Asymptotics of the list-chromatic index for multigraphs

Jeff Kahn
Department of Mathematics and RUTCOR
Rutgers University
New Brunswick NJ 08903

ABSTRACT

The *list-chromatic index*, $\chi'_l(G)$ of a multigraph G is the least t such that if $S(A)$ is a set of size t for each $A \in E := E(G)$, then there exists a proper coloring σ of G with $\sigma(A) \in S(A)$ for each $A \in E$.

The list-chromatic index is bounded below by the ordinary chromatic index, $\chi'(G)$, which in turn is at least the fractional chromatic index, $\chi'^*(G)$. In [30] we showed that the chromatic and fractional chromatic indices are asymptotically the same; here we extend this to the list-chromatic index:

Theorem $\chi'_l(G) \sim \chi'^*(G)$ as $\chi'_l(G) \to \infty$.

The proof uses sampling from "hard-core" distributions on the set of matchings of a multigraph to go from fractional to list colorings.

1 Introduction

A (proper edge-) *coloring* of a multigraph $G = (V, E)$ is a map $\sigma : E \to \Gamma$, Γ a set of "colors," such that $\sigma(A) \neq \sigma(B)$ whenever A, B are (distinct) non-disjoint edges. The *chromatic index*, $\chi'(G)$, of G is the least size of a Γ admitting such a coloring. (Most of our graph-theoretic terminology is fairly standard; see e.g. [43], but note this and some other references use "graph" where we use "multigraph.")

The *list-chromatic index*, $\chi'_l(G)$, of G is the least t such that if $S(A)$ is a set ("list") of size t for each $A \in E$, then there exists a coloring σ of G with $\sigma(A) \in S(A)$ for each $A \in E$. (This notion is due to Vizing [60] and independently to Erdős, Rubin and Taylor [12], the latter motivated by a conjecture of J. Dinitz

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(see [10]). See also [2] for an (already somewhat out-of-date) survey of some of the many recent developments on list coloring.)

Write $D(G)$ for the maximum degree in G and set

$$\Gamma(G) = \max\{\frac{|E(W)|}{\lfloor |W|/2 \rfloor} : W \subseteq V,\ 3 \leq |W| \equiv 1 \pmod 2\},$$

$$\chi'^*(G) = \max\{D(G), \Gamma(G)\}. \tag{1}$$

The reason for the ugly notation is that χ'^* is the "fractional" version of χ'; see (5) below. Fractional vs. integer is in fact our preferred point of view, but we first discuss our problem in more traditional graph-theoretic terms.

It is easy to see that one always has $\chi'_l(G) \geq \chi'(G) \geq \chi'^*(G)$. In [30] we proved asymptotic agreement of χ' and χ'^*; here we extend this to χ'_l:

Theorem 1.1 *For multigraphs G,*

$$\chi'_l(G) \sim \chi'^*(G) \quad \textit{as} \quad \chi'^*(G) \to \infty.$$

That is: given $\delta > 0$ there exists $D(\delta)$ so that for any multigraph G with $\chi'^*(G) > D(\delta)$ we have $\chi'_l(G) < (1+\delta)\chi'^*(G)$. (It doesn't matter which of χ'^*, χ'_l is required to tend to infinity, since they differ by at most a factor of 2.)

We will be fairly brief regarding background for Theorem 1.1, and refer the reader to [30] for an expanded version of the following discussion.

It was conjectured first by Goldberg [18], and later independently by Andersen [5] and Seymour [53] that

$$\chi'(G) > D(G) + 1 \Rightarrow \chi'(G) = \lceil \Gamma(G) \rceil, \tag{2}$$

so in particular

$$\chi'(G) \leq \chi'^*(G) + 1.$$

Moreover, a well-known conjecture—the "list-chromatic" or "list coloring" conjecture—seemingly first proposed by Vizing in 1975 (see e.g. [19]) states that

$$\chi'_l(G) = \chi'(G) \text{ for every multigraph } G. \tag{3}$$

So it may be that we can replace χ' by χ'_l in (2).

On the other hand, the best bound on χ' prior to [30], due to Nishizeki and Kashiwagi [47] (see also [46]), was

$$\chi'(G) \leq \max\{(11D(G) + 8)/10, \lceil \Gamma(G) \rceil\}.$$

(The proof follows an idea of Goldberg [18] with roots in the classic papers of Shannon [54] and Vizing [58], [59].) In particular,

$$\chi'(G) < 11\chi'^*(G)/10 + O(1).$$

For list-coloring, the best previous bound is

$$\chi'_l(G) \leq 9D(G)/5,$$

due to Hind [22]. In fact it does not seem easy—the reader might try—to significantly improve the trivial upper bound $\chi'_l(G) < 2D(G)$, even for *simple* graphs. The first such improvement was made by Bollobás and Harris [8]. The specialization of Theorem 1.1 to simple graphs (that is, $\chi'_l(G) \sim D(G)$) was proved in [27] (see following Conjecture 1.2), and again, via a completely different approach and with an improved error term, in [19]. (See these papers for further background.)

The list-chromatic conjecture (3) for *bipartite* G (and in particular the abovementioned Dinitz Conjecture, which is just the case $G = K_{n,n}$) was given a beautiful and wholly elementary proof by Fred Galvin [16]. Note that for a bipartite multigraph G we have $\chi'(G) = D(G)$ (see [36, 37] or e.g. [43]). Lack of such an easy description of χ' is one difficulty distinguishing the general from the bipartite case in (3). An elegant proof of $\chi'_l(K_n) \leq n$ (so in particular of (3) for K_n, n odd) was given by Häggkvist and Janssen in [19], following ideas of Alon and Tarsi [4] and Janssen [23].

Fractional vs. integer and a more general conjecture

Theorem 1.1 is part of a considerably more general conjecture first suggested in [26]. A basic theme here—perhaps first proposed in [29] (see [26]) and developed particularly in [26], [32], [29]—is that there are large, natural classes of integer programming problems for which fractional versions (i.e. linear relaxations) are good predictors of asymptotic behavior of the original problems. The conjecture proposed here would be a particularly striking illustration of this idea. (Again, see [30], in addition to the preceding references, for a little more on this.)

We need a few definitions. Recall that a *k-uniform hypergraph* is a collection (possibly with repeats) of k-subsets, called *edges*, of some finite set V of *vertices*. Graph-theoretic notions—those of interest here are degree, regularity, matching, chromatic and list-chromatic indices—extend straightforwardly to hypergraphs. (See e.g. [15].)

We write $\mathcal{M}(\mathcal{H})$, or just $\mathcal{M}$, for the set of matchings of $\mathcal{H}$. Thus an (edge) coloring is just an $f : \mathcal{M} \to \{0, 1\}$ with

$$\sum_{A \in M \in \mathcal{M}} f(M) = 1 \quad \forall A \in \mathcal{H}. \tag{4}$$

The fractional version of this is a *fractional coloring*, that is, an $f : \mathcal{M} \to \mathbf{R}^+$ satisfying (4); and we have the *fractional chromatic index*:

$$\chi'^*(\mathcal{H}) = \min\{\sum_{M\in\mathcal{M}} f(M) : f \text{ a fractional coloring of } \mathcal{H}\}. \tag{5}$$

The celebrated "Matching Polytope Theorem" of J. Edmonds ([9], or e.g. [52], as well as Section 8) is equivalent to the statement that for a multigraph G, $\chi'^*(G)$ is given by (1). Thus [30] and Theorem 1.1 give asymptotic agreement of list-chromatic, chromatic and fractional chromatic indices for multigraphs. Our hope is that, even absent an Edmonds-type determination of χ'^*, we have such agreement for hypergraphs of any fixed (or just bounded; this makes no difference) edge size:

Conjecture 1.2 *For fixed k and k-uniform $\mathcal{H}$,*

$$\chi'_l(\mathcal{H}) \sim \chi'(\mathcal{H}) \sim \chi'^*(\mathcal{H}) \quad \textit{as} \quad \chi'^*(\mathcal{H}) \to \infty.$$

(One can define a fractional list-chromatic index, but it turns out to be the same as χ'^*.)

As mentioned above, Conjecture 1.2 was first proposed in [26]. It was suggested by [49] and [27] which proved respectively $\chi'(\mathcal{H}) \sim d$ and $\chi'_l(\mathcal{H}) \sim d$ assuming (in addition to fixed edge size)

$$d(x) \sim d \quad \forall x \in V \tag{6}$$

and

$$d(x,y) < o(d) \text{ for all distinct } x, y \in V,$$

where $d(x)$ is the degree of x, $d(x,y)$ is the number of edges containing both x and y and the convergence (of $d(x)/d$ to 1) in (6) is uniform in x.

Proof preview and outline of paper

Our approach here belongs to what we may call the "incremental random" method. (See for example [1], [51] [14], [48], [49], [25], [27], [32], [34], [24], [56], [35], [29], [30], as well as [15], [26], [28] for general discussions.) This involves generating some object—in our case a coloring—in random or "semirandom" increments, sometimes (as here) followed by a final "greedy" stage. Each increment is produced according to some random process, and one needs to show that the probability of choosing a "good" increment is positive (though perhaps small, whence "semirandom").

Analysis of such procedures can be somewhat painful. Typically one needs to exercise rather tight control over various parameters and over a large number of iterations, and it is usually far from obvious that one can avoid unmanageable deterioration in the relevant estimates. The realization that this *is* (sometimes) possible was a fundamental contribution of some of the early papers in the subject (e.g. [1], [51], [14], [48], [49]; see also [26] for an overview of the method as of about five years ago.)

At present, however, having reached some appreciation of the feasibility of such analysis, we may expect a shift of emphasis. I think the basic message of the method as it stands now is this: if you can discover a procedure for generating the increments which in principle *ought* to work, then you should eventually be able to show that it *does* work (emphasis on "eventually"). But it may be—as in the present case—that the procedure which "ought to work" is not a very obvious one: it is here, at least in the eye of this beholder, that the real beauty of the subject is currently to be found.

In the next few pages we attempt a rough description of, and some rationale for, the "not very obvious" procedure we will use to prove Theorem 1.1. See also Section 8 for an alternate approach for which we are currently stuck at "ought to."

Basic iteration

One way to state a list-coloring problem for G is: we are given, for each color γ, some $E_\gamma \subseteq E$, and are required to find matchings $M_\gamma \subseteq E_\gamma$ whose union is E. (We usually arrange that the M_γ are disjoint, though of course this doesn't matter.) To prove Theorem 1.1 we must show this is possible whenever each edge belongs to $(1+\delta)\chi'^*$ of the E_γ, provided χ'^* is large enough relative to δ.

As indicated above, we construct (rather, "construct") the M_γ's in stages. Initially we set $G^0 = G$, $G^0_\gamma = G_\gamma$. At the beginning of the ith stage we will have chosen disjoint matchings $F^j_\gamma \subseteq E_\gamma$ for $\gamma \in \Gamma$ and $j \in [i-1]$, and will be left with residual multigraphs

$$G^{i-1}_\gamma = G_\gamma - \bigcup_{\gamma'} \bigcup_{j=1}^{i-1} F^j_{\gamma'} - \bigcup_{j=1}^{i-1} V(F^j_\gamma).$$

(Thus G^{i-1}_γ is obtained from G_γ by removing all edges colored (by any color) through stage $i-1$ together with vertices already covered by the portion of M_γ so far determined.) The ith iteration is then as follows.

(a) For each γ choose (independently of other colors) a *random* (see below) matching $M^i_\gamma \subseteq E^{i-1}_\gamma := E(G^{i-1}_\gamma)$.

(b) If an edge A is in one or more M^i_γ's, then assign it a color $\gamma(A)$ chosen uniformly at random from $\{\gamma : A \in M^i_\gamma\}$. Set $F^i_\gamma = \{A : \gamma(A) = \gamma\}$. (This is added to M_γ.)

We then update by deleting $\cup_{\gamma'} F^i_{\gamma'}$ and $V(F^i_\gamma)$ from G^{i-1}_γ to form G^i_γ. (The final "greedy" phase is omitted from the present discussion; see Section 3.)

Hard-core distributions

A crucial omission in the preceding description is the definition of a "random" matching. As in [32], [30], the correct notion here is that of a "hard-core," or entropy-maximizing, distribution on the set $\mathcal{M} = \mathcal{M}(G)$ of matchings of a multigraph G. This is the distribution defined from some $\lambda : E(G) \to \mathbf{R}^+$ by taking the probability of M to be proportional to $\prod\{\lambda(A) : A \in M\}$. (See Section 2.)

We will basically need two properties from our random matchings. The first is (roughly)

$$\sum_\gamma p^{i-1}_\gamma(A) \approx 1 \quad \forall A \in E^{i-1} \tag{7}$$

where $p^{i-1}_\gamma(A) = \Pr(A \in M^i_\gamma)$ (defined to be zero if $A \notin E^{i-1}_\gamma$) and $E^{i-1} = \cup_\gamma E^{i-1}_\gamma$ is the set of edges uncolored through stage $i-1$.

The second—for use in the Lovász Local Lemma (Lemma 6.1)—is some degree of (approximate) independence: behavior of the random matching at a particular place in the graph shouldn't depend very much on what happens far from that place.

Initially (7) is easy, even in the exact form

$$\sum_\gamma p^0_\gamma(A) = 1 \quad \forall A \in E. \tag{8}$$

For, setting $\chi'^* = D$ and assuming $|\{\gamma : A \in E_\gamma\}| = (1+\delta)D$ for each A, it's immediate from the definition of χ'^* that there are distributions (producing random matchings M^1_γ) for which

$$\Pr(A \in M^1_\gamma) = [(1+\delta)D]^{-1} \quad \forall A \in E_\gamma \tag{9}$$

(see Section 3). Of course (9) gives (8).

