "outneighbors" by "neighbors"). In general the walk is taken according to the (entropy maximizing) $\mathbf{x}$ realizing $h(G)$; that is, the next vertex is chosen from the as yet unseen neighbors of the current vertex with probabilities proportional to the edge weights. (We take X_0 to be an arbitrary but fixed vertex.)

We stop the walk at a time $l = n - o(n/\log n)$, and would like to show that for the stopped walk, say $X = (X_0, \ldots, X_l)$, we have, except in fairly pathological situations,

$$H(X) \geq 2h(\mathbf{x}) - n \log e - o(n) \tag{5}$$

(where H is (ordinary, binary) entropy; see [5] or [13] for entropy basics), and

$$\text{w.h.p. } X \text{ can be completed to a Hamiltonian cycle.} \tag{6}$$

Of course if we have these we are done, since (6) implies

$$\log \Psi(G) \geq H(X) - 1 - o(1).$$

The pathological situations are those in which G is close to either a complete bipartite graph or the complement thereof. When this happens, $h(\mathbf{x})$ can be shown to be not much more than $(n/2)\log(n/2)$; so we are aiming for the lower bound of Theorem 1.1, which (in the pathological cases) is not too hard to establish directly. At any rate, for the present discussion we assume G is not pathological.

To see why (5) is natural, we expand $H(X)$ via the "chain rule,"

$$H(X) = H(X_1|X_0) + \cdots + H(X_l|X_0, \ldots, X_{l-1}), \tag{7}$$

and consider the contribution of the step leaving a particular $w \in V$. Our guiding idea is that $\{X_0, \ldots, X_i\}$ looks roughly like a random (uniform) $(i+1)$-subset of V. If this is indeed the case, then an easy calculation shows that, when $w = X_{i-1}$, $H(X_i|X_1, \ldots, X_{i-1})$ should typically be about $h(w) + \log(1 - i/n)$, where $h(w) = \sum_z \mathbf{x}_{wz} \log(1/\mathbf{x}_{wz})$ (the entropy of the first step of a walk with $X_0 = w$). Thus, overall, the entropy of the walk should be about $\sum h(w) + \sum_i \log(1 - i/n) \approx \sum h(w) - n \log e$ (note we have chosen l large enough that we can safely ignore the missing $n - l$ terms). Since $\sum h(w) = 2h(\mathbf{x})$, this (suitably quantified) gives (5).

Our goal, then, is to show that this idealized behavior is not too different from what actually happens. (This will say in particular that $V \setminus \{X_0, \ldots, X_l\}$ looks roughly like a random $(n - l - 1)$-subset of V, and (6) will follow relatively easily from this.)

To keep track of how things evolve we work with a family $\mathcal{F}$ of "relevant" functions $f : V \to \Re^+$, and consider the walk to be "good" if it gives accurate samples of the relevant f's, meaning (roughly) that for each $f \in \mathcal{F}$ and i,

$$f(\{X_0, \ldots, X_i\}) \approx i f(V)/n$$

(where $f(S) = \sum_{v \in S} f(v)$). Our main task will be to show that the walk is likely to be good (Lemma 2.2).

The relevant functions are both a means of measuring the progress of the walk and, at the end, the basis for the conclusions (5) and (6). Here are two examples: $f_v(w) = \mathbf{x}_{vw}$ will be a relevant function (for each v), so that if the walk is good through time i, then the (normalization of the) restriction of $\mathbf{x}$ to $G_i := G - \{X_0, \ldots, X_{i-1}\}$ is nearly proper, implying that the stationary distribution of the ordinary $\mathbf{x}$-walk on G_i is close to uniform (which is one prerequisite for continued good behavior); and proper evolution of the f_v's and the (relevant) functions $g_v(w) := \mathbf{x}_{vw} \log(1/\mathbf{x}_{vw})$ will fairly easily give (5). (See the proof of (16) at the end of Section 2.)

The analysis that establishes good behavior is reminiscent of the celebrated "nibble" method (e.g. [18] or [14]), in that we break the walk into fairly short intervals and show that actual behavior over such an interval is (very) likely to be close to expected behavior.

We briefly sketch how this will go. We will be considering the behavior of the walk over an interval $(X_a, \ldots, X_b)$, in part by comparing it with the corresponding ordinary walk, say $Y = (X_a = Y_0, Y_1, \ldots)$, on G_a. There are three parts to the analysis.

First, the behavior of $(X_a, \ldots)$ closely follows that of Y. This is easy: if we couple the two walks so that they agree up to the first time that Y revisits a vertex, then this agreement is likely to continue up to time b, provided $b-a$ is not too large. (This is why we need to make the intervals fairly short.)

Second, Y has nearly uniform stationary distribution (a consequence, as noted above, of the assumption that X is good up to time a) and mixes (converges to stationarity) very rapidly. (This is where we need to assume G_a is not pathological).

Combining these two observations, we find that, even for q quite small, the distribution of each X_i given the walk to time $i-q$ is typically close to uniform distribution on $V \setminus \{X_1, \ldots, X_{i-q}\}$; so the *expectations* $\mathsf{E}f(X_i)$ for $f \in \mathcal{F}$ (again, given the walk to time $i-q$) are close to what we would like. (Of course nothing like this is true if we condition on the walk to time $i-1$.)

The third—and perhaps main—point is that the *actual* values of quantities $\sum\{f(X_i) : a \leq i \leq b\}$ are very likely to be close to what these expectations suggest. For this we need Azuma's inequality, applied here in a slightly atypical setting, in which the expectations in question are not fixed in advance, but are themselves functions of the evolving sequence X (see Lemma 5.3).

Technical lemmas supporting this analysis are proved in Section 5 and the analysis itself is carried out in Section 6. We begin, in Section 2, by proving Theorem 1.2 modulo the results of these later sections and Section 4, which develops the graph theory needed for (6) and the pathological cases of Theorem 1.2. Section 3 contains the proofs of Theorem 1.3 and the aforementioned bipartite version, Theorem 3.1, as well as one technical result on the values taken by an entropy-maximizing $\mathbf{x}$.

2 Main points and proof of Theorem 1.2

Here, to see more concretely where this is all headed, we want to give the proof of Theorem 1.2 modulo several results that will be proved in later sections, the most important being Lemma 2.2. We first need some preliminaries.

Two general conventions: we use $\mu(f)$ for the expectation of a function f with respect to a probability measure μ, and $a = (1 \pm \delta)b$ for "$a \in ((1 - \delta)b, (1 + \delta)b)$."

For the following definitions we suppose $G = (V, E)$ is a graph on n vertices and $\mathbf{x} : E \to \Re^+$. A *half-set* of G is a subset of V of size $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. For $A, B \subseteq V$ set $e(A, B) = |\{ab \in E : a \in A, b \in B\}|$ and $\mathbf{x}(A, B) = \sum\{\mathbf{x}_{ab} : a \in A, b \in B\}$. (Here and elsewhere we set $\mathbf{x}_{vw} = 0$ when $vw \notin E$.) Note that the last two expressions double count edges contained in $A \cap B$; we will sometimes avoid this by using $e(A) = e(A, A)/2$ and $\mathbf{x}(A) = \mathbf{x}(A, A)/2$.

For $\xi \in [0, 1]$, we say G is ξ-*Dirac* if it has minimum degree at least $(1 - \xi)n/2$, ξ-*normal* if for any two half-sets A, B we have $e(A, B) > \xi n^2$, and ξ-*special* if it is not ξ-*normal*. We also say that $\mathbf{x} : E \to \Re^+$ is ξ-*normal* if $\mathbf{x}(A, B) > \xi n$ for any two half-sets A, B, and ξ-*special* otherwise. These definitions are inspired by the "extremal condition" of [19, p.40].

Notice that if G admits a ξ-normal proper weighting $\mathbf{x}$, then G itself is ζ-normal with $\zeta = \xi/(n\|\mathbf{x}\|_\infty)$ (since for any half-sets A, B we have $\mathbf{x}(A, B) \leq e(A, B)\|\mathbf{x}\|_\infty$). We set

$$\xi = \log^{-3} n \quad \text{and} \quad \zeta = \tfrac{1}{64}\xi^4; \tag{8}$$

ξ will soon be the assumed normality of $\mathbf{x}$ and ζ the corresponding normality of G given by Lemma 3.2.

