$\to$ WMA (**)

Step 2 (lists for A)

Cl. 2 (Given (**)) whp. $\nexists$ L-coloring of $G[A,B] =: H$

Pf given L-map $\sigma : B \to \Gamma$, let

$$Q_\sigma = \{\nexists \text{ L-col'g } \tau \text{ of } H \text{ w } \tau|_B = \sigma\}$$

$$\to \{\nexists \text{ L-col. of } H\} = \cup Q_\sigma$$

SHOW: $\sum \mathbb{P}(Q_\sigma) = o(1)$ $(\to \blacksquare)$

L17

① $|\{\sigma's\}| = t^{|B|} < t^{(1+o(1))n/\sqrt{d}}$

② Given σ let F_v $(= F_\sigma(v)) = \sigma(N_v \cap B)$
forb.

$\to Q_\sigma = \{L_v \not\subseteq F_v \;\forall v \in A\}$

$\boxed{\text{MP: } v \in A \Rightarrow |F_v| \geq s/2}$

$\to \mathbb{P}(L_v \subseteq F_v) \approx 2^{-t}$ (b'day again)

$\to \mathbb{P}(Q_\sigma) \approx \exp[-2^{-t}(1-o(1))n]$

$\to \sum \mathbb{P}(Q_\sigma) < \exp[(1+o(1))nd^{-1/2}\log t - (1-o(1))n2^{-t}]$

$= o(1)$ (recall $t = \lfloor \frac{1}{2}\log_2 d - 2\log_2\log d \rfloor$)

more than needed here

$\blacksquare$

Now Saxton-Thomason

Ref: AS §1.6

- just for d-reg (for simp; gen'l similar)

Prelim & first look at containers: (G gph)

$\mathcal{I}(G) := \{$ indept sets of $G\}$, $i(G) = |\mathcal{I}(G)|$

Thm (sim. to Sapozhenko '01) $\forall \varepsilon > 0\ \exists B$

$\forall$ G simple, d-reg (on V, $|V| = n$) $\exists\ \mathcal{C} \subseteq 2^V$ s.t.

① $I \in \mathcal{I}(G) \Rightarrow \exists\, C \in \mathcal{C},\ I \subseteq C$ (→ "containers")

② $C \in \mathcal{C} \Rightarrow \boxed{|G[C]| < \varepsilon n d}$ $(*)$ (vs $|G| = nd/2$)

③ $|\mathcal{C}| < \exp[B\, n/d \log d]$

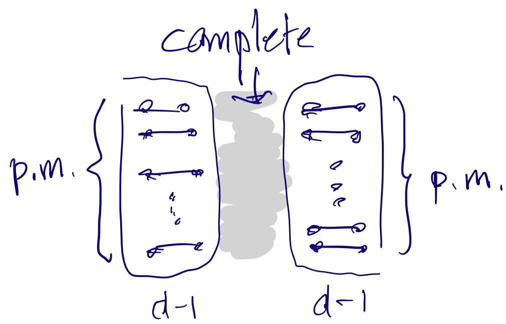
Obs $(*) \Rightarrow |C| < (1+\varepsilon)n/2$

Pf $|C|d \leq |G| + |G[C]| \leq \frac{nd}{2} + \varepsilon n d$ ☐

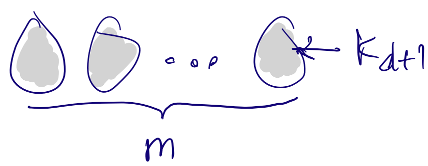
Perspective: G can't do C's indept

→ $|\{$max'l ind. sets$\}| = 2 \cdot 2^{n/4}$

(etc.)



②



mis $= (d+1)^m \approx \exp[\frac{n}{d} \log d]$

– agrees to thm but Ex:

smallest $|\mathcal{C}| \approx \exp[\frac{n}{d} \log(1/\varepsilon)]$

Pf (one ver; $\approx$ Sap) WMA d large (or absorb in $\mathcal{B}$ ...)

Say $V = [n]$, $\boxed{I \text{ given}}$, $b := \varepsilon d$

Maintain (evolving) W, T, Z disjt $\subseteq V$ (us. not all of V)

initial: $W = V$, $T = Z = \emptyset$

final: $W = \emptyset$, $T \subseteq I \subseteq T \cup Z =: C$

(W can only shrink; T can only grow; Z can do both)

do for $i = 1, \dots, n$: step i:

- $i \notin W \longrightarrow$ skip; else:
- $d_W(i) < b \longrightarrow$ move to Z
- $d_W(i) \geq b \longrightarrow$ query: $i \in I$? $\longrightarrow$

  NO: delete

  YES: move to T & delete $N(i)$ (from $W \cup Z$)

end: $C = C(I) = T \cup Z$ ($\mathcal{C} = \{C(I)\text{'s}\}$)

This does it: ① ✓

② $|G[C]| < |Z| b$ ($= \varepsilon n d$) why?

[verts of T iso in C

$i \in Z \Rightarrow |\underbrace{\{j \in Z : i < j \sim i\}}_{\text{in } W \text{ at step } i}| < b$]

③ (most int'g) T det. C (w/o ref. to I) why?

[Given T (& not I), rerun alg,

replacing "$i \in I$?" by "$i \in T$?"]

& $|T| < n/b \longrightarrow |\mathcal{C}| < \binom{n}{< n/b} < \exp\left[\frac{en}{b} \log b\right]$

Remk: need $b \leq \varepsilon d$ for ② & then want b large for "T small"