Now a general distribution satisfying (9) need not have any useful properties *other than* (9). But it turns out—see Theorem 2.1—that there is a (unique) *hard-core* distribution with (9); and this distribution, which is also the unique entropy-maximizing distribution with (9), does have substantial independence properties. (See Lemma 2.2 and Section 4.1.) So we will use these distributions for our initial choices M^1_γ, and subsequent M^i_γ's will be chosen according to the same distributions appropropriately conditioned on the outcomes of earlier iterations. (These conditional distributions are again hard-core. It is also possible, again using Theorem 2.1, to define new hard-core distributions at each stage, but the present approach seems a little more convenient.) For $i > 0$ we cannot hope to control individual $p_\gamma^i(A)$'s but—this is in fact our main task—must still manage to maintain some version of (7).

Remark. We are thus, as in [32], [30], making a rather direct randomized transition between the fractional and integer versions of our problem, exploiting the fundamental observation Theorem 2.1 (from [50], [39], [40]). Note Theorem 2.1 does not depend on Edmonds' Theorem. However we do need Edmonds' Theorem for our basic independence result, Lemma 2.2; and though I think this use, too, could be avoided, it is the failure of anything like Lemma 2.2 which appears to be the main obstacle to a proof of Conjecture 1.2.

At any rate, I believe hard-core distributions have the potential to play a major role in the further development of the probabilistic method: they exist in great generality (Theorem 2.1 is really just an example), and—as suggested by entropy-maximization—*tend* to have useful independence properties. (But this is only a tendency. Establishing such properties raises issues akin to those involved in the phase transitions of statistical physics, and should itself be of considerable interest.) So they may, as in [32], [30] and the present work, prove useful in situations where the types of distributions common in the probabilistic method, typically involving many truly independent choices, are unsuitable.

"Ought to work"

In iterating the procedure described above, we will fix the weights λ_γ (which give the initial hard-core distributions with (9)) once and for all, and at the ith iteration work with the hard-core distribution p_γ^{i-1} given by the restriction of λ_γ to E_γ^{i-1}.

To see why our approach *might* work, let us pretend for a moment that we intend to color simply by choosing matchings $M_\gamma = M_\gamma^1$ according to distributions satisfying (8). Of course choosing the M_γ's independently gives substantial overlap and leaves about a $(1/e)$-fraction of the edges uncolored (note the p_γ^0 are small).

On the other hand, if we could arrange that, in addition to satisfying (8), the M_γ were *disjoint*, then $(M_\gamma : \gamma \in \Gamma)$ would automatically be a coloring. What's perhaps surprising is that we can actually manage something close to this. Of course our parallel (incremental) construction of the M_γ's guarantees disjointness; the interesting (and central) point is that it doesn't perturb the initial hard-core distributions very much.

This is based on the simple but key observation that one can sample from a hard-core distribution in stages. Note that, viewed from a particular $\gamma \in \Gamma$, our procedure becomes:

(a) Choose $M_\gamma^i \in \mathcal{M}(G_\gamma^{i-1})$ according to λ_γ.

(b) Form F^i_γ by deleting from M^i_γ all A with $\gamma(A) \neq \gamma$. Form G^i_γ by deleting from G^{i-1}_γ all vertices of edges of F^i_γ and all edges of $\cup_{\gamma'} M^i_{\gamma'}$.

We then continue with M^{i+1}_γ chosen from $\mathcal{M}(G^i_\gamma)$ *according to* λ_γ.

Now for an $A \in E^{i-1}_\gamma \setminus M^i_\gamma$ it's easy to see, using (7) and the fact that the $p^{i-1}_\gamma(A)$'s are small, that

$$\Pr(A \notin E^i_\gamma) = 1 - \prod_{\gamma' \neq \gamma} (1 - p^{i-1}_{\gamma'}(A)) \approx 1 - e^{-1} \tag{10}$$

(see (73)). Similarly (see (74)), for $A \in M^i_\gamma$ it turns out that

$$\Pr(A \in F^i_\gamma) \approx 1 - e^{-1}. \tag{11}$$

The "sampling in stages" mentioned above (and detailed in Section 4.2) says (modulo a minor point involving edges of $M^i_\gamma \setminus F^i_\gamma$) that if we had (10), (11) exactly and *independently* for all $A \in E^{i-1}_\gamma$, then $F^i_\gamma \cup M^{i+1}_\gamma$ would be distributed as M^i_γ. So if we performed (a) with such an idealized version of (b) for $i = 1, \ldots, s$, then

$$F^1_\gamma \cup \cdots \cup F^s_\gamma \cup M^{s+1}_\gamma \tag{12}$$

would be distributed as M^1_γ.

But if the *actual* matchings (12) were so distributed, then we would be nearly done, since for large s these matchings will (as γ varies) be mostly disjoint. (They are disjoint if, e.g., we replace M^{s+1}_γ by F^{s+1}_γ.)

This, then, is the essence of the proof: by constructing in parallel, we in effect "couple" the processes producing our random matchings so as to keep the distributions of the individual M_γ's close to the original hard-core distributions p^0_γ, while forcing the matchings to be disjoint.

Implementation

For organizational purposes our basic result is Lemma 3.1, which describes what happens in a single iteration. The easy derivation of Theorem 1.1 from this is given in Section 3.

Technical supports for the proof of Lemma 3.1 are the approximate independence results of Section 4.1, martingales (Section 5), and the Local Lemma. (The combination of martingales and the Local Lemma is used in similar ways in some of the papers cited above, though choices of martingales here are perhaps less obvious than in most earlier applications. It may also be that some or all of our martingales could be replaced by the powerful ideas of Talagrand [57], the present paper having actually been written before [57] appeared.)

As suggested above, the crucial point (Lemma 3.1(b)) is maintenance of an appropriate version of (7). Martingales are first used to show that the left hand side of (7)—a sum of many small r.v.'s—is concentrated near its mean. This part of the argument, given in Section 6, requires some care, as the sum in (7) is typically affected by developments in all colors (even those for which $A \notin E_\gamma$), and some of these effects can be relatively substantial. In particular we depend here on the results of Section 4.1 (to bound the potential effects of various perturbations) and Lemma 5.3 (from [27]), an "adaptive" concentration inequality which is more delicate than more standard results along the lines of "Azuma's Inequality."

The most interesting and unusual feature of the present situation is that it's not even clear that the *expected* value of the sum in (7) is close to 1. (In antecedents, estimation of expectations is generally fairly straightforward.) This is (roughly) the content of Lemma 6.2, which is proved in Section 7 by tying ("coupling") the actual process to the idealized version. Here again martingales and the results of Section 4.1 provide technical support.

Additional terminology

For $X, Y \in V \cup E$, we use $\Delta(x, y)$ or, if necessary, $\Delta_G(X, Y)$ for distance from X to Y in G, defined in the natural way. (Thus, e.g., if $A, B \in E$ then

$$\Delta(A, B) = \min\{\Delta(x, y) : x, y \in V, x \in A, y \in B\}.)$$

As above, we write $\mathcal{M}(G)$ for the set of matchings of G. For additional graph-theoretic background see e.g. [43].

We write $a =_\delta b$ for $e^{-\delta} < a/b < e^\delta$. (In some earlier papers we used $(1 + \delta)$ instead of e^δ, but the present definition seems preferable because it composes nicely: $a =_\varepsilon b =_\delta c \Rightarrow a =_{\varepsilon+\delta} c$.)

For natural number n we write $[n]$ for $\{1, \ldots, n\}$. In general we treat large numbers as integers, trusting that the reader will agree this is preferable to cluttering the paper with essentially irrelevant $\lfloor\ \rfloor$'s.

2 Hard-core distributions

In this section we just want to give enough background to enable us to describe our basic iteration (Lemma 3.1). Further preliminaries on hard-core distributions are then given in Section 4.

As in [30], a key ingredient of the present work is the notion of a *hard-core* distribution (h.c.d.) on the set $\mathcal{M} = \mathcal{M}(G)$ for a multigraph G. This is a distribution

$p = p_\lambda$ derived from some $\lambda : E \to \mathbf{R}^+$ according to

$$w(M) = \prod_{A \in M} \lambda(A),$$

$$p(M) = w(M) / \sum_{M' \in \mathcal{M}} w(M').$$

We will also say p is the hard-core distribution given by λ on G and sometimes simply refer to M as *chosen according to* λ *on* G. In addition, we will often consider the h.c.d. given by the restriction of λ to some subgraph G' of G, and will simply refer to this distribution as *given by* λ *on* G'.

Remark. The name "hard-core" is that given to such distributions in statistical physics (e.g. [6]), where the weights λ_A are sometimes called *activities.* ("Monomer-dimer system" and "exclusion model" are also used; see e.g. [20, 21], [38], [41].) Other recent, rather diverse contexts in which hard-core distributions have proved important include [42], [50], [39, 40]. (They are called "normal populations" in [50], and in [39, 40] are not thought of as probability distributions at all, but as "canonical" convex representations of points in $\mathbf{R}^n$.)

We will make frequent (tacit) use of the observation that if M is chosen according to λ on G and, for some disjoint $F, H \subseteq E$, we condition on $\{F \subseteq M \subseteq E \setminus H\}$, then $M \setminus F$ is chosen according to λ on $G - V(F) - H$.

We write $x \prec M$ for $x \in \cup_{A \in M} A$ (that is, x is covered by the matching M). For p a probability distribution on $\mathcal{M}$, $M \in \mathcal{M}$ chosen according to p, $x_i \in V$ and $A \in E$, we write $p(\overline{x}_1, \ldots, \overline{x}_t)$ for $\Pr(x_1, \ldots, x_t \not\prec M)$, and $p(A)$ for $\Pr(A \in M)$. We also extend this notation to conditional probabilities—e.g. $p(\overline{x}|\overline{y})$—in the obvious ways.

The numbers $p(A)$ are the *marginals* of p. The marginal vectors $(p(A) : A \in E) \in \mathbf{R}^E$ are of central interest here. The set of such vectors is actually a familiar object, the *matching polytope* of G,

$$\begin{aligned} MP(G) &= \operatorname{conv}\{1_M : M \in \mathcal{M}(G)\} \\ &= \{(p(A) : A \in E) : p \text{ a probability distribution on } \mathcal{M}\}. \end{aligned}$$

(In its usual form Edmonds Theorem is a description of the matching polytope, so of those $f \in [0,1]^E$ which are marginal vectors of probability distributions on $\mathcal{M}$; see following Proposition 8.1)

Of course, the existence of a fractional coloring of total weight T is equivalent to the existence of a p with all marginals $1/T$, so that

$$\chi'^*(G) = \min\{T : \{1/T\}^E \in MP(G)\}. \tag{13}$$

Here we are interested in which f's are marginals of *hard-core* distributions. This is answered by the following result of Rabinovich, Sinclair and Wigderson [50] and Lee [39], [40].

Theorem 2.1 *If $f \in \mathbf{R}^E$, $f \geq 0$, then there is a hard-core distribution p with marginals f if and only if $f \in (1-\delta)MP(G)$ for some $\delta > 0$, and in this case p is both the unique such hard-core distribution and the (unique) entropy-maximizing distribution with marginals f.*

If we *fix* δ then the p given by Theorem 2.1 exhibits considerable approximate independence, a phenomenon which is at the heart of the present work (and of [30]), and is the reason for our interest in hard-core distributions in this context. The basic result is from [31]:

Lemma 2.2 *For any $\delta > 0$ there is a $\delta' > 0$ such that if p is a hard-core distribution whose vector of marginals lies in $(1-\delta)MP(G)$, then*

$$p(\overline{x}, \overline{y}) > \delta' \quad \forall x, y \in V.$$

The relevant consequence of Lemma 2.2 (Lemma 4.4) and a sort of complementary result (Corollary 4.7), again providing some measure of approximate independence, are given in Section 4.

3 Basic iteration

The following lemma describes what happens in a single iteration of the procedure sketched in Section 1. For its statement, suppose $G = (V, E)$ is a multigraph with, for each $\gamma \in \Gamma$, $G_\gamma = (V_\gamma, E_\gamma) \subseteq G$, $\lambda_\gamma : E_\gamma \to \mathbf{R}^+$, and p_γ the associated hard-core distribution. We will also, when convenient, regard λ_γ as defined on all of E, with $\lambda_\gamma(A) = 0$ if $A \in E \setminus E_\gamma$.

Lemma 3.1 *For each $K, \zeta > 0$ there are $\xi = \xi(K, \zeta) > 0$ and $D^* = D^*(K, \zeta)$ such that: if $D > D^*$ and*

$$\lambda_\gamma(A) \leq K/D \quad \forall \gamma \in \Gamma, A \in E_\gamma, \tag{14}$$

$$d_G(v) \leq D \quad \forall v \in V, \tag{15}$$

$$\sum_\gamma p_\gamma(A) =_\xi 1 \quad \forall A \in E, \tag{16}$$

then there are matchings $F_\gamma \subseteq E_\gamma$ so that with

$$G'_\gamma = G_\gamma - V(F_\gamma) - \cup F_{\gamma'},$$

p'_γ the hard-core distribution given by λ_γ on G'_γ and $G' = G - \cup F_\gamma$, we have

$$d_{G'}(v) \le \frac{1+\zeta}{1+\xi} e^{-1} D \quad \forall v \in V, \tag{17}$$

$$\sum_\gamma p'_\gamma(A) =_\zeta 1 \quad \forall A \in E(G'). \tag{18}$$

(The expression $(1+\zeta)/(1+\xi)$ is convenient later; of course substituting $1+\zeta$ gives an equivalent statement.)