We now specify the set $\mathcal{F}$ of relevant functions; these will be of five types,

f_v, g_v, h_v, g_{vw} and h_{wx}. The first three are defined for each $v \in V$ and are given by

$$f_v(w) = \mathbf{x}_{vw},$$

$$g_v(w) = \mathbf{x}_{vw} \log(1/\mathbf{x}_{vw}),$$

and

$$h_v = \mathbf{1}_{N(v)}$$

(where $N(v)$ is the neighborhood of v). For the g_{vw}'s and h_{wx}'s, let

$$A(w) = \{x \in V : |N(x) \setminus N(w)| > \zeta n/4\}. \tag{9}$$

We set

$$g_{vw}(x) = |N(x) \setminus N(w)| \mathbf{1}_{A(w) \setminus N(v)}(x),$$

$$h_{wx} = \mathbf{1}_{N(x) \setminus N(w)},$$

and put $g_{vw} \in \mathcal{F}$ if $e(V \setminus N(v), V \setminus N(w)) > 2\zeta n^2/3$, and $h_{wx} \in \mathcal{F}$ if $x \in A(w)$.

Set

$$\rho = \max\{\|f\|_\infty / f(V) : f \in \mathcal{F}\}$$

(note f can be any of the five types of functions in $\mathcal{F}$ and (recall) $f(S)$ is $\sum_{v \in S} f(v)$). Then (it is not hard to see that)

$$\rho \le \max\{4/(\zeta n), \|\mathbf{x}\|_\infty\} \tag{10}$$

(which will turn out to be $4/(\zeta n)$).

We next need some terminology regarding the random walk. Formally this is $X = (X_0, \ldots, X_l)$, where X_0 is some fixed vertex and, for $i \ge 1$,

$$\Pr(X_i = w | X_0, \ldots, X_{i-1}) \propto \mathbf{x}_{w, X_{i-1}} \mathbf{1}_{\{w \notin \{X_0, \ldots, X_{i-1}\}\}}.$$

(Recall $\mathbf{x}_{vw} = 0$ if $vw \notin E$. As usual, "$\propto$" means "is proportional to.") We use $X_I = (X_i : i \in I)$, $X(i) = (X_0, \ldots, X_i)$ $(= X_{\{0,\ldots,i\}})$, and $V_i = V \setminus \{X_0, \ldots, X_{i-1})$ (so $G_i = G[V_i]$).

We will break the walk into intervals of length κ; this will be of the form $n^{2/3} \log^{O(1)} n$, but we defer a more precise specification until (56). A *milestone* is an index of the form $r\kappa$, with r a nonnegative integer, and a *basic segment* is a subwalk $(X_a, \ldots, X_b)$ with a, b consecutive milestones.

We need to show that the walk is likely to satisfy $f(V_j) \approx (n - j) f(V)/n$ for each relevant f and each j. As one might expect, the accuracy of these estimates deteriorates over time; to allow for this we introduce "phases" of the walk, and a slowly relaxing sequence of "tolerances" δ_m as follows.

Set

$$c = 1/4 \quad \text{and} \quad \gamma = 1/6. \tag{11}$$

(We will need $c < 1/3$ (see 62), and $c(1-\gamma) > \gamma$ (see 12).) Let $a_0, \ldots, a_s = l$ be the sequence of milestones with $a_0 = 0$ and, for $i \geq 1$,

$$n - a_i = \lfloor (1-\gamma)(n - a_{i-1})/\kappa \rfloor \kappa$$

(that is, a_i is the first milestone for which $n - a_i \leq (1-\gamma)(n - a_{i-1})$). Thus $l = n - o(n/\log n)$ corresponds to

$$s = -(\log\log n + \omega(1))/\log(1-\gamma).$$

(It will be enough to know that $s = \Theta(\log\log n)$.) The ith *phase* of the walk is then $(X_{a_{i-1}}, \ldots, X_{a_i})$

Now set $\delta_0 = 0$ and define $\delta_1, \ldots, \delta_s$ by

$$\delta_m = [(c(1-\gamma)-\gamma)(1 - a_{m-1}/n)]^{-1}[a_{m-1}\delta_{m-1}/n + 2\kappa\rho] \tag{12}$$

for $1 \leq m \leq s$. Despite this very precise definition, we won't need to be very careful with the actual values of the δ's; for instance, it will be enough to say that (as is easily verified)

$$\delta_s < \exp[O((\log\log n)^2)]\kappa\rho. \tag{13}$$

Say that a subwalk X_I of phase m is *good* if

$$f(X_I) = (1 \pm \delta_m)\frac{|I|}{n} f(V) \tag{14}$$

for each $f \in \mathcal{F}$, and that $X(i)$ is good (or X is *good through time* i) if each basic segment contained in $X(i)$ is good. In particular, the entire walk is good if each of its basic segments is good.

Proof of Theorem 1.2

We assume throughout that G is an n-vertex Dirac graph, and that

$$\mathbf{x} \text{ is entropy maximizing} \tag{15}$$

(that is, $h(\mathbf{x}) = h(G)$). Lemma 2.2 below is the heart of the matter, but can fail to apply if $\mathbf{x}$ is sufficiently "abnormal"; in such cases Theorem 1.2 is contained in

Lemma 2.1 *If G is an arbitrary graph and $\mathbf{x}$ a φ-special proper weighting, then $h(\mathbf{x}) \le (n/2)\log(n/2) + O\left(\varphi n \log^2(1/\varphi)\right)$. If, in addition, G is Dirac and $\varphi \le 1/\log^3 n$, then $\Psi(G) \ge n!/(2+o(1))^n$.*

This is proved in Section 4.

So, with ξ as in (8), we may assume for the remainder of this discussion that $\mathbf{x}$ is ξ-normal. We then have the main point:

Lemma 2.2 *If* $\mathbf{x}$ *is* ξ*-normal then* X *is good with probability at least* $1 - o(1/\log n)$.

(The bound $o(1/\log n)$ is what the present argument requires, but there is plenty of room here and we could easily do much better.) A little reflection should convince the reader that some normality assumption is needed here. (Think of G consisting of two disjoint cliques of size $n/2$ joined by a few edges.)

So we are entitled to assume the conclusion of Lemma 2.2 and want to show this implies (5) and (6). We first verify (6), which is mainly an instance of the following graph-theoretic lemma, proved in Section 4. Recall that a graph is *Hamiltonian connected* if any two of its vertices can be joined by a Hamiltonian path.

Lemma 2.3 *If a graph* H *is* α*-Dirac and* β*-normal on* n *vertices with* $\beta > \max\{2\alpha, 13/n\}$*, then* H *is Hamiltonian connected.*

(The $13/n$ should of course be ignored.) We want to apply this to $G[V_l \cup \{X_0\}]$, so essentially need to say that G_l has the appropriate normality and "Diracity" whenever X is good. These are both special cases of properties we need to establish in the course of proving Lemma 2.2: see (47) and (49), which imply in particular that $G[V_l \cup \{X_0\}]$ satisfies the hypotheses of Lemma 2.3 with $\alpha = \delta_s$ and $\beta = \frac{1}{320}\xi^4$ $(= \omega(\alpha))$. (That G_l is $(\delta_s n/(n-l))$-Dirac is immediate from the definition of "good" applied to the functions h_v; we could get by with this, but may as well assert the stronger property since we will wind up proving it anyway.) So we have (6).

Turning to (5), we consider the expansion (7). Given $v \in V$, set

$$h_i(v) = \sum\{(\mathbf{x}_{vw}/\mu)\log(\mu/\mathbf{x}_{vw}) : w \in V_i\},$$

where we (temporarily) set $\mu = \sum\{\mathbf{x}_{vw} : w \in V_i\} = f_v(V_i)$. Thus for $v \in V_i$, $h_i(v)$ is the entropy of a random step from v (according to $\mathbf{x}$) in G_i, and we have

$$H(X_{i+1}|X_0, \ldots, X_i) = \mathsf{E}h_i(X_i).$$

Claim. *If X is good then*

$$\sum_{i=0}^{l-1} h_i(X_i) > 2h(\mathbf{x}) - n\log e - o(n). \tag{16}$$

(Note the familiar abuse: this recycles "X" as a possible value of the random object we've been calling X.)

Note that (5) follows easily from the claim when we recall (Lemma 2.2) that we have the lower bound $1 - o(1/\log n)$ on the probability that the walk is good (and the trivial upper bound $n\log n$ on $h(\mathbf{x})$).

