$\boxed{L11}$ Long remk on RW '94 : $G = G_{n,3}$ Ham. a.s.

$Y := |\{ \text{HC's in } G \}|$

FACT : $\mathrm{Var}(Y) = \Theta(\mathbb{E}^2 Y)$

$\Longrightarrow$ $\mathbb{P}(\underbrace{\text{Ham}}_{Y \neq 0}) = \Omega(1)$

▶ $\forall$ r.v. X $\mathbb{P}(X \neq 0) \geq \mathbb{E}^2 X / \mathbb{E} X^2$ (more later)

▶ gen'l : $\forall$ Y, Z r.v.'s w $Y \in \mathbb{R}$

$$\circledast \quad \mathrm{Var}(Y) = \mathrm{Var}\{ \mathbb{E}[Y|Z] \} + \mathbb{E}\{ \mathrm{Var}[Y|Z] \}$$

fns of Z

sim : (law of t. E :)

$$\circledast \quad \mathbb{E} Y = \mathbb{E}\{ \mathbb{E}[Y|Z] \} := \sum_z \mathbb{P}(Z = z)\, \mathbb{E}[Y | Z = z]$$

e.g. 1 $Y = f(Z) \to \mathbb{E}[Y|Z] = f(Z)$

$\mathrm{Var}[Y|Z] = 0$

$\to \mathrm{Var}(Y) = \mathrm{Var}(f(Z)) + 0$

e.g. 2 Y, Z ind. $\longrightarrow$ $\mathbb{E}[Y|Z] = \mathbb{E} Y$

$\mathrm{Var}[Y|Z] = \mathrm{Var}(Y)$ etc.

e.g. 3 X, Z indept, $Y = X + Z$ $\longrightarrow$

$\mathbb{E}[Y|Z] = Z + \mathbb{E} X \longrightarrow \mathrm{Var}\{ \mathbb{E}[Y|Z] \} = \mathrm{Var}\, Z$

$\mathrm{Var}(Y|Z) = \mathrm{Var}(X) = \mathbb{E}\{ \mathrm{Var}(Y|Z) \}$

Pf $\quad \mathrm{Var}\{\mathbb{E}[Y|Z]\} = \mathbb{E}\{\mathbb{E}^2[Y|Z]\} - \mathbb{E}^2\{\mathbb{E}[Y|Z]\}$

$$\overset{*}{=} \mathbb{E}\{\mathbb{E}^2[Y|Z]\} - \mathbb{E}^2 Y$$

$$\mathbb{E}\{\mathrm{Var}[Y|Z]\} = \mathbb{E}\{\mathbb{E}[Y^2|Z] - \mathbb{E}^2[Y|Z]\}$$

$$\overset{*}{=} \mathbb{E}Y^2 - \mathbb{E}\{\mathbb{E}^2[Y|Z]\}$$

Back to HC's: $*$ sugg:

 $\longleftarrow$ partition space acc. Z ($= ?$) & hope for:

most of $\mathrm{Var}(Y)$ is in $\mathrm{Var}\{\mathbb{E}[Y|Z]\}$

$\rightsquigarrow$ $\mathrm{Var}[Y|Z]$ vs. small $\rightsquigarrow$ Cheb.

& this works (!); v. roughly:

$Z = (X_1, \dots, X_\ell)$ $\quad$ (X_i as before, eventually $\ell \to \infty$)

ℓ large $\to$ $\mathrm{Var}\{\mathbb{E}[Y|Z]\} \approx \mathrm{Var}(Y)$

▷ N.B. this alone isn't enough; also need decent behavior of (most) $\mathbb{E}[Y|Z]$'s

Back to

[ref: Lyons-Peres §5.3
backgd also Grimmett,
AS §11.11]

$\circledast \quad \mathbb{P}(X\neq 0) \geq \mathbb{E}^2X / \mathbb{E}X^2$

$\mathbb{E}X^2 = \sigma_X^2 + \mu_X^2$

$\circledast$ Pf $\quad \mathbb{E}^2X = \mathbb{E}^2 X \mathbb{1}_{\{X\neq 0\}} \underset{\text{C-S}}{\leq} \mathbb{E}X^2 \, \underbrace{\mathbb{E}\mathbb{1}^2_{\{X=0\}}}_{\mathbb{P}(X\neq 0)}$ $\square$