Following some preliminaries, Lemma 3.1 is proved in Sections 6 and 7. In the rest of this section we show why it implies Theorem 1.1.

For Theorem 1.1 we must show that for any fixed $\delta > 0$,

$$\chi'_l(G) < (1+\delta)\chi'^*(G) \tag{19}$$

whenever $T := \chi'^*(G)$ is sufficiently large. That is, given $S(A) \subseteq \Gamma$ with $|S(A)| = (1+\delta)T$ for each $A \in E$, and $E_\gamma := \{A \in E : \gamma \in S(A)\}$, we must show there are matchings $M_\gamma \subseteq E_\gamma$ with

$$\cup M_\gamma = E. \tag{20}$$

For each $\gamma \in \Gamma$ let $\lambda_\gamma : E_\gamma \to \mathbf{R}^+$ be the weight function whose associated hard-core distribution p_γ has marginals

$$p_\gamma(A) = [(1+\delta)T]^{-1} \quad \forall A \in E_\gamma.$$

Since this marginal vector is in $(1+\delta)^{-1}MP(G_\gamma)$ (see (13)), we get existence of λ_γ from Theorem 2.1, while from Lemma 2.2 and the fact that $p_\gamma(A) = \lambda_\gamma(A)p_\gamma(\overline{x}, \overline{y})$, where x, y are the ends of A, we have, for each A,

$$\lambda_\gamma(A) \le K/T \tag{21}$$

for some $K = K(\delta)$. Of course we also have

$$\sum_\gamma p_\gamma(A) = 1 \quad \forall A \in E.$$

Now let s be a positive integer with $s < 2\log K$ (log is ln, 2 is fairly arbitrary) and

$$e^s > 4eK. \tag{22}$$

Set $\zeta_0 = 0$, $\zeta_s = 1$ (this value, too, is quite arbitrary), and for $i = s, \ldots, 2$,

$$\zeta_{i-1} = \xi(K, \zeta_i)$$

(ξ as in Lemma 3.1).

Let $T^* = D^*(K, \zeta_1)$ and assume T $(= \chi'^*(G))$ satisfies

$$T > e^s T^*. \tag{23}$$

Set $T_0 = T$ and for $i = 1, \ldots, s$,

$$T_i = \frac{1+\zeta_i}{1+\zeta_{i-1}} e^{-1} T_{i-1} = (1+\zeta_i)e^{-i}T.$$

Let $G^0 = G$, $G^0_\gamma = G_\gamma$, $E^0 = E(G^0)$, $E^0_\gamma = E(G^0_\gamma)$.

For $i = 1, \ldots, s$ we may apply Lemma 3.1 (with $\zeta = \zeta_i$, K as in (21) and $D = T_{i-1}$) to find matchings $F^i_\gamma \subseteq E^{i-1}_\gamma$ so that, with

$$G^i_\gamma = G^{i-1}_\gamma - V(F^i_\gamma) - \cup_{\gamma'} F^i_{\gamma'}, \qquad E^i_\gamma = E(G^i_\gamma),$$

$$G^i = G^{i-1} - \cup_\gamma F^i_\gamma, \qquad E^i = E(G^i),$$

λ^i_γ the restriction of λ_γ to E^i_γ, p^i_γ the corresponding hard-core distribution, and $d^i(v)$ the degree of v in G^i, we have

$$d^i(v) \le T_i \qquad \forall v \in V,$$

$$\sum_\gamma p^i_\gamma(A) =_{\zeta_i} 1 \qquad \forall A \in E^i. \tag{24}$$

(Notice that by (21), $\lambda^{i-1}_\gamma(A) \le \lambda_\gamma(A) \le K/T \le K/T_{i-1}$, which gives hypothesis (14) of Lemma 3.1; and by (23), $T_{i-1} \ge T_s \ge D^*(K, \zeta_1) \ge D^*(K, \zeta_i)$—that is, we may assume the last inequality—so that the lemma does indeed apply.)

Finally we need to find matchings $F^{s+1}_\gamma \subseteq E^s_\gamma$ with $\cup_\gamma F^{s+1}_\gamma = E^s$. This can be done greedily:

For each $A \in E^s$ we have (by (24)) $\sum_\gamma p^s_\gamma(A) > e^{-1}$ and, using (21),

$$p^s_\gamma(A) \le \lambda_\gamma(A) \le K/T$$

(the first inequality is trivial; see (26)), so that, by (22),

$$|\{\gamma : A \in E^s_\gamma\}| > e^{-1}T/K > 4e^{-s}T = 2T_s.$$

On the other hand degrees in G^s are at most T_s, so a greedy coloring suffices at this point.

□

4 More on hard-core distributions

Throughout this section we take $G = (V, E)$ with $\lambda : E \to \mathbf{R}^+$ and write p or p_G for the corresponding h.c.d. on $\mathcal{M} = \mathcal{M}(G)$. Similarly for $H \subseteq G$, p_H is the h.c.d. given by λ on H. We write f_p for the vector $(p(A) : A \in E)$ of marginals of p.

Recall that if x, y are the ends of A then

$$p(A) = \lambda(A)p(\overline{x}, \overline{y}). \tag{25}$$

This gives the trivial but useful bound (already used above)

$$p(A) \leq \lambda(A). \tag{26}$$

Let $\lambda_{xy} = \sum\{\lambda(A) : x,\, y \text{ are the ends of } A\}$. In view of (25) we have

$$p(\overline{x}) + \sum_{y \sim x} \lambda_{xy} p(\overline{x}, \overline{y}) = 1,$$

which, when divided by $p(\overline{x})$, gives the basic identity

$$p(\overline{x}) = [1 + \sum_{y \sim x} \lambda_{xy} p(\overline{y}|\overline{x})]^{-1} \tag{27}$$

Corollary 4.1 *If $D(G) \leq D$ and $\lambda(B) \leq \lambda$ for all $B \in E$, then for all $x \in V$, $p(\overline{x}) \geq [1 + \lambda D]^{-1}$*

This gives something like a converse of (26):

Corollary 4.2 *If $D(G) \leq D$ and $\lambda(B) \leq \lambda$ for all $B \in E$, then for any $A \in E$, $p(A) \geq [1 + \lambda D]^{-2}\lambda(A)$.*

Proof. Taking x, y to be the ends of A and applying Corollary 4.1 to both $p(\overline{x})$ and $p(\overline{y}|\overline{x})$, we have $p(\overline{x}, \overline{y}) \geq [1 + \lambda D]^{-2}$ and the result follows from (25).

□

4.1 Approximate independence

Here we give what we need in the way of (approximate) independence properties of h.c.d.'s.

A first easy consequence of Lemma 2.2, observed in [30], is

Corollary 4.3 *For all $\delta > 0$ there exists $K(\delta)$ such that if $f_p \in (1-\delta)MP(G)$, then*

$$\sum_y \lambda_{xy} \le K(\delta) \qquad \forall x \in V.$$

This will allow us in what follows to assume that our various weight functions λ satisfy

$$\sum_y \lambda_{xy} \le K \quad \forall x \in V \tag{28}$$

for some constant K. The desired independence is then given by the following result from [30] (see Lemma 3.2; the present statement is a little stronger than that given in [30] but is what's actually proved there.)

Lemma 4.4 *For all K and $\xi > 0$ there exists $\Delta = \Delta_1(K, \xi)$ so that for any λ satisfying (28) and M drawn from $\mathcal{M}$ according to λ, the following are true.*

(a) *For any $v \in V$ and Q any specification of the restriction of M to $\{B \in E : \Delta(v, B) \ge \Delta\}$,*

$$p(\overline{v}|Q) =_\xi p(\overline{v}).$$

(b) *Similarly, for any $A \in E$ and Q any specification of the restriction of M to $\{B \in E : \Delta(A, B) \ge \Delta\}$,*

$$p(A|Q) =_\xi p(A).$$

Lemma 4.4 says that probabilities $p(\overline{v})$, $p(A)$ are not much affected by what happens far from v or A; actually in what follows we will only work with $p(A)$.

We will also need to know something about the effects of perturbations which take place closer to A. For this it's convenient to work with Godsil's [17] notion of the *path-tree* $T(G, v)$ associated with a graph G and $v \in V(G)$. (This is called a *tree of walks* in [17]; "path-tree" is from [43].)

The vertices of $T = T(G, v)$ are the paths of G which begin at v. Two vertices of T are adjacent if one is a maximal proper subpath of the other. We write v for the singleton path (v) and take it to be the root of T. In what follows we will actually be interested in some $A \ni v$, say with other end w, and will also write A for the edge $\{(v), (v, A, w)\}$ of T.

We take π to be the natural projection from T to G; thus $\pi((y_0, A_1, y_1, \ldots, y_l)) = y_l$ and $\pi(\{(y_0, \ldots, y_{l-1}), (y_0, , \ldots, y_{l-1}, A_l, y_l)\}) = A_l$. If G is weighted by λ, then we also regard T as weighted by λ, with $\lambda_B = \lambda_{\pi(B)}$.

Path-trees turn out to capture considerable information about matchings, and to be in some respects easier to work with than the original graph. (Again see

[17] or the exposition in [43]. See also [33] for a more sophisticated application of path-trees than that given here.) For our purposes the relevant connection is given by

Lemma 4.5 *With notation as above,* $p_G(\overline{v}) = p_{T(G,v)}(\overline{v})$*, and for any* $A \ni v$*,* $p_G(A) = p_{T(G,v)}(A)$*.*

As observed in see [33], the first statement is an immediate consequence of the main result of [17]. (Actually [17], [33] only discuss unweighted graphs, but their arguments extend without modification to the more general case.) The second statement is then easily seen to follow via $p(A) = \lambda(A)p(\overline{v}, \overline{w}) = \lambda(A)p(\overline{v})p(\overline{w}|\overline{v})$.

□

For T a tree weighted by λ and $Y, Z \in V(T) \cup E(T)$, let

$$\psi(Y, Z) = \psi_T(Y, Z) = \prod\{\lambda(B) : B \in E(P(Y, Z))\},$$

where $P(Y, Z)$ is the shortest path having an end in each of Y, Z. (So e.g. $\psi(Y, Z) = 1$ if Y, Z share a vertex.)

Our basic complement to Lemma 4.4, proved below, is

Lemma 4.6 *For any weighted (by* λ*) tree* T*,* $A \in E(T)$ *and* $x \in V(T)$*,*

$$|p(A) - p(A|\overline{x})| \le \lambda(A)\psi(A, x).$$

For T_1, T_2 subtrees of a tree T, and A some distinguished edge of T, set

$$V(T_1)\Delta^* V(T_2) = \{x \in V(T_1)\Delta V(T_2) : P(A, x) - x \subseteq T_1 \cap T_2\}$$

if $A \in T_1 \cap T_2$ and $V(T_1)\Delta^* V(T_2) = \emptyset$ otherwise. Repeated application of Lemma 4.6 yields a basic tool in what follows:

Corollary 4.7 *With notation as above,*

$$|p_{T_1}(A) - p_{T_2}(A)| \le \lambda(A) \sum\{\psi(x) : x \in V(T_1)\Delta^* V(T_2)\}.$$

For the proof of Lemma 4.6 we need some additional notation. For $F \subseteq E$, set

$$w(F) = \prod_{A \in F} \lambda(A)$$

(thus extending the w defined earlier on $\mathcal{M}$), and for $\mathcal{F} \subseteq 2^E$, set

$$w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w(F).$$

Proof of Lemma 4.6. Suppose the ends of A are v, w, with w the end closer to x. Let $\mathcal{P}$ be the set of paths having one end x and containing A, and $\mathcal{Q}$ the set of paths with one end v and *not* containing A.

Set $\mathcal{M}(A) = \{M \in \mathcal{M} : A \in M\}$ and define $\mathcal{M}(\overline{x})$, $\mathcal{M}(A, \overline{x})$ analogously. Then

$$\begin{aligned} |p(A) - p(A|\overline{x})| &= |\frac{w(\mathcal{M}(A))}{w(\mathcal{M})} - \frac{w(\mathcal{M}(A,\overline{x}))}{w(\mathcal{M}(\overline{x}))}| \\ &= \frac{|w(\mathcal{M}(A))w(\mathcal{M}(\overline{x})) - w(\mathcal{M}(A,\overline{x}))w(\mathcal{M})|}{w(\mathcal{M})w(\mathcal{M}(\overline{x}))}. \end{aligned} \tag{29}$$

The numerator is

$$|\sum\{w(M_1)w(M_2) : A \in M_1, x \not\sim M_2\} - \sum\{w(M_1)w(M_2) : A \in M_1, x \not\sim M_1\}| \tag{30}$$

(where we always take $M_i \in \mathcal{M}$), which reduces to

$$\begin{aligned} &\sum_{P \in \mathcal{P}} w(P)w^2(\mathcal{M}(G - V(P))) \\ &\qquad = \lambda(A)\psi(A,x) \sum_{Q \in \mathcal{Q}} w(Q)w^2(G - V(P(v,x) \cup Q)). \end{aligned} \tag{31}$$

For the reduction, note that each term $w(M_1)w(M_2)$ appearing in (30) is of the form

$$\prod_B \lambda(B)^{f(J,B)} \tag{32}$$

where the multisubset J of E is the union of two matchings and $f(J,B)$ is the multiplicity of B in J. Let $c(J)$ be the number of nonsingleton components of J not containing A or x and not consisting of two vertices joined by an edge B with $f(J,B) = 2$. If A, x are not in the same component of J, then the monomial (32) appears $2^{c(J)}$ times in each of the two sums in (30). If A, x are in the same component $P \in \mathcal{P}$, then (32) appears $2^{c(J)}$ times in one of the sums and not at all in the other; namely, it appears in the first sum if $P(w,x)$ is even and in the second if $P(w,x)$ is odd. This is easily seen to give the left hand side of (31). (The equality in (31) is an immediate consequence of the obvious bijection $\mathcal{Q} \to \mathcal{P}$.)