Proof of Claim. Write δ for δ_s. Given i, set $t = t(i) = \lceil i/\kappa \rceil \kappa$ (the first milestone not preceding i); then (trivially) $h_i(v) \geq h_t(v)$ for any v (recall $h_t(v)$ was defined for $v \in V$). Temporarily fix v and set $\mu = f_v(V_t)$. Then

$$h_t(v) = \sum \{(\mathbf{x}_{vw}/\mu) \log(\mu/\mathbf{x}_{vw}) : w \in V_t\} = \mu^{-1} g_v(V_t) + \log \mu. \tag{17}$$

On the other hand, since X is good, we have

$$\mu = (1 \pm \delta)(n-t)/n \quad \text{and} \quad g_v(V_t) > (1-\delta)h(v)(n-t)/n$$

(note $h(v) = \sum_z \mathbf{x}_{vz} \log(1/\mathbf{x}_{vz}) = g_v(V)$), implying

$$h_t(v) > \frac{1-\delta}{1+\delta}\, h(v) + \log \frac{n-t}{n} + \log(1-\delta).$$

Thus we have

$$\begin{aligned} \sum_{i=0}^{l-1} h_i(X_i) &> \frac{1-\delta}{1+\delta} \sum_{i=0}^{l-1} h(X_i) + \sum_{i=0}^{l-1} \log \frac{n-t(i)}{n} + l \log(1-\delta) \\ &> (1-2\delta) \sum_{v \in V} h(v) - (n-l) \log n - n \log e + l \log(1-\delta) \\ &= 2h(\mathbf{x}) - n \log e - o(n). \end{aligned}$$

■

3 Weights

In this section we are mainly concerned with proving Theorem 1.3 and the following related result.

Theorem 3.1 *If G is bipartite on $X \cup Y$ with $|X| = |Y| = n$ and all degrees at least $d \geq n/2$, then $h(G) \geq n \log d$.*

Before turning to these we establish one technical point that will be useful when we consider the mixing speed of the ordinary random walk associated with an optimal $\mathbf{x}$ (see Lemma 5.2).

Lemma 3.2 *If $\mathbf{x}$ satisfies* (15) *and is ξ-normal, then*

$$\mathbf{x}_e \in (2^{-10}\xi^7 n^{-1}, 64\xi^{-3}n^{-1}) \quad \forall e \in E.$$

Proof. We first observe that for any even closed walk $(v_0, \ldots, v_{2k-1}, v_{2k})$ with $\mathbf{x}_i := \mathbf{x}_{v_{i-1}v_i}$ (and $v_{2k} = v_0$), we have

$$\mathbf{x}_1\mathbf{x}_3 \cdots \mathbf{x}_{2k-1} = \mathbf{x}_2\mathbf{x}_4 \cdots \mathbf{x}_{2k}, \tag{18}$$

since otherwise, for an appropriate nonzero ε, we could increase $h(\mathbf{x})$ by adding ε to $\mathbf{x}_1, \ldots, \mathbf{x}_{2k-1}$ and subtracting it from $\mathbf{x}_2, \ldots, \mathbf{x}_{2k}$ (so the change on a repeated edge will be some multiple of ε).

We first rule out very large weights in $\mathbf{x}$ and then very small ones. For the large weights, suppose $\mathbf{x}_{uv} = T/n$ with $T > 64\xi^{-3}$. Set $S = T^{1/3}$ and let $A = \{w \sim u : \mathbf{x}_{uw} < S/n\}$, $B = \{w \sim v : \mathbf{x}_{vw} < S/n\}$. Then $|A|, |B| > n/2 - n/S$, so ξ-normality of $\mathbf{x}$ implies $\mathbf{x}(A, B) > \xi n - 2n/S$. It follows that there are $a \in A, b \in B$ with $\mathbf{x}_{ab} > (4/n^2)(\xi n - 2n/S) > 2\xi/n$. But then the cycle (u, v, b, a) violates (18) (namely $\mathbf{x}_{uv}\mathbf{x}_{ab} > 2T\xi/n^2 > S^2/n^2 > \mathbf{x}_{ua}\mathbf{x}_{vb}$), so we have $\|\mathbf{x}\|_\infty \le 64\xi^{-3}/n$.

Now suppose that for some $uv \in E$ we have $\mathbf{x}_{uv} < \varepsilon/n$. Choose w, z with $\mathbf{x}_{uw}, \mathbf{x}_{vz} > 1/n$ and set $A = N(w), B = N(z)$. Then ξ-normality of $\mathbf{x}$ gives $a \in A, b \in B$ with $\mathbf{x}_{ab} > 4\xi/n$. But we also know that $\mathbf{x}_{wa}, \mathbf{x}_{zb} < 64\xi^{-3}/n$, so, again appealing to (18), we have

$$(64\xi^{-3})^2\varepsilon/n^3 > \mathbf{x}_{uv}\mathbf{x}_{wa}\mathbf{x}_{zb} = \mathbf{x}_{uw}\mathbf{x}_{vz}\mathbf{x}_{ab} > 4\xi/n^3,$$

and $\varepsilon > 2^{-10}\xi^7$. ■

Note in particular that for $\mathbf{x}$ as in Lemma 3.2—as it will be in Section 6—we have $\|\mathbf{x}\|_\infty < 4/(\zeta n)$, so that (10) becomes

$$\rho \le 4/(\zeta n). \tag{19}$$

We now turn to Theorem 1.3. We will prove something a little stronger: any graph of minimum degree $d \ge n/2$ admits a proper edge weighting $\mathbf{x}$ with

$$\sum_{e \in E} \mathbf{x}_e^2 \le \frac{n}{2d}. \tag{20}$$

To see that this is stronger, note that (2) is the same as

$$\sum_e \frac{2\mathbf{x}_e}{n} \log \mathbf{x}_e \leq \log(1/d). \tag{21}$$

On the other hand, if (20) holds, then Jensen's inequality (with $\sum \mathbf{x}_e = n/2$) gives

$$\sum_e \frac{2\mathbf{x}_e}{n} \log \mathbf{x}_e \ \leq \ \log \sum_e \frac{2\mathbf{x}_e^2}{n} \ \leq \ \log(1/d).$$

Lemma 3.3 *Under the hypotheses of Theorem 1.3 there is a vertex weighting $\mathbf{u} : V \to \mathbf{R}^+$ for which the edge weighting $\mathbf{x}$ given by*

$$\mathbf{x}_{vw} = \mathbf{u}_v + \mathbf{u}_w \tag{22}$$

is proper and satisfies (20)*.*

Proof. If $G = K_{n/2,n/2}$ then we may take $\mathbf{u}_v = 1/n$ for all v; so we may assume this is not the case, and in particular G is nonbipartite.

Let M be the $(V \times E)$ vertex-edge incidence matrix of G (that is, the (v, e)-entry of M is 1 if $v \in e$ and 0 otherwise), $A = MM^t$, and $\mathbf{u}$ the unique solution to

$$A\mathbf{u} = \mathbf{1} \tag{23}$$

(where $\mathbf{1}$ is the all 1's vector; note G connected and nonbipartite implies A is nonsingular). We then define $\mathbf{x}$ by (22)—that is, $\mathbf{x} = M^t\mathbf{u}$—and note that

$\sum_{e \ni v} \mathbf{x}_e = 1$ for each v (this is equivalent to (23)). The key to showing that $\mathbf{x}$ (is nonnegative and) satisfies (20) is

$$\mathbf{u} \geq \mathbf{0}. \tag{24}$$

We first derive (20) from this. We have

$$\sum_{e \in E} \mathbf{x}_e^2 = \mathbf{u}^t M M^t \mathbf{u} = \mathbf{u}^t \mathbf{1} = \sum_{v \in V} \mathbf{u}_v;$$

so we need to say the last sum is at most $n/(2d)$, which is true:

$$n/2 = \sum_e \mathbf{x}_e = \sum_v \mathbf{u}_v d(v) \geq d \sum \mathbf{u}_v, \tag{25}$$

where we used (24) (and $d(v) \geq d$) for the inequality. $\square$

Proof of (24). Assume for a contradiction that $N := \{x \in V | \mathbf{u}_x < 0\} \neq \emptyset$, and set $P = \{x \in V | \mathbf{u}_x > 0\}$. Write α_v for $-\mathbf{u}_v$ (we will use this only when $v \in N$). For any $y \in N$,

$$\sum_{x \in P} \mathbf{u}_x \geq \sum \{\mathbf{u}_x : y \sim x \in P\} = 1 + \alpha_y d(y) + \sum \{\alpha_x : y \sim x \in N\}.$$

Setting $d'(y) = d_N(y)$ and averaging over $y \in N$ gives

$$\sum_{x \in P} \mathbf{u}_x \geq 1 + \frac{1}{|N|} \sum_{y \in N} \alpha_y (d(y) + d'(y)).$$

Inserting this in the inequality

$$d \geq n/2 = \sum \mathbf{x}_e = \sum_{x \in P} \mathbf{u}_x d(x) - \sum_{y \in N} \alpha_y d(y) \tag{26}$$

and using the lower bound on degrees, we have

$$d \geq d(1 + \frac{1}{|N|} \sum_{y \in N} \alpha_y (d(y) + d'(y)) - \sum_{y \in N} \alpha_y d(y); \tag{27}$$

that is,

$$0 \geq \frac{d}{|N|} \sum_{y \in N} \alpha_y (d(y) + d'(y)) - \sum_{y \in N} \alpha_y d(y). \tag{28}$$