$\circledast \quad \longleftrightarrow \quad \mathbb{P}(X=0) \leq \sigma^2/(\mu^2+\sigma^2)$

$\longrightarrow$ improves Cheb: $\mathbb{P}(X=0) \leq \sigma^2/\mu^2$

earlier: $\sigma \ll \mu \longrightarrow$ little diff.

next appl: can only hope for $\mathbb{P}(X\neq 0) > c \quad (>0)$

$\longrightarrow \quad \circledast$: need $\mathbb{E}X^2 = O(\mu^2)$

vs. Cheb: nothing if (e.g.) $\mathbb{E}X^2 \geq 2\mu^2$

$\boxed{\text{appl}}$: perc. on trees (not vs. "probc. methods...")

backgd: G: loc. fin., conn. ($\Rightarrow$ ctble)

canon. ex: $\mathbb{Z}^d$

(bond) perc. on G:

$\mathbb{P}(e \text{ open}) = p \quad \forall e \in E = E(G)$ ind'ly
present

(e.g. $G_{n,p} \longleftrightarrow$ perc. on K_n)

theoretical (v. briefly) $(\Omega, \mathcal{F}, \mathbb{P})$ — or...

- $\Omega = \{0,1\}^E$ $(E = E(G))$ (or c. more gen'l)

  $\omega = 1_{\{\text{opens}\}}$ "edge config"

- $\mathcal{F}$: σ-field gen by events $\{\omega_e = 1\}$

  - cylinder: $C(\Lambda, Z) = \{\omega : \omega_\Lambda \in Z\}$ $\bar{\omega}$
    - $\Lambda \subseteq E$ finite
    - $Z \subseteq \{0,1\}^\Lambda$ } (default)
    - $\omega_\Lambda = (\omega_e : e \in \Lambda)$

[C cyl. $\Leftrightarrow$ $\exists$ fin $\Lambda \subseteq E$ s.t. $\{\omega \in C\}$ det by ω_Λ]

E.g. in $\mathbb{Z}_d$: $\{\underline{0} \longleftrightarrow \{x : \|x\|_\infty = N\}\}$ cyl.

but $\{\underline{0} \longleftrightarrow (1,0,\dots,0)\}$ is not

- $\exists!$ (prob) m. $\mu = \mu_p$ on $(\Omega, \mathcal{F})$ s.t.

$$\mu(C(\Lambda, Z)) = \sum_{\eta \in Z} \prod_{e \in \Lambda} p^{\eta_e} (1-p)^{1-\eta_e} \qquad \forall \Lambda, Z$$

- Ev. $A \in \mathcal{F}$ can be approx'd by cylinders, i.e.

  $\forall \varepsilon > 0$ $\exists$ cyl C s.t. $\mu(A \Delta B) < \varepsilon$

e.g. $A = \{\underline{0} \longleftrightarrow (1,0,\dots,0)\}$ approx'd by

$B_N := \{\underline{0} \longleftrightarrow (1,0,\dots,0)$ in $\{x : \|x\|_\infty \leq N\}\}$

$B_N \uparrow A$ (i.e. $B_1 \subseteq B_2 \subseteq \cdots$ & $\cup B_N = A$)

$\Longrightarrow$ $\mu(A \setminus B_N) \longrightarrow 0$

(back to perc)

open cluster : cpt of {open edges}

- a central Q: when is there an i.o.c. ?
  i = inf.

- FACT ($\subseteq$ Kolmogorov's 0-1 law)

$$\forall p \quad \mu_p(\exists \text{ i.o.c.}) \in \{0,1\}$$

({∃ i.o.c.} "tail event")

- critical prob:

$$p_c = p_c(G) = \sup\{p : \mu_p(\exists \text{ i.o.c.}) = 0\}$$

nondec. in p

$$\left(= \inf\{p : \mu_p(\exists \text{ i.o.c.}) = 1\}\right)$$