But for any $Q \in \mathcal{Q}$, the summand $w(Q)w^2(G - V(P(v,x) \cup Q))$ in (31) is at most $w^2(\mathcal{M}(G - V(P(w,x)))$, which is less than the denominator in (29). The lemma follows.

□

4.2 Generation in stages

As indicated in Section 1, we will choose the F_γ's of Lemma 3.1 by first choosing, for each γ, $M_\gamma \in \mathcal{M}_\gamma$ according to p_γ, and then deleting some edges that are in other $M_{\gamma'}$'s. A simple but key idea here is that we can sample from a hard-core distribution in stages. This procedure, an idealized version of our actual situation as well as the basis for its analysis, is as follows.

Suppose $p = p_\lambda$ is a hard-core distribution on $\mathcal{M} = \mathcal{M}(G)$, and for each $A \in E = E(G)$, let $q_A \in [0, 1]$. Choose $M \in \mathcal{M}$ via:

(a) Choose $N \in \mathcal{M}$ according to p.

(b) Let $\{X_A : A \in E\}$ be chosen independently (of each other and of N) from $\{0, 1\}$ according to $\Pr(X_A = 1) = q_A$. Set

$$F = \{A \in N : X_A = 1\}, \qquad H = \{A \in E \setminus N : X_A = 1\}.$$

(c) Choose $M \in \mathcal{M}$ according to p conditioned on $\{F \subseteq M \subseteq E \setminus H\}$.

Remark. Notice that (c) is equivalent to

(c′) Choose $K \in \mathcal{M}(G - V(F) - H)$ according to λ and set $M = F \cup K$.

Lemma 4.8 *(With notation as above) M is chosen according to p.*

Corollary 4.9 *(With notation as above) for each $A \in E$,*

$$\Pr(A \in M | A \notin F \cup H) = p(A).$$

Proof of Corollary. By Lemma 4.8, $\Pr(A \in M) = p(A)$. But we also have

$$\begin{aligned}\Pr(A \in M) &= \Pr(A \in F) + \Pr(A \notin F \cup H) \Pr(A \in M | A \notin F \cup H) \\ &= p(A)q_A + (1 - q_A) \Pr(A \in M | A \notin F \cup H),\end{aligned}$$

and the Corollary follows.

□

Proof of Lemma 4.8. Set

$$\mathcal{Q} = \{(F, H) : F \in \mathcal{M}, H \subseteq E \setminus F\},$$

the set of possibilities for the pair (F, H). For $Q = (F, H) \in \mathcal{Q}$, $M \in \mathcal{M}$, write $M \sim Q$ if $F \subseteq M \subseteq E \setminus H$. Notice that

$$\Pr(M|Q) = \begin{cases} p(M)[\sum\{p(M') : M' \sim Q\}]^{-1} & \text{if } M \sim Q \\ 0 & \text{otherwise.} \end{cases}$$

(Here and below, expressions $\Pr(\cdot)$ refer to (a)-(c) in the obvious ways. As usual we write $p(M)$ for the probability of M under p.)

The main point is that $\Pr(Q|N)$ doesn't depend on N (provided $N \sim Q$): if we set

$$f(Q) = \prod_{A \in F \cup H} q_A \prod_{A \notin F \cup H} (1 - q_A),$$

then

$$\Pr(Q|N) = \begin{cases} f(Q) & \text{if } N \sim Q \\ 0 & \text{otherwise.} \end{cases}$$

Combining these two observations, we have

$$\begin{aligned} \Pr(M) &= \sum_{Q \sim M} \Pr(Q) \Pr(M|Q) \\ &= \sum_{Q \sim M} \sum_{N \sim Q} p(N) f(Q) \frac{p(M)}{\sum_{M' \sim Q} p(M')} \\ &= p(M) \sum_{Q \sim M} f(Q) \\ &= p(M). \end{aligned}$$

□

5 Concentration

In this section we briefly review the necessary martingale background. The discussion here is contained in that of [27] (to which we refer for proofs). See also, e.g., [45], [44], [7] for further information.

Our notation mainly follows [45]. Briefly, we are given a probability space based on a finite set Ω, and a sequence of equivalence relations $\{\equiv_i\}_{i=0}^m$ on Ω, each refining the preceding one. We take $\mathcal{A}_i$ to be the set of atoms of the partition corresponding to $\equiv_i$, and $\mathcal{F}_i$ the associated Boolean algebra. For a random variable $X : \Omega \to \mathbf{R}$ we set $X_i = E[X|\mathcal{F}_i]$ (that is, for each $\omega \in H \in \mathcal{A}_i$, $X_i(\omega) = E[X|H]$). Then $\{X_i\}$ is a martingale with respect to $\{\mathcal{F}_i\}$ (i.e. $E[X_i|\mathcal{F}_{i-1}] = X_{i-1}$). Finally, set $Z_i = X_i - X_{i-1}$ for $i = 1, \ldots, m$ and $Z = \sum Z_i$. The sequence $\{Z_i\}$ is a *martingale difference sequence* (with respect to $\{\mathcal{F}_i\}$); that is,

$$E[Z_i|\mathcal{F}_{i-1}] = 0.$$

In this paper we will always take $\Omega = \prod_{i=1}^m \Omega_i$ for some finite probability spaces Ω_i, and

$$\omega \equiv_i \omega' \quad \text{iff} \quad \omega_j = \omega'_j \quad 1 \le j \le i. \tag{33}$$

In particular, we will always have $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and X constant on each $H \in \mathcal{A}_m$ (actually, elements of $\mathcal{A}_m$ are singletons), so that $X_0 = E[X]$, $X_m = X$ and $Z = X - E[X]$. We write $\omega \sim_i \omega'$ (with $\omega, \omega' \in \Omega$) if $\omega_j = \omega'_j \quad \forall j \neq i$.

Typical of the type of inequality we need, though not strong enough for most of our applications, is the following "bounded differences inequality" (the name, as far as I know, is from [44]).

Lemma 5.1 *(With notation as above) suppose that (for some $c_1, \ldots, c_m$),*

$$|X(\omega) - X(\omega')| \le c_i$$

whenever $\omega \sim_i \omega'$. Then for any $\lambda > 0$,

$$\Pr(|X - E[X]| \ge \lambda) \le 2\exp[-2\lambda^2/(\sum c_i^2)].$$

Bounds of this type are derived via "Chernoff's method" (Markov's inequality applied to $E[e^{\eta Z}]$ for a suitable η); for example:

$$E[e^{\eta Z}] \le e^{\eta^2 T/2} \ \ \forall |\eta| \le S \quad \Rightarrow \quad Pr(|Z| > \lambda) < 2e^{-\lambda^2/2T} \ \ \forall 0 \le \lambda \le ST. \tag{34}$$

The main issue is thus bounding $E[e^{\eta Z}]$. We will need the following two results from [27] (see Lemmas 3.9 and 3.4 respectively).

Lemma 5.2 *Suppose W_α ($\alpha = 0, 1 \ldots$) are the possible values of ω_i, with $\Pr(\omega_i = W_\alpha) = q_\alpha$. Suppose further that*

$$|X(\omega) - X(\omega')| \le c_\alpha \le c$$

whenever $\omega, \omega' \in H_{i-1}$, $\omega \sim_i \omega'$, $\omega_i = W_0$ and $\omega'_i = W_\alpha$. Then

$$E[e^{\eta Z_i}|H_{i-1}] \le \exp[8\eta^2 c \sum_{\alpha \neq 0} q_\alpha c_\alpha]$$

$\forall |\eta| c \le 1$.

(The restriction $\alpha \neq 0$ in the sum is just for emphasis, since we can take $c_0 = 0$.) Lemma 5.2 (in conjunction with (36) below) does significantly better than Lemma 5.1 when q_0 is close to 1.

Lemma 5.3 *With notation as above, and with H_j understood to range over $\mathcal{A}_j$ for each j,*

$$E[e^{\eta Z}] \leq \max\left\{\prod_{i=1}^{m} E[e^{\eta Z_i}|H_{i-1}] : H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{m-1}\right\}. \tag{35}$$

(This was the main martingale contribution of [27]. Earlier results used instead

$$E[e^{\eta Z}] \leq \prod_{i=1}^{m} \max\{E[e^{\eta Z_i}|H_{i-1}] : H_{i-1} \in \mathcal{A}_{i-1}\}. \tag{36}$$

Lemma 5.3 can be stronger than results based on (36) when, roughly speaking, the influence of a particular ω_j on X varies widely from one evolution $H_0 \supseteq H_1 \supseteq \cdots \supseteq H_m$ to another. See e.g. the proof of (52) below.)

6 Proof of Lemma 3.1

We first describe a random procedure for generating matchings F_γ. Once we have properly understood the probable local behavior of the procedure, we can invoke the Lovász Local Lemma [11] to finish.

We use the Local Lemma in the form (see [13] or e.g. [3, p.55])

Lemma 6.1 *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space, and Σ a graph on vertex set $[n]$ with all degrees at most d such that for any $i \in [n]$ and $S \subseteq \{j \in [n] \setminus \{i\} : i \not\sim_\Sigma j\}$,*

$$\Pr(A_i | \wedge_{j \in S} \overline{A}_j) \leq p. \tag{37}$$

Then if $ep(d+1) \leq 1$, $\Pr(\wedge_{i \in [n]} \overline{A}_i) > 0$.

Though (37) is all that's needed for the conclusion of Lemma 6.1, in our situation much more will actually be true; namely the probability of A_i will not be significantly affected by any conditioning involving events A_j with $j \not\sim_\Sigma i$. (See Lemma 6.3.)

The F_γ's are obtained as follows.

(1) For each γ choose M_γ according to p_γ and independently of other choices. (Note we're recycling: the present M_γ is not that of (20).)

(2) For each $A \in \cup M_\gamma$, choose $\gamma(A)$ uniformly (and independently of other choices) from $\{\gamma : A \in M_\gamma\}$, and set $F_\gamma = \{A : \gamma(A) = \gamma\}$ $(\subseteq M_\gamma)$.

We then define G'_γ, p'_γ and G' as in the statement of Lemma 3.1. In what follows we write $d(v)$ for $d_G(v)$ and $d'(v)$ for $d_{G'}(v)$.

Of the conclusions of the lemma, (18) is by far the more difficult to handle. As noted below, it's easy to see that

$$\Pr(A \notin \cup F_\gamma) = \Pr(A \notin \cup M_\gamma) \approx e^{-1}$$

for each A, so that we typically have $d'(v) \approx e^{-1}d(v)$. To derive (17) (via Lemma 6.1) we then need to say something about concentration of $d'(v)$; this will be an easy application of the results of Section 5. We also need to say—a consequence of Lemma 4.4—that behavior at v is nearly independent of behavior far from v.

For (18), the results of Section 5 will again give concentration (of $\sum_\gamma p'_\gamma(A)$ on $\{A \notin \cup M_\gamma\}$), though the analysis in this case is more delicate, and in particular requires Lemma 5.3. What's unusual here is that *expectation* causes some difficulties.

What we will show is that each $p'_\gamma(A)$, conditioned on survival of A, has expectation close to $p_\gamma(A)$; that is (roughly),

$$E[p'_\gamma(A) | A \notin \cup M_{\gamma'}] \approx p_\gamma(A). \tag{38}$$

As observed in Section 1, this is perhaps the most interesting part of our analysis.

(There's a slight additional complication because—again for use in the Local Lemma—we need (38) to hold regardless of what happens far from A. This is the reason for the more general hypothesis (39) in Lemma 6.2 below.)

Take G, G_γ etc. to be as in the discussion preceding Lemma 3.1, and M_γ, p'_γ etc. as described above.

Lemma 6.2 *For each K and $\zeta > 0$ there are $\xi = \xi_1(K, \zeta) > 0$, $D_1 = D_1(K, \zeta)$ and $\Delta = \Delta_2(K, \zeta)$ for which the following is true. Fix $\gamma \in \Gamma$ and $A \in E_\gamma$, let $D > D_1$, and assume the hypotheses of Lemma 3.1 with* (16) *replaced by*

$$\textstyle\sum_{\gamma'} p_{\gamma'}(B) =_\xi 1 \quad \text{for all } B \in E_\gamma \text{ with } \Delta_G(A, B) \leq \Delta. \tag{39}$$

Then

$$E[p'_\gamma(A) | A \notin \cup M_{\gamma'}] =_\zeta p_\gamma(A).$$

This is proved in Section 7. In the rest of the present section we show that it implies Lemma 3.1.

Assume the conditions of Lemma 3.1 with $\xi = \frac{1}{2}\xi_1(K, \zeta/2)$ (trivial remark: this will be much smaller than ζ) and $D^* \geq D_1(K, \zeta/2)$ large enough to justify the estimates below.

We would like to apply Lemma 6.1 to obtain the conditions (17), (18); however, the precise interdependence of these events seems a bit subtle, making it convenient to work with a modified p'_γ, as follows.