Combining this with the trivial

$$d'(y) \geq d(y) - (n - |N|) \geq d(y) - 2d + |N|, \tag{29}$$

we have

$$0 \geq \frac{d}{|N|} \sum_{y \in N} \alpha_y (2d(y) - 2d + |N|) - \sum_{y \in N} \alpha_y d(y),$$

or

$$0 \geq \left(\frac{2d}{|N|} - 1 \right) \sum_{y \in N} \alpha_y (d(y) - d). \tag{30}$$

Since $2d > |N|$ and $d(y) \geq d$ for each y, it must be the case that all inequalities used above are actually equalities. In particular, $n = 2d$ (see (26)); $d(z) = d$ for each $z \in P \cup N$ (see (27) and (30)); and $E(N, V \setminus N)$ ($:= \{yz : y \in N, z \in V \setminus N\}$) $\subseteq E$ (see (29)). Then each of $|N|, |V \setminus N|$ is at most $d = n/2$, so in fact each is exactly $n/2$; the neighborhood of each $x \in P$ is precisely N; and $P \cup N = V$, since v with $\mathbf{u}_v = 0$ would have no neighbors in P, so lie in no edges of positive weight. So, finally, we find that, contrary to assumption, $G = K_{n/2,n/2}$. ■

Proof of Theorem 3.1. This is similar to the proof of Theorem 1.3 and we focus on the differences. The analogue of (20) is

$$\sum_{e\in E} \mathbf{x}_e^2 \leq \frac{n}{d}\ . \tag{31}$$

The derivation of Theorem 3.1 from this is as before, except the 2's should be removed from (21) and the following display. The statement of Lemma 3.3 is unchanged (except that the hypotheses are now those of Theorem 3.1 and (20) becomes (31)).

For the proof of this version of Lemma 3.3 we again first exclude one easy case, namely that $G = 2 \cdot K_{n/2,n/2}$ (two disjoint copies of $K_{n/2,n/2}$). This allows us to assume that G is connected.

We define M and A as before, but note that now $\text{rank}(A) = \text{rank}(M) = 2n-1$, the (right) kernel of A being $\langle \mathbf{w}\rangle$, where $\mathbf{w}$ is 1 on X and -1 on Y. Since $\langle \mathbf{1}, \mathbf{w}\rangle = 0$, we have $\mathbf{1} \in \text{row}(A)$, and it is now enough to show that (24) holds for *some* $\mathbf{u}$ satisfying (23). (Derivation of (31) from this is the same as that of (20) from (24), except we should replace $n/2$ by n in (25).)

The solution set of (23) is a translate of $\langle \mathbf{w}\rangle$, so, assuming for a contradiction that this contains no nonnegative vectors, we may take $\mathbf{u}$ to be the unique member satisfying

$$\sum \alpha_x = \sum \alpha_y, \tag{32}$$

where we again adopt the convention $\alpha_v = -\mathbf{u}_v$ used only when $\mathbf{u}_v < 0$, and now reserve x and y for vertices of X and Y respectively.

We define P and N as before, set $X_P = X \cap P$ and define X_N, Y_P and Y_N similarly. We also use $d'(v)$ as before, so (29) is replaced by

$$d'(x) \geq d(x) - (n - |Y_N|) \geq d(x) - 2d + |Y_N|, \tag{33}$$

and the analogous statement for the $d'(y)$'s.

The first two inequalities from the proof of (24) are now pairs of inequalities: for $y \in Y_N$,

$$\sum \mathbf{u}_x \geq \sum \{\mathbf{u}_x : y \sim x \in X \setminus X_N\} = 1 + \alpha_y d(y) + \sum \{\alpha_x : y \sim x \in X_N\},$$

whence

$$\sum \mathbf{u}_x \geq 1 + \frac{1}{|Y_N|} \left[\sum \alpha_y d(y) + \sum \alpha_x d'(x) \right],$$

and similarly with the roles of X and Y reversed. Thus

$$\begin{aligned} 2d \geq n &= \sum \mathbf{u}_x d(x) - \sum \alpha_x d(x) + \sum \mathbf{u}_y d(y) - \sum \alpha_y d(y) \\ &\geq 2d + \left(\frac{d}{|Y_N|} - 1 \right) \sum \alpha_y d(y) + \left(\frac{d}{|X_N|} - 1 \right) \sum \alpha_x d(x) \\ &\qquad + \frac{d}{|Y_N|} \sum \alpha_x d'(x) + \frac{d}{|X_N|} \sum \alpha_y d'(y). \end{aligned} \tag{34}$$

Then cancelling the $2d$'s and using (33) gives

$$0 \geq \sum \alpha_x f(x) + \sum \alpha_y g(y), \tag{35}$$

where, e.g.,

$$f(x) = d(x)\left(\frac{d}{|X_N|} - 1\right) + \frac{d}{|Y_N|}(d(x) - 2d + |Y_N|),$$

which (by subtracting and adding $(d^2/|X_N| + d^2/|Y_N| - d)$) we may rewrite as

$$(d(x) - d)\left(\frac{d}{|X_N|} + \frac{d}{|Y_N|} - 1\right) + d^2\left(\frac{1}{|X_N|} - \frac{1}{|Y_N|}\right). \tag{36}$$

Noting that the first term here is nonnegative, and inserting in (35), we have, using (32),

$$0 \geq \sum \alpha_x d^2\left(\frac{1}{|X_N|} - \frac{1}{|Y_N|}\right) + \sum \alpha_y d^2\left(\frac{1}{|Y_N|} - \frac{1}{|X_N|}\right) = 0.$$

So again, all of our inequalities are actually equalities, and then an analysis similar to our earlier one leads to the contradiction $G = 2 \cdot K_{n/2,n/2}$. ∎

4 Graph theory

Our purpose in this section is to give proofs of Lemmas 2.1 and 2.3. These are preceded by a few easy preliminaries plus Lemma 4.4, which does most of the work for Lemma 2.1.

Lemma 4.1 *Suppose G is an α-Dirac, ϑ-special graph with $\alpha < o(\vartheta)$. Then there is a half-set $A \subseteq V(G)$ such that one of $e(A, A)$, $e(A, \overline{A})$ is less than $4\vartheta n^2$.*

Proof. Let A and B be half-sets with $e(A, B) < \vartheta n^2$, and $C = A \cup B$. For any $v \in C$,

$$d_C(v) \geq (1-\alpha)n/2 - |\overline{A \cup B}| = (1-\alpha)n/2 - |A \cap B|.$$

(Strictly speaking, we only know $\big||\overline{A \cup B}| - |A \cap B|\big| \leq 1$, but we will always ignore this irrelevant annoyance.) Thus

$$|A \cap B|\,((1-\alpha)n/2 - |A \cap B|) \leq \sum_{v \in A \cap B} d_C(v) \leq e(A, B) \leq \vartheta n^2,$$

which, since $\alpha < o(\vartheta)$, implies that either $|A \cap B| < 3\vartheta n$ or $|A \cap B| > (.5 - 3\vartheta)n$. In the first case, $e(A, \overline{A}) \leq e(A, B) + 2(n/2) \cdot 3\vartheta n < 4\vartheta n^2$, and in the second, $e(A, A) \leq e(A, B) + 2(n/2) \cdot 3\vartheta n < 4\vartheta n^2$. ∎

We also need a version of Lemma 4.1 for weights. Here and in what follows we will make frequent use of the fact, an easy consequence of Jensen's inequality, that for any $\mathbf{x} : S \to \mathbf{R}^+$,

$$h(\mathbf{x}) \leq \mathbf{x}(S) \log \frac{|S|}{\mathbf{x}(S)}. \tag{37}$$

Lemma 4.2 *If G is an (arbitrary, n-vertex) graph and $\mathbf{x}$ a proper, ϑ-special edge weighting with $h(\mathbf{x}) \geq \frac{n}{2}\log\frac{n}{2}$, then there is a half-set A such that one of $\mathbf{x}(A,A)$, $\mathbf{x}(A,\overline{A})$ is at most $O(\vartheta n \log\frac{1}{\vartheta})$.*

Proof. Let A and B be half-sets for which $\mathbf{x}(A,B) \leq \vartheta n$. Because $\mathbf{x}$ is proper, $\mathbf{x}(A\cap B, V) = |A\cap B| =: an$. Also, $\mathbf{x}(A\cap B, A\cup B) \leq \mathbf{x}(A,B) \leq \vartheta n$. Thus, since $\mathbf{x}(A\cap B, \overline{A\cup B}) + \mathbf{x}(A\cap B, A\cup B) = \mathbf{x}(A\cap B, V)$, we have $\mathbf{x}(A\cap B, \overline{A\cup B}) \geq an - \vartheta n$, which, since $\mathbf{x}(\overline{A\cup B}, V) = an$, implies $\mathbf{x}(\overline{A\cup B}, \overline{A\cap B}) \leq \vartheta n$.

