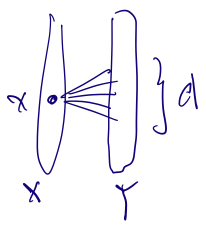
L-16

RECAP $|S_v| = t = cd/\log d$ (c TBA)



$\sigma : T \longrightarrow \prod S_y$ u.a.r.

$A_x = \{ S_x \subseteq \sigma(N_x) \} \longrightarrow \mathbb{P}(A_x) \le$?

WMA $d_x = d$, $S_y = S_x \ \forall y \sim x$

- note just coupon collector

  $\mathbb{E}$ # to see all $\sim t \ln t$ (exact: $\sum_{i=1}^{t} t/i$ — why?)

  ours: $d \approx c^{-1} t \ln t \overset{\text{sugg}}{\rightsquigarrow} \mathbb{P}(A_x)$ small if $c > 1$

RECAP

$\gamma \in S_x \longrightarrow \mathbb{P}(\gamma \notin \sigma(N_x)) = (1 - 1/t)^d \longrightarrow$

$\mathbb{P}(A_x) \overset{?}{\le} \prod_{\gamma \in S_x} \mathbb{P}(\gamma \in \sigma(N_x))$

$= \left[1 - (1 - 1/t)^d \right]^t =: p$

$\xrightarrow{\text{take } c > 1} e(d^2 - d + 1)p$ tiny (t large)

[arith: $(1 - 1/t)^d \approx d^{-1/c}$ ($1 - \alpha > e^{-(\alpha + \alpha^2)}$ for small α)

$\longrightarrow p < \exp[-d^{\varepsilon}]$]

▷ what about ? ? more gen'l:

$Z = \{\sigma_1, \ldots, \sigma_d\}$, σ_i's indept $\in S$, $B \subseteq S$

$$\Longrightarrow \quad \mathbb{P}(Z \supseteq B) \leq \prod_{j \in B} \mathbb{P}(j \in Z)$$

(events $\{j \in Z\}$ are negatively correlated)

• ETS $\forall A \subseteq S \ \ni\ i \in S \setminus A$

$$\mathbb{P}(i \in Z \mid A \subseteq Z) \leq \mathbb{P}(i \in Z) \qquad (*)$$

[obvious? More than nec. but excuse to mention:]

▷ events $\mathcal{Q}, \mathcal{B}$ are

pos. corr. ($\mathcal{Q} \uparrow \mathcal{B}$) if $\mathbb{P}(\mathcal{Q}\mathcal{B}) \geq \mathbb{P}(\mathcal{Q})\mathbb{P}(\mathcal{B})$

$$\Longleftrightarrow \quad \mathbb{P}(\mathcal{Q} \mid \mathcal{B}) \geq \mathbb{P}(\mathcal{Q}) \quad (\text{if } \mathbb{P}(\mathcal{B}) > 0)$$

neg. corr. ($\mathcal{Q} \downarrow \mathcal{B}$) sim.

e.g. Harris: w.r.t. any prod. meas on $\{0,1\}^n$

$$\mathcal{Q} \uparrow \mathcal{B} \quad \forall \text{ incr. } \mathcal{Q}, \mathcal{B}$$

e.g. (check) $\mathcal{Q} \downarrow \mathcal{B}$ $(\Leftrightarrow \mathcal{B} \downarrow \mathcal{Q})$ $\Leftrightarrow$ $\mathcal{Q} \uparrow \overline{\mathcal{B}}$

now let $\mathcal{Q} = \{A \subseteq \sigma(N_x)\}$, $\mathcal{B} = \{i \in \sigma(N_x)\}$

want $\mathcal{Q} \downarrow \mathcal{B}$

$$(*) \Longleftrightarrow \mathbb{P}(A \subseteq Z \mid i \in Z) \leq \mathbb{P}(A \subseteq Z)$$

$$\Longleftrightarrow \mathbb{P}(A \subseteq Z \mid i \notin Z) \geq \mathbb{P}(A \subseteq Z)$$

and this is obv. ⊠

▶ Thm 1 (Alon '00): $\forall G$ $\chi_\ell(G) \geq (1/2 - o(1)) \log_2 \delta_G$

▶ Thm 2 (Saxton-Thomason '15; AS §1.6): ... $(1-o(1)) \log_2 \delta_G$

- we'll show for d-reg
- as. tight ($\leftrightarrow K_{dd}$)
- $\max\{\chi_\ell(G): \text{bip}, \Delta \leq d\} \in (\Omega(\log d), \Theta(d/\log d))$