Let $\Delta_1 = \Delta_1(K, \xi)$ (see Lemma 4.4). For each $A \in E$ fix an end v_A of A, and let $T(A) = T(G, v_A)$. In addition, for each $E_\gamma \ni A$ let $T'_\gamma(A)$ be the subtree of $T(G'_\gamma, v_A)$ whose vertices are those at distance at most Δ_1 from A. (So in particular each $T'_\gamma(A)$ is a subtree of $T(A)$.) We regard A, v_A as elements of $T(A)$, $T'_\gamma(A)$ in the usual way.

Let $p''_\gamma(A) = \Pr(A \in M)$, with M drawn from $\mathcal{M}(T'_\gamma(A))$ according to λ_γ. By Lemma 4.5 and our choice of Δ_1, we have (for any A, γ)

$$p''_\gamma(A) =_\xi p'_\gamma(A). \tag{40}$$

(This estimate is stronger than necessary; the choice of Δ_1 is dictated by (49) below.)

Now for $v \in V$, $A \in E$, define r.v.'s

$$X_v = d'(v),$$

$$X_A = \sum_\gamma p''_\gamma(A)$$

and events

$$\begin{aligned} T_v &= \{X_v > \frac{1+\zeta}{1+\xi} e^{-1} D\} \\ T_A &= \{A \notin \cup M_\gamma, X_A \neq_{\zeta-\xi} 1\}. \end{aligned}$$

We will apply Lemma 6.1 to show that there there exist F_γ ($\gamma \in \Gamma$) for which none of these events occurs; that is,

$$\Pr(\bigwedge_{v \in V} \overline{T}_v \wedge \bigwedge_{A \in E} \overline{T}_A) > 0. \tag{41}$$

This with (40) gives Lemma 3.1.

Set $\Delta_2 = \Delta_2(K, \zeta/2)$ (see Lemma 6.2) and $\Delta = 2\Delta_1 + \Delta_2$. Form the graph Σ with $V(\Sigma) = V \cup E$ and

$$Y \sim_\Sigma Z \quad \text{iff} \quad \Delta_G(Y, Z) \leq \Delta.$$

Then

$$d_\Sigma(Y) < D^{\Delta+2} \quad \forall Y \in V(\Sigma). \tag{42}$$

On the other hand, we will show that for any $Y, Y_1, \ldots, Y_m \in V(\Sigma)$ with $Y \not\sim_\Sigma Y_i$ for $1 \leq i \leq m$,

$$\Pr(T_Y | \wedge_{i=1}^m \overline{T}_{Y_i}) < D^{-\omega(1)} \tag{43}$$

which with (42) gives (41) via Lemma 6.1.

What we actually show is that T_Y is unlikely under *any* conditioning that involves only edges far from Y:

Lemma 6.3 *Let Q be any specification of the restrictions of all M_γ's and F_γ's to $\{B \in E : \Delta(Y, B) > \Delta_1 + \Delta_2\}$. Then*

$$\Pr(T_Y | Q) < D^{-\omega(1)}. \tag{44}$$

This gives (43), since for $Y_i \not\sim_\Sigma Y$, T_{Y_i} depends only on which Q (as in Lemma 6.3) is produced by our random procedure. (This was our primary reason for introducing the $p''_\gamma(A)$'s.)

Proof. With Q as in the statement of the lemma, let X be X_Y conditioned on Q and $Z = X - E[X]$.

We first consider the case $Y = v$. (This doesn't require Lemma 6.2.) By (16) and our choice of Δ_1,

$$\sum_\gamma p_\gamma(A|Q) =_{2\xi} 1 \quad \forall A \ni v. \tag{45}$$

Thus, for each $A \ni v$,

$$\begin{aligned} \Pr(A \notin \cup M_\gamma | Q) &= \prod_\gamma (1 - p_\gamma(A|Q)) \\ &=_{O(1/D)} \exp[-\textstyle\sum_\gamma p_\gamma(A|Q)] \\ &=_{3\xi} e^{-1} \end{aligned}$$

(the first estimate follows easily from (45) and the fact that $p_\gamma(A|Q) \leq \lambda_\gamma(A) \leq K/D$ (see (14), (26)); 3ξ in the second estimate could be replaced by $e^{2\xi} - 1$), and it follows that, say,

$$E[X] =_{4\xi} e^{-1} d(v) \leq e^{-1} D \tag{46}$$

(using (15) for the inequality).

To show concentration, suppose $\Gamma = \{\gamma_1, \ldots\}$ and for each i, set

$$\omega_i = \begin{cases} A & \text{if } v \in A \in M_{\gamma_i} \\ \emptyset & \text{if } M_{\gamma_i} \not\ni v. \end{cases}$$

Then X depends only on $\omega = (\omega_1, \ldots)$. (Actually X only depends on those ω_i for which some $A \ni v$ is in E_{γ_i}, but other colors are automatically excluded below.) So we may write $X = X(\omega)$, and, obviously (with notation as in Section 5),

$$|X(\omega) - X(\omega')| \leq 1 \quad \forall \omega \sim \omega'. \tag{47}$$

On the other hand, setting

$$p_i = \Pr(\omega_i \neq \emptyset) = \sum_{A \ni v} p_{\gamma_i}(A|Q),$$

we have (using (45), though nothing so precise is needed here)

$$\sum p_i = \sum_{A \ni v} \sum_{\gamma} p_\gamma(A) =_{2\xi} d(v) \leq D. \tag{48}$$

Combining this with (47), (35) (or (36)) and Lemma 5.2 (with $W_0 = \emptyset$, $c_\alpha = c = 1$) followed by (34), we have, for any $D \geq \lambda = \omega(\sqrt{D \log D})$,

$$\Pr(Z > \lambda) < 2 \exp[-\frac{\lambda^2}{32 \sum p_i}] < D^{-\omega(1)}.$$

This, with (46), gives (44). (Strictly speaking, this application of (34) requires $\sum p_i = \omega(\sqrt{D \log D}\ )$; but if this fails then $d(v) < e^{-1}D$ (see (48)) and T_v is impossible.)

We now turn to the case $Y = A$. For any B with $\Delta(A, B) \leq \Delta_2$ and any γ, we have (by our choice of Δ_1)

$$p_\gamma(B|Q) =_\xi p_\gamma(B), \tag{49}$$

so that by (16),

$$\sum_\gamma p_\gamma(B|Q) =_{2\xi} 1.$$

Thus after conditioning on Q we are still in the situation described by Lemma 6.2, which allows us to conclude that

$$E[\sum_\gamma p'_\gamma(A)|Q, A \notin \cup M_\gamma] =_{\zeta/2} \sum p_\gamma(A|Q) =_{2\xi} 1,$$

and consequently, because of (40),

$$E[X|A \notin \cup M_\gamma] =_{\zeta/2+3\xi} 1. \tag{50}$$

We now write Q' for $Q \wedge \{A \notin \cup M_\gamma\}$, X' for X conditioned on $\{A \notin \cup M_\gamma\}$ (so X_A conditioned on Q') and Z' for $X' - E[X']$. By (50) we will have (44) if we can show, e.g.,

$$\Pr(|Z'| > \xi) < D^{-\omega(1)} \tag{51}$$

(Actually (51) holds with ξ replaced by any $\omega(\sqrt{(\log D)/D}\ )$; see (53).)

For the (unfortunately rather long) proof of (51) we regard X' as defined on the product space

$$\Omega = \prod_\gamma \Omega_\gamma,$$

where an element ω_γ of Ω_γ consists of M_γ chosen from $\mathcal{M}_\gamma$ according to λ_γ conditioned on Q', together with independent r.v.'s $\alpha_\gamma(B)$ for $B \in M_\gamma$, each uniform on $[0, 1]$.

Given such ω_γ's, we set

$$F_\gamma = \{B \in M_\gamma : \alpha_\gamma(B) = \max\{\alpha_{\gamma'}(B) : B \in M_{\gamma'}\}\}$$

(ignoring the zero-probability event $\{\exists B$ and $\gamma \neq \gamma'$ with $\alpha_\gamma(B) = \alpha_{\gamma'}(B)\}$). This clearly gives our original distribution on $((M_\gamma, F_\gamma) : \gamma \in \Gamma)$ conditioned on Q', so, as stated above, we may regard $\prod \Omega_\gamma$ as the space underlying X'.

We will show using Lemma 5.3 that for appropriate constant C we have

$$E[e^{\eta Z'}] \leq e^{C\eta^2/D} \quad \forall |\eta| \leq D/C. \tag{52}$$

According to (34) this gives

$$\Pr(|Z'| > \beta) < 2e^{-\beta^2 D/(4C)} \tag{53}$$

for any $0 \leq \beta \leq 2$, so in particular gives (51).

Proof of (52). Let $\Gamma = \{\gamma_1, \ldots, \gamma_{m+n}\}$ with $\{\gamma : A \in E_\gamma\} = \{\gamma_1, \ldots, \gamma_m\}$, and write $\omega = (\omega_1, \ldots, \omega_{m+n})$ for our random element of Ω. We generally replace γ_j by j in subscripts (yielding Ω_j, λ_j, M_j, α_j, G_j, E_j, F_j, G'_j, T'_j and p''_j), but write p_j for p_{γ_j} *conditioned on* Q'.

Set $X'_i = E[X'|\omega_1, \ldots, \omega_i]$ and $Z'_i = X'_i - X'_{i-1}$ (so $\sum Z'_i = Z'$). Further notation ($\mathcal{A}_i$, H_i, $\omega \sim_j \omega'$ etc.) follows Section 5.

To show (52) via Lemma 5.3 we need bounds on the individual terms

$$E[e^{\eta Z'_j}|\omega_1, \ldots, \omega_{j-1}]. \tag{54}$$

For $j \in [m]$ we will use a bound $\exp[O(\eta^2 D^{-2})]$ which will not depend on ω (see (65)). For $j > m$ valid general bounds (independent of ω) on (54) are too weak to give (52), but we can give a better bound depending on $\omega_1, \ldots, \omega_m$. (Roughly, the bound will be larger when more edges of E_j are in the matchings M_i, $i \in [m]$.)

As in Section 5, our bounds on (54) will derive from bounds on $|X'(\omega) - X'(\omega')|$ for appropriate $\omega \sim \omega'$. These bounds in turn will be based on similarity of the versions of the trees $T'_\gamma(A)$ corresponding to ω and ω', and will be consequences of Corollary 4.7.

For the rest of this section we write T for $T(A)$, T_i for $T(G_i, v_A)$, and $G'_i(\omega)$, $T'_i(\omega)$ and $p''_i(A|\omega)$ for the values of G'_i, $T'_i(A)$ and $p''_i(A)$ corresponding to $\omega \in \Omega$. We write $\psi_i(\cdot)$ for $\psi_{T_i}(A, \cdot)$ and set $\psi(\cdot) = (K/D)^{\Delta_T(A,\cdot)}$ (see Section 4.1 for ψ), so in particular we have $\psi_i(\cdot) \leq \psi(\cdot)$ whenever $\psi_i(\cdot)$ is defined.

Let $\omega_j = (M_j(\omega), \alpha_j(\omega))$. (So $\alpha_j(\omega)$ is a sequence of r.v.'s, one for each $B \in M_j(\omega)$.)

Recalling that

$$X'(\omega) - X'(\omega') = \sum_{i=1}^{m} (p''_i(A|\omega) - p''_i(A|\omega')),$$

we want to assess the impact of ω_j on the probabilities $p''_i(A|\omega)$ ($i \in [m]$).

In case $i = j$, an adequate bound is given by (14) and (26):

$$|p''_j(A|\omega) - p''_j(A|\omega')| \leq K/D \tag{55}$$

(for arbitrary ω, ω').

Suppose now that we are given $i \in [m]$, $j \in [m+n] \setminus \{i\}$, and $\omega \sim_j \omega'$. For convenience, set $M_j(\omega) = M_j$, $M_j(\omega') = M'_j$, write α_i for $\alpha_i(\omega) = \alpha_i(\omega')$, and set

$$F'_i = \{B \in M_i : \alpha_i(B) = \max\{\alpha_k(B) : k \in [m] \setminus \{j\}, B \in M_k\}\}. \tag{56}$$

(We may think of F'_i as the tentative value of F_i given by $(\omega_k : k \in [m] \setminus \{j\})$.)

In what follows we use $\sum'$ to mean we sum over $B \in E(T)$ with $\Delta_T(A, B) \leq \Delta_1$. We assert that, with π the projection from T to G,

$$|p''_i(A|\omega) - p''_i(A|\omega')| \leq \lambda_i(A) {\sum}' \{\psi(B)[1_{\{\pi(B) \in F'_i\}} + \lambda_i(B)] : \pi(B) \in M_j \Delta M'_j\}. \tag{57}$$

(Recall we conventionally take $\lambda_i(B) = 0$ if $B \notin E_i$.)

To see this, first notice that it follows from the definition of G'_i that

$$V(G'_i(\omega)) \Delta V(G'_i(\omega')) \subseteq V(F'_i \cap (M_j \Delta M'_j)) \tag{58}$$

and

$$\{C \in E(G'_i(\omega))\Delta E(G'_i(\omega')) : C \cap (V(G'_i(\omega))\Delta V(G'_i(\omega'))) = \emptyset\}$$
$$\subseteq E_i \cap (M_j \Delta M'_j). \tag{59}$$

On the other hand, $x \in V(T)$ can only be in $\Delta^* := V(T'_i(\omega))\Delta^* V(T'_i(\omega'))$ if either

(a) $\pi(x) \in V(G'_i(\omega))\Delta V(G'_i(\omega'))$ or

(b) x is the end further from A of some $B \in E(T_i)$ for which

$$\pi(B) \in E(G'_i(\omega))\Delta E(G'_i(\omega')) \text{ and } \pi(B) \cap (V(G'_i(\omega))\Delta V(G'_i(\omega'))) = \emptyset.$$

(If the last condition fails then either x satisfies (a) or is not in Δ^* because the other end of B—that closer to A—satisfies (a).)