Let $E_1 = E(A\setminus B)\cup E(B\setminus A)\cup E(A\cap B, \overline{A\cup B})$ (where $E(X)$ is the set of edges contained in X) and $E_2 = E\setminus E_1$. We have

$$\sum_{e\in E_1}\mathbf{x}_e\log\frac{1}{\mathbf{x}_e} + \sum_{e\in E_2}\mathbf{x}_e\log\frac{1}{\mathbf{x}_e} = h(\mathbf{x}) \geq \frac{n}{2}\log\frac{n}{2}. \tag{38}$$

Let $b = \min\{a, \frac{1}{2} - a\}$. We are done if we can show that $b = O(\vartheta\log\frac{1}{\vartheta})$, since

$$\min\left\{\mathbf{x}(A,A), \mathbf{x}(A,\overline{A})\right\} \leq \mathbf{x}(A,B) + bn.$$

We have $|E_1| \leq (n/2 - bn)n/2$ and $|E_2| \leq n^2/2$; so, setting $\lambda n = \mathbf{x}(E_2) \leq \mathbf{x}(\overline{A\cup B}, \overline{A\cap B}) + \mathbf{x}(A,B) \leq 2\vartheta n$ and using (37), (38), we have

$$\frac{n}{2}\log\frac{n}{2} \leq h(\mathbf{x}) \leq (\frac{n}{2} - \lambda n)\log\frac{(n/2-bn)}{1-2\lambda} + \lambda n\log\frac{n}{2\lambda}.$$

Dividing the left and right sides of this inequality by $n/2$ gives

$$\log\frac{n}{2} \leq (1-2\lambda)\log(n/2-bn) + 2\lambda\log(n) + H(2\lambda),$$

or

$$0 \leq (1-2\lambda)\log(1-2b) + 2\lambda + H(2\lambda) \leq -(1-4\vartheta)2b + O(\vartheta\log\frac{1}{\vartheta})$$

and the lemma follows. ∎

Proposition 4.3 *If G is bipartite on $A \cup B$ with $|A| = |B| = n$ and $\delta(G) > (n+1)/2$, then G contains a Hamiltonian (a,b)-path for each $a \in A$ and $b \in B$.*

Proof. This is a simple consequence of a theorem of Moon and Moser [15], stating that a bipartite graph with parts of size m and minimum degree greater than $m/2$ is Hamiltonian. In the present case this says that $G - \{a,b\}$ contains a Hamiltonian cycle C. But then, since $|C| = 2n-2$ and $N(A) \cap C$, $N(A) \cap C$ are disjoint sets of size greater than $(n-1)/2$, there are consecutive vertices of C belonging to $N(a)$ and $N(b)$. This gives the required path. ∎

Lemma 4.4 *Suppose G is a Dirac graph containing a half-set A for which $\max\{e(A) + e(\overline{A}), e(A,\overline{A})\} > (1/4 - o(\log^{-2} n))n^2$. Then if there is a Hamiltonian path joining $x, y \in V(G)$, there are at least $n!/(2+o(1))^n$ such paths.*

The proof of this is somewhat similar to some of the arguments in [19].

Proof. We let $B = \overline{A}$ and first suppose that $e(A)+e(B) > (1/4-o(\log^{-2} n))n^2$. Set $e(A) + e(B) = (1/4 - \lambda)n^2$; so $\lambda < o(\log^{-2} n)$.

We slightly modify the partition so that all degrees within the parts are fairly large. Let $A^b = \{v \in A \mid d_A(v) < \frac{1}{4}n\}$ and $B^b = \{w \in B \mid d_B(w) < \frac{1}{4}n\}$. Since $e(A) + e(B) \geq \frac{n^2}{4} - \lambda n^2$, we have $\binom{|A|}{2} - e(A) \leq \lambda n^2$, so $|A^b| < 4\lambda n$. Similarly, $|B^b| < 4\lambda n$. Let $A' = (A \setminus A^b) \cup B^b$ and $B' = (B \setminus B^b) \cup A^b$. Thus $A' \cup B'$ is again a partition of V and

$$d_{A'}(v) > \frac{n}{4} - 4\lambda n \ \ \forall\, v \in A',\ d_{B'}(w) > \frac{n}{4} - 4\lambda n \ \ \forall\, w \in B'.$$

Also, $e(A') \geq e(A) - \frac{n}{4} \cdot |A^b| \geq \binom{|A|}{2} - 2\lambda n^2 \geq \binom{|A'|}{2} - 5\lambda n^2$, and similarly $e(B') \geq \binom{|B'|}{2} - 5\lambda n^2$.

We give the proof for $x \in A'$ and $y \in B'$; the argument for x, y on the same side of the partition is similar. Since there is a Hamiltonian (x, y)-path, there is an edge ab disjoint from $\{x, y\}$ with $a \in A'$ and $b \in B'$. For the conclusion of the lemma it's enough to show that there are at least $(n/(2e + o(1))^{n/2}$ Hamiltonian (x, a)-paths in $G[A']$ and at least this many Hamiltonian (b, y)-paths in $G[B']$; these assertions are instances of

Claim Let H be a graph with $|V(H)| = m$, $\delta(H) = \omega(\sqrt{\lambda} m)$, $|E(H)| >$

$\binom{m}{2} - \lambda m^2$ and $\lambda < o(\log^{-2} m)$. Then, for any $u, v \in V(H)$, there are at least $(m/(e+o(1))^m$ Hamiltonian (u,v)-paths.

(Note we could require a stronger lower bound on δ, but assume only what's needed for the claim. Note also that the λ used here is slightly different from the one introduced above.)

Proof. Set $Q = \{v \in V(H) : m - d(v) \geq \sqrt{\lambda} m\}$. The lower bound on $e(H)$ gives $|Q| \leq 2\sqrt{\lambda} m$. It is also easy to see, using $\delta(G) = \omega(\sqrt{\lambda} m)$ and, again, the lower bound on $e(G)$ that any two vertices of H can be connected by $\omega(\sqrt{\lambda} m)$ independent paths of length 3 (where, recall, *independent* means no path contains an internal vertex of another).

It follows that for some $u' \notin Q$ there is a (u, u')-path P of length $3|Q|$ containing $Q \setminus \{v\}$ and not containing v. Let v' be any neighbor of v not in P (so $v' \notin Q$). Now $H' := H - v - (V(P) \setminus \{u'\})$ is a graph on $m - 3|Q|$ vertices with

$$\delta(H') \geq m - 3|Q| - \sqrt{\lambda} m \geq m - 7\sqrt{\lambda} m. \tag{39}$$

The desired bound then follows from the observation that H' contains at least $(m/(e+o(1)))^m$ Hamiltonian (u', v')-paths, since (a) there are at least $(\delta(H') - 1)_l$ paths of length $l := (1 - 15\sqrt{\lambda})m$ beginning at u' and not containing v' (where $(x)_l = x(x-1)\cdots(x-l+1)$)—note $(\delta(H') - 1)_l >$

$(m/(e+o(1))^m$ by (39) and our assumption on λ—and (b) each of these can be completed to a Hamiltonian (u',v')-path, this according to a theorem of Ore [17], stating that a graph F with $\delta(F) > |V(F)|/2$ is Hamiltonian connected. □

Now suppose $e(A,B) = (1/4-\lambda)n^2$ with $\lambda < o(\log^{-2})n^2$. We first modify the partition so that all degrees across the parts are fairly large. Let $A^b = \{v \in A \mid d_B(v) < \frac{1}{4}n\}$ and $B^b = \{w \in B \mid d_A(w) < \frac{1}{4}n\}$. Since $e(A,B) \geq \frac{n^2}{4} - \lambda n^2$, we have $|A||B| - e(A,B) \leq \lambda n^2$, so $|A^b|, |B^b| < 4\lambda n$. Let $A' = (A \setminus A^b) \cup B^b$ and $B' = (B \setminus B^b) \cup A^b$. Thus $A' \cup B'$ is again a partition of V and

$$d_{B'}(v) > \frac{n}{4} - 4\lambda n \;\; \forall\, v \in A',\; d_{A'}(w) > \frac{n}{4} - 4\lambda n \;\; \forall\, w \in B', \tag{40}$$

and $e(A',B') \geq e(A,B) - \frac{n}{4} \cdot (|A^b| + |B^b|) \geq |A'||B'| - 3\lambda n^2$.