▶ truth? (guess ≈ l.b. ??)

Pf of Alon
- warmup for S-T
- not his first try ...

given G, $\delta_G \geq d$ ($|V| = n$, say)

want L_v's of size $t \sim \frac{1}{2} \log_2 d$ s.t. $\nexists$ L-col'g

Let $t \approx \frac{1}{2} \log_2 d$ even (unimp) TBA, $s = t^2$

$\Gamma = [s]$

$\boxed{L_v \text{ unif } \in \binom{\Gamma}{t}}$ $(v \in V)$ indept

show whp $\nexists$ L-col'g (for suitable t)

Step 1 $B \sim \text{Bin}(V, 1/\sqrt{d})$

& choose L_v's for $v \in B$

– waiting on the rest

▶ gen'l princ: conserve randomness

v good if ① $v \notin B$

② $\forall T \in \binom{\Gamma}{s/2}$ $\boxed{\exists\, w \in B \cap N_v,\ L_w \subseteq B}$ ⊛

$A = \{$gd v's$\}$

Cl. 1 whp $\boxed{\begin{array}{l}\text{(a) } |B| < (1+o(1))\, n/\sqrt{d} \\ \text{(b) } |A| > n - o(n)\end{array}}$ ⊛⊛ (more than nec.)

Preview: Step 2: choose L_v's for $v \in A$ $\rightarrow$
need L-col'g of A compat. w some L-col'g σ of B
& ② $\rightsquigarrow$ compat. hard $\underline{\forall \sigma}$ (why?)

Pf (a) Ex. (Chernoff)

(b) T survives at v if ⊛ fails

$v \notin A \Leftrightarrow v \in B$ or some T sur. at v $\rightarrow$

ETS $\mathbb{P}(\exists T \text{ sur. at } v) = o(1)$

why? $(\mathbb{E}|V \setminus A| = o(1)$ + Markov$)$

$w \sim v$: $\mathbb{P}(w \text{ kills } T \text{ at } v) \overset{⊛}{\approx} d^{-1/2}\, 2^{-t}$

⊛ "birthday paradox" & reason for t^2 (here & below — is common)
Ex: 2^{-t} overest. by $\sim \sqrt{e}$

$\rightarrow \mathbb{P}(T \text{ sur. at } v) \approx (1 - d^{-1/2}\, 2^{-t})^{d_v} \quad d_v \geq d$

$< \exp[-\sqrt{d}\, 2^{-t}]$

$\rightarrow \mathbb{P}(\exists T \text{ sur at } v) < 2^{t^2} \exp[-\sqrt{d}\, 2^{-t}]$

$= o(1)$ if $t = \lfloor \tfrac{1}{2}\log_2 d - 2\log_2 \log_2 d \rfloor$

$\boxed{\text{DO THIS}}$

$\rightarrow$ WMA

<u>Step 2</u> (lists for A)

<u>Cl. 2</u> (Given) whp. $\nexists$ L-coloring of $G[A,B] =: H$

<u>Pf</u> given L-map $\sigma : B \to \Gamma$, let

$$Q_\sigma = \{\exists \text{ L-col'g } \tau \text{ of } H \text{ w } \tau|_B = \sigma\}$$

$$\rightarrow \{\exists \text{ L-col. of } H\} = \bigcup Q_\sigma$$

<u>SHOW:</u> $\sum \mathbb{P}(Q_\sigma) = o(1)$ $\quad(\rightarrow)$