Now if (a) holds (and $x \in \Delta^*$), then by (58) x is the end closer to A of some $B \in \pi^{-1}(F'_i \cap (M_j \Delta M'_j))$ (and we have $\psi_i(x) = \psi_i(B) \le \psi(B)$). Also, by (59), B as in (b) must lie in $\pi^{-1}(M_j \Delta M'_j)$ and, moreover, $\psi_i(x) = \psi_i(B)\lambda_i(B) \le \psi(B)\lambda_i(B)$.

Since no B in the preceding paragraph corresponds to more than one x (and since, by the definition of $T'_i(A)$, $x \in \Delta^*$ implies $\Delta_T(A, x) \le \Delta_1$), it follows that $\lambda_i(A)\sum\{\psi_i(x) : x \in \Delta^*\}$, which by Corollary 4.7 is an upper bound on the left hand side of (57), is also a lower bound on the right hand side.

□

Now, still fixing $j \in [m+n]$ (and $\omega \sim_j \omega'$), but letting i vary, set, for each B,

$$h(B) = \frac{K}{D}\psi(B)[|\{i \in [m] \setminus \{j\} : \pi(B) \in F'_i\}| + \sum_{i \in [m]\setminus\{j\}} \lambda_i(B)]. \tag{60}$$

Then summing (59) on $i \in [m] \setminus \{j\}$ (using $\lambda_i(A) \le K/D$) and combining with (55) gives

$$|X'(\omega) - X'(\omega')| \le \sum{}'\{h(B) : \pi(B) \in M_j \Delta M'_j\} + (K/D) \cdot 1_{\{j \in [m]\}}. \tag{61}$$

We now turn to bounding the terms (54). First observe that the expression

$$|\{i \in [m] \setminus \{j\} : \pi(B) \in F'_i\}| + \sum_{i \in [m]\setminus\{j\}} \lambda_i(B)$$

in (60) is at most $O(1)$: the first term is at most 1 (by the definition of F_i'), while for the second we have, using (14)-(16) and Corollary 4.2,

$$\sum \lambda_i(B) \le (K+1)^2 \sum p_{\gamma_i}(\pi(B)) = O(1). \tag{62}$$

Thus

$$h(B) = O(D^{-1}\psi(B)). \tag{63}$$

On the other hand, since each $x \in V(T_i)$ is in at most two edges of $\pi^{-1}(M_j \Delta M_j')$, we have

$$\sum{}' \{\psi(B) : \pi(B) \in M_j \Delta M_j'\} \le 2 \sum \{\psi(x) : \Delta_T(A, x) \le \Delta_1\} < O(1)$$

(since by (15) the second sum is at most $2\sum_{i=0}^{\Delta_1} K^i$).

Thus (61) implies that

$$|X'(\omega) - X'(\omega')| < O(1/D) \quad \forall \omega \sim_j \omega', \tag{64}$$

whence, according to Lemma 5.2 (with W_0 arbitrary and just using $c = O(1/D)$ and $\sum q_\alpha \le 1$),

$$E[e^{\eta Z_j'} | \omega_1, \ldots, \omega_{j-1}] \le \exp[O(\eta^2 D^{-2})] \quad \forall |\eta| c \le 1 \text{ and } \omega \in \Omega. \tag{65}$$

This is the bound we will use if $j \in [m]$.

For larger j we need to be more careful. Here we fix $\omega_1, \ldots, \omega_m$ (equivalently fix $H_m \in \mathcal{A}_m$) with associated matchings $M_1, \ldots, M_m$ and "tentative" $F_1', \ldots, F_m'$ given by (56). (More precisely, they are given by (56) with M_k's and α_k's corresponding to an arbitrary $\omega \in H_m$.)

Now for given $j > m$, let $\emptyset = M_j^0, M_j^1, \ldots$ be the possible values of M_j, and set $p_j(M_j^\alpha) = q_j^\alpha$.

Suppose $\omega, \omega' \in H_m$ with $\omega \sim_j \omega'$, $M_j(\omega) = M_j^0$ $(= \emptyset)$ and $M_j(\omega) = M_j^\alpha$. Then by (61) we have

$$|X'(\omega) - X'(\omega')| < \sum{}' \{h(B) : \pi(B) \in M_j^\alpha\} =: c_\alpha.$$

Moreover, setting $c = \max\{c_\alpha : \alpha \ne 0\}$, we have by (64) (more precisely, by the discussion leading to (64)),

$$c < O(1/D). \tag{66}$$

This, according to Lemma 5.2, gives

$$\begin{aligned} E[e^{\eta Z_j'} | \omega_1, \ldots, \omega_{j-1}] &< \exp[8\eta^2 c \sum q_j^\alpha c_\alpha] \\ &= \exp[O(\eta^2 D^{-1} \sum_\alpha q_j^\alpha \sum{}' \{h(B) : \pi(B) \in M_j^\alpha\})] \end{aligned} \tag{67}$$

whenever $|\eta|c \le 1$.

But, still fixing H_m and now letting j run over $m+1,\dots,m+n$, we have (justification follows)

$$\begin{aligned}\sum_j\sum_\alpha q_j^\alpha \sum{}'\{h(B):\pi(B)\in M_j^\alpha\}) &= \sum{}' h(B)\sum_j p_j(\pi(B)) \\ &= O(\sum{}' h(B)) \\ &= O(D^{-1}\sum{}'\psi(B)) \qquad (68)\\ &= O(1). \qquad (69)\end{aligned}$$

Because: $\sum_j p_j(\pi(B)) = O(1)$ follows from (62) and (26) (note it is not simply contained in (16), since $p_j(\pi(B))$ is not the same as $p_{\gamma_j}(\pi(B))$); (68) is from (63); and (69) follows from

$$\sum{}'\psi(B) \le D\sum\{\psi(x):\Delta_T(A,x)\le\Delta_1\}.$$

Inserting the above bound in (67) we have for any $\omega\in H_m$ (but note H_m was arbitrary) and $|\eta|c\le 1$,

$$\prod_{j=m+1}^{m+n} E[e^{\eta Z_j'}|\omega_1,\dots,\omega_{j-1}] < \exp[O(\eta^2/D)],$$

and, finally, this together with (65) (used for $j\in[m]$) and Lemma 5.3 gives (52).

□

7 Proof of Lemma 6.2

Throughout this section we fix $\gamma\in\Gamma$ and $A\in E_\gamma$.

We may assume, to avoid trivialities in what follows, that K (in Lemma 6.2) is a bit large (e.g. $K>10$ is probably enough) and ζ a bit small.

Let $\zeta_1 = \frac{1}{7}\zeta K^{-2}$, $\zeta_2 = \frac{1}{3}\zeta_1$, and set $\Delta = \Delta_1(K,\zeta_2)$ (see Lemma 4.4 for Δ_1). Fix $\xi>0$ with

$$\xi < \zeta^2/(400K^{\Delta+5}) \qquad (70)$$

and take D_1 large enough so that $D>D_1$ justifies the estimates below. (Of course one can—and we do—just think of Δ, ξ as constants with Δ large relative to K, ζ, and ξ small relative to Δ, and then let $D\to\infty$.)

Note that, as earlier, (14) and (26) imply $p_\gamma(B)\le K/D$ for all B, γ, and the same will be true for any conditional (hard-core) distribution $p_\gamma(\cdot|Q)$, since this is again based on the weights λ_γ. We will use this observation repeatedly, usually without further mention.

It's convenient to encode the process which generates the sets $F_{\gamma'} \cap E_\gamma$ as follows. Let $A = A_0, A_1, \ldots, A_m$ be some ordering of E_γ. Set

$$W_i = \{\gamma' \neq \gamma : A_i \in M_{\gamma'}\},$$

$$X_i = \begin{cases} 1_{\{W_i \neq \emptyset\}} & \text{if } A_i \notin M_\gamma \\ 1_{\gamma(A_i)=\gamma} & \text{if } A_i \in M_\gamma. \end{cases}$$

Set $W = (W_1, \ldots, W_m)$, $X = (X_1, \ldots, X_m)$ and $Q = (M_\gamma, X)$.

We are mainly interested in A_i for which $\Delta_G(A, A_i) \leq \Delta$ (as in (39)). Call such A_i, and also their indices i, *relevant*, and assume the relevant i's are $1, \ldots, k$. We also call a vertex x relevant if $\Delta_G(A, x) \leq \Delta$, so in particular any A_i containing a relevant vertex is itself relevant.

Viewed "from γ" our process first chooses M_γ and then, depending on what happens for $\gamma' \neq \gamma$, discards some edges of E_γ (possibly including some from M_γ) and takes F_γ to consist of the surviving edges of M_γ. We would like to say that these deletions—which are described by the X_i's—occur fairly independently, enough so to allow us to model Q by a process of the type described in Section 4.2.

To begin, notice that for each i,

$$\Pr(A_i \notin E'_\gamma | A_i \notin M_\gamma) = 1 - \prod_{\gamma' \neq \gamma} (1 - p_{\gamma'}(A_i)), \tag{71}$$

while, setting $Z_i = |\{\gamma' \neq \gamma : A_i \in M_{\gamma'}\}|$,

$$\Pr(A_i \in F_\gamma | A_i \in M_\gamma) = E[\frac{1}{Z_i + 1}]. \tag{72}$$

It follows, using (39), that for relevant i we have, say,

$$\Pr(A_i \notin E'_\gamma | A_i \notin M_\gamma) \ =_{2\xi} \ 1 - e^{-1} \tag{73}$$

$$\Pr(A_i \in F_\gamma | A_i \in M_\gamma) \ =_{2\xi} \ 1 - e^{-1} \tag{74}$$

(Verification is left to the reader. Note that Z_i is approximately Poisson with mean $\sum_{\gamma' \neq \gamma} p_{\gamma'}(A_i)$, and that for Z Poisson with mean ϑ, $E[(Z+1)^{-1}] = \vartheta^{-1}(1 - e^{-\vartheta})$.)

Set $q = 1 - e^{-1}$. We will prove Lemma 6.2 by "coupling" Q with an idealized version $Q^* = (M^*, X^*)$ of the type described in Section 4.2 (note (73), (74) describe something similar to step (b) of that construction):

(a) Choose $M^* \in \mathcal{M}_\gamma$ according to p_γ.

(b) For each i with $A_i \in E_\gamma$, choose X_i^* (independently of M^* and of other X_j^*'s) according to

$$\Pr(X_i^* = 1) = q, \qquad \Pr(X_i^* = 0) = 1 - q.$$

Set

$$F^* = \{A_i \in M^* : X_i^* = 1\}, \quad H^* = \{A_i \in E_\gamma \setminus M^* : X_i^* = 1\},$$
$$G^* = G_\gamma - V(F^*) - H^*, \quad E^* = E(G^*),$$

and let p^* be the hard-core distribution given by λ_γ on G^*.

To see where this is heading, recall that by Corollary 4.9,

$$E[p^*(A)|A \in E^*] = p_\gamma(A). \tag{75}$$

Thus, setting $\mathcal{Q} = \{Q\}$, $\mathcal{Q}^* = \{Q^*\}$, we will have something like Lemma 6.2 provided our coupling—a probability measure on $\mathcal{Q} \times \mathcal{Q}^*$—is mostly concentrated on pairs (Q, Q^*) for which $p'_\gamma(A|Q)$ and $p^*(A|Q^*)$ are close. (See (76) for a precise statement. In what follows we use Lemma 4.8 directly rather than Corollary 4.9; nonetheless, it is the corollary that guides our thinking.)

Note. The expressions $p'_\gamma(A|Q)$, $p^*(A|Q^*)$ are a little redundant, since, e.g., $p'_\gamma(A)$ is a function of Q; still, their use seems to add some clarity in what follows.

A coupling

Fix $M_\gamma = M^*$. (We will always take $M_\gamma = M^*$, so our interest is really in the coupling of X and X^*. Probabilities in what follows are conditioned on M_γ. This has no effect on W, but does slightly influence X in that $\Pr(X_i = 1)$ depends on whether $A_i \in M_\gamma$ (though not on any other information from M_γ).)

Given W, X, we choose X^* as follows. Set

$$q_i = q_i(W) = \Pr(X_i = 1|W_0, \ldots, W_{i-1})$$

and:

(i) if $q_i \geq q$, let the Bernoulli r.v. $\varepsilon_i = \varepsilon_i(W)$ be given by

$$\Pr(\varepsilon_i = 1) = \frac{q_i - q}{q_i}$$

(independently of all other random choices), and set $X_i^* = X_i(1 - \varepsilon_i)$;

(ii) if $q_i < q$, let ε_i be Bernoulli with

$$\Pr(\varepsilon_i = 1) = \frac{q - q_i}{1 - q_i}$$

and set $X_i^* = X_i + (1 - X_i)\varepsilon_i$.

That is: in (i) we obtain X_i^* from X_i by substituting 0 for 1 with probability ε_i in case $X_i = 1$, and similarly in (ii) with the roles of 0 and 1 reversed. This evidently gives for all W, i and $X_0^*, \ldots, X_{i-1}^*$,

$$\Pr(X_i^* = 1 | W, X_0^*, \ldots, X_{i-1}^*) = q,$$

and it follows that (M^*, X^*) has the distribution described in (a) and (b) above.