We next remove a short path from G to equalize the sizes of A' and B' . We may assume that $|A'| \leq |B'|$. Again we just do the case $x \in A'$, $y \in B'$, the argument for x, y on the same side of the partition being similar.

By assumption, G has a Hamiltonian (x,y)-path, say P. Let S be some $(|B'|-|A'|)$-subset of $E(P) \cap E(B')$. The graph with vertex set B' and edge set S is a disjoint union of paths (some of them single vertices). Let P_y be the path containing y and y' the other end of this path (so possibly $y' = y$).

Let G' be the subgraph of G obtained by deleting $(E(A) \cup E(B)) \setminus S$. By (40), any $p, q \in B'$ can be joined by $\omega(\lambda n)$ independent paths of length 4 in G'. Since $|B'| - |A'| \le 8\lambda n$, it follows that for some $x' \in A'$ there is an (x, x')-path $P(x)$ in G' of length $O(\lambda n)$ such that $E(P(x)) \cap E(P) = S \setminus E(P_y)$. Thus $G'' := G' - V(P(x) \setminus \{x'\}) - (V(P_y) \setminus \{y'\})$ is bipartite with parts $A'' := V(G'') \cap A'$ and $B'' := V(G'') \cap B'$, and $|A''| = |B''| \ge n/2 - O(\lambda n)$. The following claim then says that there are at least $n!/(2 + o(1))^n$ Hamiltonian (x', y')- paths in G'', which of course gives the same number of Hamiltonian (x, y)-paths in G.

Claim Let H be a bipartite graph on $U \cup W$ with $|U| = |W| = m$, $e(H) > (1 - \lambda)m^2$ and $\delta(H) = \omega(\sqrt{\lambda} m)$, where $\lambda < o(\log^{-2} m)$. Then, for any $u \in U$ and $w \in W$, there are at least $(1 - o(1))^m \cdot (m!)^2$ Hamiltonian (u, w)-paths.

Proof. Let $Q = \{v \in V(H) : m - d(v) \ge \sqrt{\lambda} m\}$. The lower bound on $e(H)$ gives $|Q| \le 2\sqrt{\lambda} m$. It is also easy to see, using $\delta(H) = \omega(\sqrt{\lambda} m)$ and, again, the lower bound on $e(H)$, that any two vertices of H can be connected by $\omega(\sqrt{\lambda} m)$ independent paths of length at most 4. It follows that for some $u' \in U \setminus Q$, there is a (u, u')-path P_u of length exactly $4|Q|$ containing all vertices of $Q \setminus \{w\}$ and not containing w. Let P_w be a path of length 2 disjoint from P_u joining w and some $w' \in W \setminus Q$.

Then $H' := H - (V(P_u) \setminus \{u'\}) - (V(P_w) \setminus \{w'\})$ is bipartite on $2m - 4|Q| - 2$ vertices, with $m - 2|Q| - 1$ vertices on each side of the bipartition and $\delta(H') \geq m - \sqrt{\lambda}m - 2|Q| - 1 \geq m - 5\sqrt{\lambda}m - 1$. We then proceed as in the previous case: H' contains at least $((\delta(H') - 1)_l)^2 > (m/(e + o(1))^{2m}$ Hamiltonian (u', w')-paths, since (a) there are at least this many paths of length $2l := 2(1 - 10\sqrt{\lambda})m$ beginning at u' and not containing w', and (b) Proposition 4.3 says that each of these can be completed to a Hamiltonian (u', w')-path; and of course this gives the desired (u, w)-paths in H. ■

Proof of Lemma 2.1.

By Lemma 4.2, there is a half-set A such that one of $\mathbf{x}(A, \overline{A})$, $\mathbf{x}(A, A)$ is less than $O(\varphi n \log(1/\varphi))$. We consider only the first case; the second can be handled similarly. Setting $\mathbf{x}(A, \overline{A}) = xn$, we have, again using (37),

$$\begin{aligned} h(\mathbf{x}) &\leq xn \log \frac{n^2/4}{xn} + (1/2 - x)n \log \frac{n^2/4}{(1/2 - x)n} \\ &= (n/2)(\log(n/2) + H(2x)) < (n/2)\log(n/2) + O(\varphi n \log^2 \frac{1}{\varphi}). \end{aligned}$$

For the second assertion we show $e(A) + e(\overline{A})$ is large and apply Lemma 4.4 (and Dirac's Theorem). By Lemma 1.3 we have $h(\mathbf{x}) \geq (n/2)\log(n/2)$. On the other hand, setting $e(A) + e(\overline{A}) = (1 - \lambda)n^2/4$ and using (37), we have

$$h(\mathbf{x}) \leq (n/2 - xn) \log \frac{(1-\lambda)n^2/4}{n/2 - xn} + xn \log \frac{n^2/4}{xn}.$$

Combining and rewriting (along lines similar to those in the proof of Lemma 4.2) gives $\lambda < H(2x)/(1-2x) < o(\log^{-2} n)$, as desired. (So we just needed $\varphi < o(1/(\log^2 n \log\log n))$.) ■

Proof of Lemma 2.3.

We first show that for any given $x, y \in V(G)$ $(x \neq y)$, $G' := G - \{x, y\}$ is Hamiltonian. To begin, we assert that G' is 2-connected. To see this, suppose instead that A, B are disjoint components of $G' - v$. Since G is α-Dirac, each of A, B has size at least $(1-\alpha)n/2 - 2$; so in G we have $e(A, B) > \beta n^2 - 2(\alpha n/2 + 2)n/2 > 0$, a contradiction.

Now let $C = (v_1, \ldots, v_m)$ be a longest cycle in G'. According to a(nother) theorem of Dirac [8] we have $m \geq 2[(1-\alpha)n/2 - 2]$. (Recall Dirac's theorem says that a 2-connected graph of minimum degree k has a cycle of length at least $\min\{2k, n\}$.)

For $w \in V(G) \setminus V(C)$, let $S(w) = \{v_i \,|v_{i-1} \in N(w)\}$ (with subscripts interpreted modulo m). If $v \in V(G') \setminus V(C)$, then $|S := S(v)| \geq (1-\alpha)n/2 - (n - m - 1) \geq n/2 - (3\alpha n/2 + 3)$, implying, by our normality assumption, $e(S, S) > (\beta - 3\alpha/2)n^2 - 3n \geq 1$. This gives a contradiction: if $v_i v_j \in E(S)$

then

$$(v, v_{j-1}, v_{j-2}, \dots, v_{k+1}, v_k, v_j, v_{j+1}, \dots, v_{k-1})$$

is a longer cycle than C. So C is indeed a Hamiltonian cycle of G'.

Finally the desired Hamiltonian (x, y)-path is obtained from C by a similar argument (rerouting through some $v_j v_k \in E(S(x), S(y))$). ■

5 Mixing and concentration

Here we establish a few basic points regarding our self-avoiding walk (recall these were outlined near the end of Section 1).

Recall that the *variation distance* between probability measures μ and ν on a set S is

$$\|\mu - \nu\| = \sup\{|\mu(T) - \nu(T)| : T \subseteq S\}$$

and satisfies

$$\|\mu - \nu\| = \inf\{\Pr(X \neq Y)\}, \tag{41}$$

the infimum over coupled random variables (r.v.'s) X and Y having laws μ and ν respectively (see e.g. [9, p.119]). We will abuse this: for X, Y r.v.'s and π a probability measure, we use (i) $\|X - Y\|$ and (ii) $\|X - \pi\|$ for the variation distances between (i) the laws of X and Y, and (ii) the law of X and π.