Analysis of the coupling

Let us write μ for the coupling; that is, $\mu(Q, Q^*)$ is the probability that the process described above produces the pair (Q, Q^*).

Call $(Q, Q^*) \in \mathcal{Q} \times \mathcal{Q}^*$ *good* if

$$|p'_\gamma(A|Q) - p^*(A|Q^*)| \leq \zeta_1 \lambda_\gamma(A)$$

and *bad* otherwise. Our main task will be to show

$$\Pr((Q, Q^*) \text{ is bad}) < \zeta_1 + o(1). \tag{76}$$

We first show that this gives Lemma 6.2. Here it's convenient to extend p'_γ and p^* to $E_\gamma \setminus E'_\gamma$ and $E_\gamma \setminus E^*$ (respectively) in the natural way:

$$p'_\gamma(A|Q) = \begin{cases} 1 & \text{if } A \in F_\gamma \text{ under } Q \\ 0 & \text{if } A \notin F_\gamma \cup E'_\gamma \text{ under } Q, \end{cases}$$

$$p^*(A|Q^*) = \begin{cases} 1 & \text{if } A \in F^* \text{ under } Q^* \\ 0 & \text{if } A \notin F^* \cup E^* \text{ under } Q^*. \end{cases}$$

Notice that with this convention, Lemma 4.8 gives

$$E[p^*(A|Q^*)] = p_\gamma(A). \tag{77}$$

(The left hand side of (77) is the probability that A is in a matching chosen according to p_γ conditioned on the (random) event $\{F^* \subseteq M \subseteq E \setminus H^*\}$; and by Lemma 4.8 this matching has the same distribution as one chosen according to p_γ.)

Now consider the difference

$$\begin{aligned} |E[p'_\gamma(A)] - p_\gamma(A)| &= |E[p'_\gamma(A|Q)] - E[p^*(A|Q^*)]| \\ &\leq \sum_Q \sum_{Q^*} \mu(Q, Q^*) |p'_\gamma(A|Q) - p^*(A|Q^*)|. \end{aligned} \tag{78}$$

Since $p'_\gamma(A|Q)$, $p^*(A|Q^*)$ are never more than $\lambda_\gamma(A)$, the terms $|p'_\gamma(A|Q)-p^*(A|Q^*)|$ in (78) are bounded above by $\lambda_\gamma(A)$ in all cases, and by $\zeta_1\lambda_\gamma(A)$ if (Q,Q^*) is good.

It follows, using (76) and Corollary 4.2, that the right hand side of (78) is at most

$$\begin{aligned}\zeta_1\lambda_\gamma(A)+\lambda_\gamma(A)\Pr((Q,Q^*)\text{ is bad}) &< (2\zeta_1+o(1))(K+1)^2p_\gamma(A)\\ &=: \alpha p_\gamma(A), \qquad (79)\end{aligned}$$

so that

$$E[p'_\gamma(A)] =_\beta p_\gamma(A), \qquad (80)$$

where $\beta=-\log(1-\alpha)\approx\alpha$.

Now let

$$\mathcal{Q}_0=\{A\in E'_\gamma\}=\{A\not\in M_\gamma, X_0=0\},$$

so that

$$E[p'_\gamma(A)]=\Pr(A\in F_\gamma)+\Pr(\mathcal{Q}_0)E[p'_\gamma(A)|\mathcal{Q}_0].$$

For the "known" terms on the right hand side here we have the estimates

$$\Pr(A\in F_\gamma)=_{2\xi}p_\gamma(A)(1-e^{-1})$$

(by (74)) and

$$\Pr(\mathcal{Q}_0)=\prod_{\gamma'}(1-p_{\gamma'}(A))=_{2\xi}e^{-1}.$$

Combining these with (80) and rearranging, we find that, say,

$$E[p'_\gamma(A)|\mathcal{Q}_0]=_{3\beta+4\xi}p_\gamma(A)$$

(provided β, ξ are a bit small). Thus we have Lemma 6.2.

□

Proof of (76). Again fix $M_\gamma=M^*$. Note that we may ignore the case $A\in M_\gamma$, since this occurs with probability only $p_\gamma(A)\le K/D$. So we assume henceforth that $A\not\in M_\gamma$.

Once M_γ is fixed, Q (resp. Q^*) is determined by X (resp. X^*), so we write $p'_\gamma(A|X)$ and $p^*(A|X^*)$ in what follows.

For the proof of (76) we first show that with probability close to 1 all relevant $q_i(W)$ are close to their expected values, which in turn, according to (73), (74), are close to q. This allows us to confine our attention to W's for which

$$q_i(W)\approx q \quad\text{for all relevant } i.$$

We then show that for any such W, $p^*(A|X^*)$ is likely to be close to $p'_\gamma(A|X)$.

Set $q_i^0 = E[q_i(W)] = \Pr(X_i = 1)$ and recall that the "relevant" indices are $1, \ldots, k$.

Lemma 7.1 *There is a constant C so that with probability $1 - o(1)$*

$$|q_i(W) - q_i^0| < C\sqrt{(\log D)/D} \qquad \forall i \in [k]. \tag{81}$$

Proof. Fix $i \in [k]$. We would like to use the results of Section 5 to show that $q_i(W)$ is concentrated near its mean, q_i^0. This requires a little care, since the number of relevant j's can be something like D^Δ and it can happen that changing a single W_j changes $q_i(W)$ by $\Omega(1/D)$.

But, setting $\{\gamma' \neq \gamma : A_i \in E_{\gamma'}\} = \{\gamma_1, \ldots, \gamma_s\}$ and $\omega_j = \omega_j(W) = M_{\gamma_j} \cap \{A_1, \ldots, A_{i-1}\}$, we see that $q_i(W)$ depends only on $\omega_1, \ldots, \omega_s$. Thus, setting

$$Y_j = Y_j(\omega_1, \ldots, \omega_s) = E[q_i(W)|\omega_1, \ldots, \omega_{j-1}],$$

we find that $(Y_j)_{j=0}^s$ is a martingale with $Y_0 = E[q_i(W)] = q_i^0$, $Y := Y_s = q_i(W)$ and, we assert,

$$\omega \sim_j \omega' \quad \Rightarrow \quad |Y(\omega) - Y(\omega')| \leq \lambda_{\gamma_j}(A_i).$$

To see this note that $q_i(W)$ is given for $A_i \notin M_\gamma$ (resp. $A_i \in M_\gamma$) by the right hand side of (71) (resp. (72)) with $p_{\gamma'}(A_i)$ replaced by $p_{\gamma'}(A_i|W_0, \ldots, W_{i-1})$. (Though they don't appear explicitly, the $p_{\gamma'}(A_i)$ determine the right hand side of (72). Of course γ' may now be taken to run over $\gamma_1, \ldots, \gamma_s$.) But then, using the fact that $p_{\gamma_j}(A_i|W_0, \ldots, W_{i-1})$ is always between 0 and $\lambda_{\gamma_j}(A_i)$, it's easy to see that changing ω_j changes $q_i(W)$ by at most $\lambda_{\gamma_j}(A_i)$. (In case $A_i \in M_\gamma$ this can be seen without any calculation if we recall that $q_i(W) = \Pr(\gamma(A_i) = \gamma|W_0, \ldots, W_{i-1})$.)

The lemma now follows from Lemma 5.1: Letting $c_j = \lambda_{\gamma_j}(A_i)$, we know $c_j \leq K/D$ and, by Corollary 4.2, (14), (15) and (39),

$$\sum c_j \leq (1+K)^2 \sum p_{\gamma_j}(A_i) < (1+K)^2(1+\xi),$$

so that

$$\sum c_j^2 < \frac{(1+K)^2(1+\xi)K}{D} =: \frac{T}{D}.$$

Thus by Lemma 5.1,

$$\Pr(|q_i(W) - q_i^0| \geq C\sqrt{(\log D)/D}\ ) < 2\exp[-(2C^2 \log D)/T].$$

On the other hand, $k < D^{\Delta+2}$ (by (15)), so we have Lemma 7.1 for any $C > \sqrt{(\Delta+2)T/2}$.

□

For the rest of our discussion we fix W satisfying (81). By Lemma 7.1 we will have (76) if we can show that for each such W

$$\Pr(|p^*(A|X^*) - p'_\gamma(A|X)| > \zeta_1\lambda_\gamma(A)) < \zeta_1 \tag{82}$$

Proof of (82). We will show that (82) holds even if, in addition to M_γ and W, we also fix X.

Let us first review our situation. We are given $M_\gamma \not\ni A$, $W = (W_0, \dots, W_k)$ and $X = (X_0, \dots, X_k) \in \{0,1\}^{k+1}$. We choose $X^* = (X_0^*, \dots, X_k^*)$ randomly from $\{0,1\}^{k+1}$ by changing each X_i to $1 - X_i$ with probability ε'_i given by

$$\varepsilon'_i = \begin{cases} \varepsilon_i = \varepsilon_i(W) & \text{if either } q_i = q_i(W) > q \text{ and } X_i = 1 \text{ or } q_i < q \text{ and } X_i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

these changes made independently. We then define

$$G'_\gamma = G_\gamma - V(\{A_i \in M_\gamma : X_i = 1\}) - \{A_i \in M_\gamma : X_i = 0\} - \{A_i \notin M_\gamma : X_i = 1\},$$

$$G^* = G_\gamma - V(\{A_i \in M_\gamma : X_i^* = 1\}) - \{A_i \notin M_\gamma : X_i^* = 1\},$$

and write p', p^* for the hard-core distributions given by λ_γ on G', G^*.

Notice that the ε'_i are small: by (81) together with (73) or (74) (depending on whether $A_i \in M_\gamma$), we have for relevant i (we omit the easy calculation)

$$0 \le \varepsilon'_i < 4\xi. \tag{83}$$

We would like to show, based on this fact, that G'_γ, G^* are usually close enough to allow us to conclude that $p^*(A|X^*) \approx p'_\gamma(A|X)$. Here, as earlier, it turns out to be easier to work, not directly with G'_γ, G^*, but with the associated trees of walks.

Fix an end v of A, set $T = T(G_\gamma, v)$ (weighted by λ_γ), with $\pi : T \to G_\gamma$ the usual projection, and write v, A also for the natural elements of T (those whose projections are the elements v and A of G_γ). Set $\psi(\cdot) = \psi_T(A, \cdot)$.

Let T_Δ be the subtree of T whose vertex set is $\{x \in V(T) : \Delta_T(A, x) \le \Delta\}$. (The edges of T_Δ are thus those $B \in E(T)$ with $\Delta_T(A, B) \le \Delta - 1$.) Notice that

$$\text{if } B \in E(T) \text{ contains a vertex of } T_\Delta, \text{ then } \pi(B) \text{ is relevant.} \tag{84}$$

Set

$$\begin{aligned} T' &= T(G'_\gamma, v) \cap T_\Delta, \\ T^* &= T(G^*, v) \cap T_\Delta. \end{aligned}$$

(Note that T', T^* may be empty since v may be covered by F_γ and/or F^*.)

Our choice of Δ (for which see the beginning of this section) implies, via Lemma 4.5 and (26), that

$$\begin{aligned} |p_{T'}(A|X) - p'_\gamma(A|X)| &\leq \zeta_2\lambda_\gamma(A) \\ |p_{T^*}(A|X^*) - p^*(A|X^*)| &\leq \zeta_2\lambda_\gamma(A), \end{aligned}$$

where we again retain the redundant specifications of X, X^* for clarity. (Here we used the fact that if $x =_\varepsilon y$, then $|x-y| \leq \varepsilon \max\{x,y\}$.)

Thus (82) is implied by

$$\Pr(|p_{T^*}(A|X^*) - p_{T'}(A|X)| > (\zeta_1 - 2\zeta_2)\lambda_\gamma(A)) < \zeta_1, \tag{85}$$

for which we will again use Corollary 4.7.

Set $\Delta^* = V(T')\Delta^* V(T^*)$. By Corollary 4.7 we know that

$$|p_{T^*}(A|X^*) - p_{T'}(A|X)| \leq \rho\lambda_\gamma(A), \tag{86}$$

where

$$\rho = \rho(X, X^*) = \sum\{\psi(x) : x \in \Delta^*\}.$$

We will show

$$E[\rho] \leq \zeta_1(\zeta_1 - 2\zeta_2), \tag{87}$$

whence, by Markov's inequality,

$$\Pr(\rho > \zeta_1 - 2\zeta_2) < \zeta_1,$$

which with (86) gives (85).

Proof of (87). For each $B \in E(T_\Delta) \setminus \{A\}$, let the ends of B be y_B and z_B, with y_B the end nearer A, and for $x \in V(T_\Delta)$, let B_x be the edge nearest A containing x.

We will show that for any $x \in V(T_\Delta)$ *not* satisfying

$$\pi(B_x) \in M_\gamma, \tag{88}$$

we have

$$\Pr(x \in \Delta^*) < 8\xi. \tag{89}$$

This gives (87) as follows. For x satisfying (88) we have

$$\psi(x) = \lambda_\gamma(B_x)\psi(y_{B_x}) \leq \frac{K}{D}\psi(y_{B_x}),$$

and so, since each y is y_{B_x} for at most one such x,

$$\sum\{\psi(x):\pi(B_x)\in M_\gamma\}\le \frac{K}{D}\sum\{\psi(y):y\in V(T_\Delta)\}<K^{\Delta+2}/D.$$

(For the second inequality we used (14) and (15), which give $\sum\psi(y)\le 2\sum_{i=0}^{\Delta}K^i < K^{\Delta+1}$.)