Lemma 5.1 *Let G be a graph, $v \in V(G)$, $\mathbf{x} : E(G) \to \mathbf{R}^+$, and $p_0 = \max\{\mathbf{x}_{wy}/\sum_z \mathbf{x}_{wz}\}$ (the largest of the transition probabilities for the ordinary $\mathbf{x}$-walk). Let $X_0, \ldots$ and $Y_0, \ldots$ be, respectively, the self-avoiding $\mathbf{x}$-walk and the ordinary $\mathbf{x}$-walk starting from v. Then (for any q)*

$$\Pr(X_q \neq Y_q) \leq q^2 p_0.$$

Proof. If the two walks agree to time $i-1$, say $X_j = Y_j = w_j$ for $j = 0, \ldots, i-1$, then we may couple at the next step so that

$$\Pr(X_i \neq Y_i) = \Pr(Y_i \in \{w_0, \ldots, w_{i-1}\}) \leq i p_0.$$

Thus

$$\Pr(X_q \neq Y_q) \leq \sum_{i=1}^{q} \Pr(X_i \neq Y_i | X_j = Y_j,\ j = 0, \ldots, i-1) < q^2 p_0.$$

∎

Lemma 5.2 *Let H be an α-Dirac, β-normal graph on m vertices, and $\mathbf{x}$ an edge weighting of H with $\mathbf{x}_{wy}/\sum_z \mathbf{x}_{wz} \geq \tau/m$ for each $wy \in E(H)$. Then for any $v \in V(H)$, $(Y_0 = v, \ldots)$ the ordinary $\mathbf{x}$-walk from v on H, and π the* *stationary distribution of this walk, we have, for any* $\eta > 0$, $\|Y_q - \pi\| < \eta$ *provided* $\beta > \alpha/4$ *and*

$$q > \tau^{-2}(\beta - \alpha/4)^{-1} 2 \ln(1/\eta). \tag{42}$$

Proof. Let w be any vertex, $(Z_0, \ldots)$ the ordinary $\mathbf{x}$-walk from w, and $A \subseteq V$ any half-set. Our assumptions on H imply

$$\begin{aligned} \Pr(Z_2 \in A) &\geq (\tau/m)^2 e(N(w), A) \\ &> (\tau/m)^2[\beta m^2 - (m/2 - |N(w)|)|A|] > \tau^2(\beta - \alpha/4). \end{aligned}$$

Thus for $(Z_0, \ldots)$, $(Z_0,' \ldots)$ the walks started from any $w, w' \in V$, we have

$$\|Z_2 - Z_2'\| = \max_{|B| \geq n/2} |\Pr(Z_2 \in B) - \Pr(Z_2' \in B)| < 1 - \tau^2(\beta - \alpha/4).$$

(The restriction $|B| \geq n/2$ is allowed because the expression being maximized is the same for B as for $\overline{B}$.) So by (41) we can arrange $\Pr(Z_2 = Z_2') > \tau^2(\beta - \alpha/4)$.

We then couple the walk $(Y_0, \ldots)$ from v with the stationary walk, say $(W_0, \ldots)$, in the usual way (Y_i and W_i are equal if $Y_{i-1} = W_{i-1}$ and otherwise independent), to obtain

$$\|Y_q - \pi\| \leq \Pr(Y_q \neq W_q) < (1 - \tau^2(\beta - \alpha/4))^{q/2} < \exp[-\tau^2(\beta - \alpha/4)q/2],$$

and the lemma follows. ∎

Lemma 5.3 *Let $X_0, \dots$ be random variables taking values in a set V, $f : V \to \mathbf{R}$, and $a = \|f\|_\infty$. Then for any $t > 0$ and positive integers b and q,*

$$\Pr(|\sum_{k=0}^{b}\{f(X_{k+q}) - \mathsf{E}[f(X_{k+q})|X_0, \dots, X_k]\}| > ta\sqrt{qb}) < 2qe^{-t^2/2}.$$

Proof. We prove the slightly stronger assertion that for each $r \in \{0, \dots, q-1\}$ and s,

$$\Pr(|\sum_{i=1}^{s}\{f(X_{iq+r}) - \mathsf{E}[f(X_{iq+r})|X_0, \dots, X_{(i-1)q+r}]\}| > ta\sqrt{s}) < 2e^{-t^2/2}. \quad (43)$$

(It is stronger because $\sqrt{s_0} + \cdots + \sqrt{s_{q-1}} \le \sqrt{q(s_0 + \cdots + s_{q-1})}$.) To see this, let $W_0 = (X_0, \dots, X_r)$, $W_i = (X_{(i-1)q+r+1}, \dots, X_{iq+r})$ if $i \ge 1$, and $Z_i = f(X_{iq+r}) - \mathsf{E}[f(X_{iq+r})|W_0, \dots, W_{i-1}]$ (the summand in (43)). Then $(Z_1, \dots, Z_s)$ is a martingale difference sequence with respect to $(W_0, \dots, W_s)$ (that is, $\mathsf{E}[Z_i|W_0, \dots, W_{i-1}] = 0$), and $|Z_i| \le a$, so (43) follows from Azuma's inequality (see e.g. [1]). ∎

6 Proof of Lemma 2.2

We will use *with very high probability* (w.v.h.p.) to mean with probability at least $1 - n^{-\omega(1)}$. (This is convenient since we will have to multiply failure probabilities by terms of the form $n^{O(1)}$, but, as mentioned following Lemma 2.2, there is nothing delicate about these estimates.)

In this section we complete the proof of Theorem 1.2 by proving the following strengthening of Lemma 2.2.

Lemma 6.1 *If (*$\mathbf{x}$ *is* ξ*-normal and) the walk* X *is good through milestone* $(p-1)\kappa$*, then w.v.h.p. it is good through milestone* $p\kappa$

(where we assume $p\kappa \leq l$).

Proof of Lemma 6.1.

Supposing the "current segment," $R = [(p-1)\kappa, p\kappa]$, is contained in phase m, let μn be the end of phase $m-1$, $(\mu+\nu)n = (p-1)\kappa$, $I = [\mu n]$ and $Q = [\mu n, (\mu+\nu)n]$ (so $R = [(\mu+\nu)n, (\mu+\nu)n+\kappa]$).

We assume the walk is good through the end of Q and want to show that for any $f \in \mathcal{F}$,

$$\text{w.v.h.p. } X_R \text{ is good wrt } f \tag{44}$$

(meaning, of course, that (14) holds with I replaced by R). Of course if we have this we are done, since $|\mathcal{F}| < O(n^2)$.

There are two parts to proving (44). We first need to show, for each $j \in R$, that the residual graph G_j is nearly (A) normal and (B) Dirac, and that (C) the stationary distribution of the $\mathbf{x}$-walk on G_j is nearly uniform. Once we have these properties we may apply the results of Section 5 to obtain (44).

Note that (A)-(C) are *deterministic* statements: they apply to any fixed good $X((p-1)\kappa)$ and an arbitrary $X(j-1)$ extending it.

Fix $j \in R$. That X is good through the end of Q implies, for any $g \in \mathcal{F}$,

$$(1-\mu(1+\delta_{m-1})-\nu(1+\delta_m))g(V)-\kappa\|g\|_\infty <$$

$$g(V_j) < (1-\mu(1-\delta_{m-1})-\nu(1-\delta_m))g(V),$$

or, recalling $\rho = \max\{\|h\|_\infty/h(V) : h \in \mathcal{F}\}$ and setting $\vartheta = \mu\delta_{m-1}+\nu\delta_m+\kappa\rho$ (and relaxing slightly),

$$g(V_j) = (1-\mu-\nu \pm \vartheta)g(V), \tag{45}$$

or (relaxing a little further)

$$g(V_j) = (1 \pm c\delta_m)\frac{n-j}{n}g(V) \tag{46}$$

where c is given by (11). (That (45) implies (46) follows from—and was the reason for—the definition of δ_m given in (12). Since the coefficient of $g(V)$ on the r.h.s. of (46) is

$$(1 \pm c\delta_m)(1-\mu-\nu-\lambda/n) = 1-\mu-\nu-\lambda/n \pm c\delta_m(1-\mu-\nu-\lambda/n),$$

where $\lambda = j-(\mu+\nu)n$, we want

$$\mu\delta_{m-1}+\nu\delta_m+\kappa\rho < c\delta_m(1-\mu-\nu-\lambda/n)-\lambda/n,$$

which, since $\nu < \gamma(1-\mu)$ (see the specification of the sequence $\{a_i\}$), is implied by

$$\mu\delta_{m-1} + \kappa\rho < (c(1-\gamma)-\gamma)(1-\mu)\delta_m - (1+c\delta_m)\lambda/n,$$

which follows from (12).)

We now turn to the properties (A)-(C); of these, only (A) requires any effort, and we save it for last. For the remainder of our discussion we set $\delta_m = \delta$ and recycle $m := n - j$.

(B) Applying (46) to the h_v's gives

$$d_j(v) := h_v(V_j) > (1-c\delta)(n-j)/2 \quad \forall v \in V_j \ ; \tag{47}$$

that is, G_j is $(c\delta)$-Dirac.

(C) Applying (46) to the f_v's gives

$$\pi_j(v) \ \ (= f_v(V_j)/\sum_{v' \in V_j} f_{v'}(V_j)) \ \ = (1 \pm c'\delta)/(n-j) \quad \forall v \in V_j, \tag{48}$$

with $c' = 2c(1-c\delta)^{-1} \approx 2c$.

(A) As already mentioned in Section 2, (15) and the ξ-normality of $\mathbf{x}$ imply, via Lemma 3.2, that G is ζ-normal with ζ as in (8) ($\zeta = \frac{1}{64}\xi^4$). We will show

$$G_j \text{ is } (\zeta/6)\text{-normal.} \tag{49}$$

Proof. Suppose for a contradiction that A, B are half-sets of G_j with $e(A, B) < \zeta m^2/6$. Then there are $v \in A$, $w \in B$ with $|N(v) \cap B|, |N(w) \cap A| < \zeta m/3$, implying $d_j(v), d_j(w) < (1/2 + \zeta/3)m$ and, by (46),

$$d(v), d(w) < (1 - c\delta)^{-1}(1/2 + \zeta/3)n. \tag{50}$$

Also, since G is Dirac (and again using (46)), we have

$$d_j(v), d_j(w) > (1 - c\delta)m/2. \tag{51}$$

Let $A' = V \setminus N(w)$, $B' = V \setminus N(v)$, $A'' = A' \cap V_j$ $(= V_j \setminus N(w))$ and $B'' = B' \cap V_j$. We have $|A''| < (1 + c\delta)m/2$ (by (51)) and $|A \setminus A''| = |N(w) \cap A| < \zeta m/3$, so $|A'' \setminus A| < (\zeta/3 + c\delta/2)m$; and similarly $|B'' \setminus B| < (\zeta/3 + c\delta/2)m$. Thus

$$\begin{aligned} e(A'', B'') &< e(A, B) + 2(\zeta/3 + c\delta/2)m(1 + c\delta)m/2 \\ &< [\zeta/6 + (\zeta/3 + c\delta/2)(1 + c\delta)]m^2 < (\zeta/2 + \delta)m^2. \end{aligned} \tag{52}$$

On the other hand, we can bound $e(A'', B'')$ from below as follows. By (50) we have $|A'|, |B'| > (1/2 - \zeta/3)n$, so that

$$e(A', B') > \zeta n^2 - 2(\zeta/3)n \cdot n/2 = (2\zeta/3)n^2. \tag{53}$$

Thus we have (with explanations to follow)

$$e(A'', B'') = e(V_j \setminus N(v), V_j \setminus N(w))$$

$$
\begin{aligned}
&= \sum\{|(N(x)\cap V_j)\setminus N(w)| : x\in V_j\setminus N(v)\}\\
&> (1-c\delta)\frac{m}{n}\sum\{|N(x)\setminus N(w)| : x\in (V_j\setminus N(v))\cap A(w)\} \quad (54)\\
&= (1-c\delta)\frac{m}{n}\ g_{vw}(V_j)\\
&> (1-c\delta)^2(m/n)^2 g_{vw}(V) \quad (55)\\
&> (1-c\delta)^2(m/n)^2[e(A',B')-\zeta n^2/8]\\
&> (13\zeta/24-\delta)m^2,
\end{aligned}
$$

contradicting (52). (For (54) we applied (46) to the functions h_{wx} for $x\in A(w)$ (see (9) for $A(w)$); for (55) we used relevance of the function g_{vw} (which follows from (53)) and, again, (46).) □

Given these preliminaries we turn to (44). We will soon want to apply Lemma 5.2 with $H=G_j$, so need to see what values are allowed for the parameters in this lemma. Recalling that $\mathbf{x}$ is ξ-normal, we find that Lemma 3.2 together with (46) applied to the f_v's allows us to take $\tau=(1+c\delta)^{-1}\xi^7/2$. (In more detail: Lemma 3.2 and (46) give, respectively, $\mathbf{x}_{wy}\geq \xi^7/(2n)$ and $\sum\{\mathbf{x}_{wz} : z\in V_j\} < (1+c\delta)(n-j)/n$.) Legal values of α and β are given by (B) and (A): we may take $\alpha=c\delta$ and $\beta=\zeta/6$.

We then take $\eta=n^{-2}$ (say) and set q equal to the right hand side of (42), so in particular $q = \log^{O(1)} n$. We can now, finally, specify κ:

$$\kappa = Cn^{2/3}(q\log^2 n)^{1/3} \tag{56}$$

with C a largish constant (actually $C > 4^{2/3}$ will be enough). So as promised earlier, $\kappa = n^{2/3}\log^{O(1)} n$. Note also that $\beta \gg \alpha$ (see (13), (19) and (11)).

Fix $f \in \mathcal{F}$ and set $\|f\|_\infty = a$. We want to show that w.v.h.p.

$$\sum_{i\in R} f(X_i) = (1\pm\delta)\frac{\kappa}{n}f(V). \tag{57}$$

Let $r = (\mu+\nu)n$ (the last index of Q). We may rewrite the l.h.s. of (57) as

$$\sum_{j=r}^{r+q-1} f(X_j) + \sum_{j=r}^{r+\kappa-q} f(X_{j+q}).$$

We don't have much control over the terms in the first sum, but there are not very many of them and we will more or less ignore them (see (63)). We show that each term in the second sum has expectation close to $f(V)/n$ and apply Lemma 5.3.

Fix $j \in (r, r+\kappa-q)$ and $X(j)$ extending $X(r)$. Let $Y_0,\ldots$ be the (ordinary) $\mathbf{x}$-walk from X_j on V_j. We have

$$\begin{aligned}|\mathsf{E}[f(X_{j+q})|X(j)] - f(V)/n| &\leq |\mathsf{E}[f(X_{j+q})|X(j)] - \mathsf{E}f(Y_q)| \\ &+ |\mathsf{E}f(Y_q) - \frac{f(V_j)}{n-j}| + |\frac{f(V_j)}{n-j} - \frac{f(V)}{n}|. \end{aligned}\tag{58}$$

We may bound the terms on the r.h.s. of (58) as follows.

For the first term we use Lemma 5.1. The transition probabilities of the $\mathbf{x}$-walk on G_j are at most

$$\frac{\|\mathbf{x}\|_\infty}{\min_v f_v(V_j)} \leq \frac{64}{\xi^3 n}\frac{n}{(1-c\delta)(n-j)} = \frac{64}{\xi^3(1-c\delta)(n-j)}$$

(with the bound on $\|\mathbf{x}\|_\infty$ coming from Lemma 3.2). If we temporarily write λ for the last expression, this gives, using Lemma 5.1,

$$\begin{aligned} |\mathsf{E}[f(X_{j+q})|X(j)] - \mathsf{E}f(Y_q)]| &= |\sum_w [\Pr(X_{j+q}=w|X(j)) - \Pr(Y_q=w)]f(w)| \\ &\leq \lambda q^2 a. \end{aligned} \tag{59}$$

For the second term on the r.h.s. of (58) we have

$$\begin{aligned} |\mathsf{E}f(Y_q) - \frac{f(V_j)}{n-j}| &\leq a\|Y_q - \pi_j\| + |\pi_j(f) - \frac{f(V_j)}{n-j}| \\ &< an^{-2} + c'\delta f(V_j)/(n-j) && (60) \\ &< c''\delta f(V)/n && (61) \end{aligned}$$

with c' as in (48) and (say) $c'' = c'(1+2c\delta)$ $(< 2c + o(1))$. For (60) we used Lemma 5.2 and (48), and for (61) we used (46).

Finally, for the last term in (58) we have, again using (46),

$$|\frac{f(V_j)}{n-j} - \frac{f(V)}{n}| < \frac{c\delta}{n}f(V).$$

Thus, summarizing, the l.h.s. of (58) is at most

$$(3c + o(1))\delta f(V)/n. \tag{62}$$

On the other hand Lemma 5.3 gives

$$\Pr(|\sum_{j=r}^{r+\kappa-q} \{f(X_{j+q}) - \mathsf{E}[f(X_{j+q})|X(j)]\}| > ta\sqrt{q(\kappa-q)}) < 2qe^{-t^2/2}.$$

So taking $t = \log n$ (we need $\omega(\sqrt{\log n})$) and combining with (the discussion culminating in) (62), we find that w.v.h.p.

$$\begin{aligned} |\sum_{j\in R} f(X_j) - \frac{\kappa}{n} f(V)| &\leq qa + (\kappa - q)(3c + o(1))\delta f(V)/n + \log n\sqrt{q(\kappa-q)}\, a \\ &< \delta\kappa f(V)/n, \end{aligned} \tag{63}$$

where the second inequality follows easily from our choice of parameters. (Noting that $\delta > 2\kappa\rho$ (see (12)) we find that the r.h.s. of (63) is at least

$$2\kappa^2 \rho f(V)/n \;\geq\; 2\kappa^2 a/n \;>\; n^{1/3} a$$

(whereas $qa < a \log^{O(1)} n$). The main point (and reason for our choice of κ) is that—again since $a \leq \rho f(V)$ and $\delta > 2\kappa\rho$—the last term in the first line of (63) is at most $C^{-3/2}\delta\kappa f(V)/n$.) ∎

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