Thus (89) implies

$$\begin{aligned} E[\rho] &\le K^{\Delta+2}D^{-1}+\sum\{\Pr(x\in\Delta^*)\psi(x):\pi(B_x)\notin M_\gamma\} \\ &\le (K/D+8\xi)K^{\Delta+1} \ < \ \zeta_1(\zeta_1-2\zeta_2) \end{aligned}$$

(using (70) for the last inequality).

□

Proof of (89). Set $H_1=G'_\gamma$, $H_2=G^*$, $V(H_i)=V_i$ and $E(H_i)=E_i$. Notice that

$$V_1\Delta V_2=V(\{A_i\in M_\gamma:X_i^*\ne X_i\}), \tag{90}$$

while

$$\begin{aligned} &\{A'\in E_1\Delta E_2:A'\cap(V_1\Delta V_2)=\emptyset\} \\ &\qquad\subseteq\{A_i:X_i^*\ne X_i\}\cup\{A_i\in M_\gamma:X_i^*=X_i=0\}. \end{aligned} \tag{91}$$

(The last term turns out to be irrelevant.)

Now $x\in V(T_\Delta)$ can only belong to Δ^* if either

$$\pi(x)\in V_1\Delta V_2 \tag{92}$$

or

$$A_i:=\pi(B_x)\in E_1\Delta E_2 \text{ and } A_i\cap(V_1\Delta V_2)=\emptyset. \tag{93}$$

But according to (90), (92) is only possible if there is a $B\ni x$ with $A_i:=\pi(B)\in M_\gamma$ (note there is at most one such B since M_γ is a matching) and, moreover, $X_i^*\ne X_i$.

Similarly, because we assume x does not satisfy (88), (91) says that (93) requires $X_i^*\ne X_i$.

Thus, in view of (84), (83) implies that each of (92), (93) occurs with probability at most 4ξ (for any given x). So we have (89).

□

8 Epilogue: another approach

In this section we briefly discuss an alternate approach to Theorem 1.1. Our primary interest here—since we really prefer the procedure used above—is in the rather intriguing possibility that a process like that we now describe might be susceptible of analysis.

Context

The parallel construction of matchings used above was suggested by the coloring procedure of [27]. (A basic step of that procedure consisted of tentatively assigning each as yet uncolored A a random element, $\tau(A)$, from its list of still-legal colors, and then making this assignment permanent if no edge meeting A was also (tentatively) assigned $\tau(A)$. This idea had further striking consequences in [34], [24].)

In contrast, the randomized procedures for *ordinary* coloring used in [49], [30] are sequential. Here, for coloring edges with about T colors—i.e. covering with about T matchings—a basic step consists of choosing, for appropriate small constant ϑ, about ϑT of the matchings randomly and *independently*, and then deleting their union from the hypergraph or multigraph under consideration. (The definition of random matching is not too important for the moment; [49] uses some version of "random greedy" and [30] uses hard-core distributions.)

The requirement that we choose many matchings simultaneously—this is the "Rödl nibble"—is crucial here, roughly because it makes local behavior of one step of the procedure (e.g. the number of edges used at any given vertex) highly predictable, and this allows use of the Local Lemma to keep the process on track. On the other hand, a little reflection should convince the reader that there is no sensible generalization to list-colorings that involves choosing matchings in groups. What we would like to propose here is a procedure for choosing only one matching at a time.

Something analogous happens in [56], which analyzes the natural random greedy procedure for choosing large matchings in regular hypergraphs of fixed edge size, large degree and small "codegrees" ([51], [14], [48], [49], [27] deal with such hypergraphs). But the difficulties associated with colorings are more formidable than those for matchings. (A crucial difference is that for matchings it's enough that each vertex do well *on average* (that is, be likely to be covered), whereas for colorings we cannot afford to do badly *anywhere*.) So it would, I think, be extremely interesting if analysis of what we propose here could actually be carried out. (Though we will not do so, it is not hard to formulate an analogous proposal for an alternate proof of the main result of [27].)

Preprocessing

We are (again) trying to show that for any fixed $\delta > 0$, we have $\chi'_l(G) < (1+\delta)\chi'^*(G)$ whenever $\chi'^*(G)$ is sufficiently large. For the present approach it's convenient to make arrangements for the greedy phase beforehand. (The device employed here could also be used in [27] and in the preceding proof of Theorem 1.1.)

We assert that it is enough to show that for each positive ε and η, if $|S(A)| = (1+\varepsilon)T$ for all $A \in E$, and $T := \chi'^*(G)$ is sufficiently large, then there is a partial coloring which colors all but at most ηT of the edges at each vertex; that is, there are $Y \subseteq E$ and a proper $\sigma : E \setminus Y \to \Gamma$ such that

$$d_Y(v) < \eta T \quad \forall v \in V \tag{94}$$

and

$$\sigma(A) \in S(A) \quad \forall A \in E \setminus Y. \tag{95}$$

For suppose this is true and suppose we are given lists $S(A) \subseteq \Gamma$ of size $(1+\delta)T$ (with $\delta > 0$ fixed and T large). Fix $\varepsilon, \eta > 0$ with $\varepsilon < \delta/4$ and

$$\eta < \varepsilon^2/2. \tag{96}$$

Using the Local Lemma we can choose, for each $v \in V$, $\Gamma^0(v) \subseteq \Gamma$ so that, with

$$\begin{aligned} S^0(A) &= S(A) \cap \Gamma^0(v) \cap \Gamma^0(w), \\ S^*(A) &= S(A) \setminus (\Gamma^0(v) \cup \Gamma^0(w)) \end{aligned}$$

whenever $A \in E$ has ends v, w, we have

$$\begin{aligned} |S^0(A)| &= (1+o(1))\varepsilon^2|S(A)|, \\ |S^*(A)| &= (1+o(1))(1-\varepsilon)^2|S(A)| \quad (> (1+\varepsilon)T) \end{aligned}$$

for all $A \in E$. (Just choose the $\Gamma^0(v)$ independently from Γ according to $\Pr(\gamma \in \Gamma^0(v)) = \varepsilon$ independently for all $\gamma \in \Gamma$, and apply the Local Lemma.)

Then, by assumption, we can find Y, σ satisfying (94) and (95) with S^* in place of S; and a greedy coloring of Y using the lists S^0—this exists by (96)—completes σ to a full coloring, since σ cannot use a color from $S^0(A)$ at any edge meeting A.

□

Procedure

We need one little observation which limits the constraints we have to check in verifying that our procedure stays on track:

Proposition 8.1 *If $f \in \mathbf{R}^E$ satisfies $f \geq 0$,*

$$\sum_{v \in A \in E} f(A) \leq (1-\varepsilon)^2 \quad \forall v \in V, \quad \textit{and}$$

$$\sum_{W \supseteq A \in E} f(A) \leq 1-\varepsilon \qquad \forall W \subseteq V \ \textit{with}\ |W| \leq \varepsilon^{-1} \ \textit{and}\ G[W] \ \textit{connected},$$

then $f \in (1-\varepsilon)MP(G)$.

(Here, finally, it seems we should state Edmonds' Theorem: it says that a non-negative $f \in \mathbf{R}^E$ is in $MP(G)$ if and only if $\sum_{A \ni v} f(A) \leq 1$ for all $v \in V$ and $\sum_{A \subseteq W} f(A) \leq \lfloor |W|/2 \rfloor$ for all $W \subseteq V$.)

We omit the easy proof of Proposition 8.1. (It is about the same as the proof of Proposition 4.1 in [30].)

□

Now suppose we are given lists $S(A)$ of size $(1+\varepsilon)T$ ($T = \chi_l'^*(G)$ large), and want to produce Y, σ satisfying (94), (95).

We need a little notation. Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ and $E_i = \{A \in E : \gamma_i \in S(A)\}$. Set (a crucial quantity)

$$s^i(A) = |S(A) \cap \{\gamma_j : j \geq i\}|.$$

Let $\mathcal{W}$ be the set of W's appearing in Proposition 8.1:

$$\mathcal{W} = \{W \subseteq V : |W| \leq \varepsilon^{-1},\ G[W] \text{ connected}\}.$$

Finally, to avoid introducing another parameter, suppose $(1+\eta)^2 < 1+\varepsilon$.

Now set $E_1' = E_1$, $Y_0 = \emptyset$, and for $i = 1, \ldots$ do:

I. Set (for each A)

$$f^i(A) = \frac{1}{s^i(A)} 1_{\{A \in E_i'\}}.$$

If there exists $v \in V$ with

$$\sum_{A \ni v} f^i(A) > (1+\eta)^{-2}$$

or

$$|E(v) \cap (\bigcup_{j<i} Y_j)| \geq \eta T$$

(where $E(v) = \{A \in E : A \ni v\}$), then say the event T_v has occurred and terminate the procedure.

Similarly, if there exists $W \in \mathcal{W}$ with

$$\sum_{A \subseteq W} f^i(A) > (1+\eta)^{-1} \lfloor |W|/2 \rfloor,$$

then say T_W has occurred and terminate. (So we allow several of the events T_v, T_W to occur simultaneously.)

II. Otherwise, by Proposition 8.1, we have $f^i \in (1+\eta)^{-1} MP(E_i')$ (with $MP(E_i')$ and, in the next sentence, $\mathcal{M}(E_i')$ defined in the obvious ways). Let p^i be the hard-core distribution on $\mathcal{M}(E_i')$ with marginals f^i. Choose M_i according to p^i and set

$$Y_i = \{A \in E \setminus \bigcup_{j<i} (M_j \cup Y_j) \setminus M_i : s^i(A) < \eta T\},$$

$$E'_{i+1} = E_{i+1} \setminus \bigcup_{j \le i} (M_j \cup Y_j).$$

Rationale

If none of the events T_v, T_W occurs, then clearly we do get the desired Y ($= \cup Y_i$) and σ. We assert, moreover, that each T_v, T_W occurs with probability $\exp[-\Omega(T)]$. To see this (in outline), let

$$E^i = E \setminus \bigcup_{j<i} (M_j \cup Y_j),$$

and for $A \in E$,

$$q^i(A) = \frac{1}{s^i(A)} 1_{\{A \in E^i\}}.$$

(So q^i extends f^i to all edges which survive to stage i. We should really modify these definitions and the following discussion to cover the possibility that we terminate at or before stage i; but just about any sensible convention will take care of this, and we won't worry about it here.)

Now for $A \in E_i'$ we have

$$\begin{aligned} E[q^{i+1}(A)] \ &\le \ (1 - f^i(A)) \frac{1}{s^{i+1}(A)} \ = \ (1 - \frac{1}{s^i(A)}) \frac{1}{s^i(A) - 1} \\ &= \frac{1}{s^i(A)} \ = \ q^i(A) \end{aligned}$$

(the inequality derives from the possibility that A is put in Y_i), and for $A \in E^i \setminus E'_i$,

$$E[q^{i+1}(A)] \;=\; \frac{1}{s^{i+1}(A)} \;=\; \frac{1}{s^i(A)} \;=\; q^i(A).$$

Thus, for any given $v \in V$, $\{X_i := \sum_{A \ni v} q^i(A)\}_{i \geq 1}$ is a supermartingale ($E[X_{i+1}|X_i] \leq X_i$); so in particular

$$E[X_i] \leq X_1 = \frac{d_G(v)}{(1+\varepsilon)T} \leq (1+\varepsilon)^{-1},$$

and one can show, using the results of Section 5 (and recalling that $(1+\eta)^2 < 1+\varepsilon$)

$$\Pr(\exists i \;\; X_i > (1+\eta)^{-2}) < \exp[-\Omega(T)]. \tag{97}$$

In addition we have, for any A, $\Pr(A \in Y) < \eta/(1+\varepsilon)$, whence

$$E[d_Y(v)] < \frac{\eta T}{1+\varepsilon},$$

and we can again use martingales to show

$$\Pr(d_Y(v) > \eta T) < \exp[-\Omega(T)].$$

This together with (97) gives $\Pr(T_v) < \exp[-\Omega(T)]$, and $\Pr(T_W) < \exp[-\Omega(T)]$ is shown similarly (more easily, since we don't have to worry about Y).

□

So we would like to conjecture that one can use the Local Lemma—or at least something in the same spirit—applied to the events T_v, T_W, to show

$$\Pr(\bigwedge_{v \in V} \overline{T}_v \wedge \bigwedge_{W \in \mathcal{W}} \overline{T}_W) > 0. \tag{98}$$

(And of course we could strengthen this to say that the probability in (98) is at least $\exp[-n^{O(1)} e^{-\Omega(T)}]$, again expressing the idea that we have something like independence.)

The problem, of course, is independence. We would like to say that for sufficiently large (relative to ε, η) constant Δ, the probability of T_v (or T_W) cannot be much increased by any conditioning on events $\overline{T}_{v'}$, $\overline{T}_{W'}$ with v', W' at distance greater than Δ from v (or W). (Actually we only need $\Delta = o(T/(\log T))$, though I doubt that this helps. Notice that the restriction $W \in \mathcal{W}$ imposes bounds

$T^{O(\Delta)}$ on degrees in the graph Σ used in the Local Lemma; this is the reason for Proposition 8.1.)

We have at least arranged that if no T_v, T_W occurs through stage i, then, using Corollary 4.3 and Lemma 4.4, we do have good independence behavior in p^i. Effects *between* i's are presumably more subtle; nonetheless, I believe they too are small, and can eventually be understood well enough to allow completion of a proof of Theorem 1.1 along the lines suggested here.

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