Pf of Azuma

Obs: W $\mathbb{R}$-val'd r.v., $\mathbb{E}W=0$, $|W|\le a$ $\Rightarrow$

$$\mathbb{E}e^W \le \frac{1}{2}(e^{-a}+e^a) \le e^{a^2/2}$$ saw w Chernoff

EX: h cvx $\Rightarrow$ $\mathbb{E}h(W) \le \frac{1}{2}(h(-a)+h(a))$

Azuma (as usual):

$$\mathbb{P}(Z\ge t) = \mathbb{P}(e^{sZ}\ge e^{st}) \le e^{-st}\,\mathbb{E}e^{sZ}$$

MAIN: $\mathbb{E}e^{sZ} \le \exp[s^2 \sum c_k^2/2]$ ($\leadsto$ QED)

Pf Ind. on k to show

$$\mathbb{E}\, e^{s(Z_1+\cdots+Z_k)} \le \exp[s^2(c_1^2+\cdots+c_k^2)/2]$$

$k=1$: Obs. ($W=sZ_1$; note $\mathbb{E}W=0$)

$\boxed{\text{L23}}$

$k>1$:

$$\text{l.h.s.} = \mathbb{E}\prod_{i=1}^k e^{sZ_i} \qquad (\text{vs. } \ne \prod_{i=1}^k \mathbb{E}e^{sZ_i})$$

$$= \mathbb{E}\{\mathbb{E}[\prod^k e^{sZ_i} \mid Z_1,\dots,Z_{k-1}]\}$$

$$= \mathbb{E}\{e^{s(Z_1+\cdots+Z_{k-1})}\,\mathbb{E}[e^{sZ_k}\mid Z_1,\dots,Z_{k-1}]\}$$ why?

$$\le e^{s^2c_k^2/2}\,\mathbb{E}\,e^{s(Z_1+\cdots+Z_{k-1})}$$ why?

$$\overset{\text{ind.}}{\le} e^{s^2(c_1^2+\cdots+c_k^2)/2}$$

∎

Ex. 1 (modulo a claim) $G = G_{n,1/2}$, $X = \chi(G)$: how concentrated?

Remk: recall $\alpha(G) \begin{cases} \geq k & \text{if } \mu(n,k) := \binom{n}{k} 2^{-\binom{k}{2}} \to \infty \\ < k & \text{if } \mu(n,k) \to 0 \end{cases}$

$\xrightarrow{\text{e.g.}}$ for "most" n $\exists\, k = k(n)$ s.t. $\alpha(G) = k$ whp (!)

(& $\underset{=}{\forall} n$ $\alpha(G)$ conc. on ≤ 2 val's)

For $k \in [n-1]$ let $Y_k = G \cap \{kl : l > k\}$ $\quad$ ($V(G) = [n]$)

$$\left(\& \quad X_k = \mathbb{E}[X \mid \underbrace{Y_1, \dots, Y_k}_{\text{all info on edges mtg } [k]}] \right)$$

[A-S: "vertex exposure mart." := mart. w.r.t. these Y_k's]

Claim $|Z_k| \leq 1$

$\xrightarrow{\text{Azuma}}$

Shamir-Spencer '87: $\mathbb{P}(|X - \mathbb{E}X| > t) < 2\exp\left[-\frac{t^2}{2(n-1)}\right]$

$\chi(G_{n,1/2})$ "conc. in $O(\sqrt{n})$"

[precisely (see below): conc. in $\omega\sqrt{n}$ for any $\omega = \omega(n) \to \infty$]

Strange: says nothing ab. $\mathbb{E}X$ ... but soon led to

Bollobás '88: $\chi(G_{n,1/2}) \sim \frac{n}{2\log_2 n}$ whp

"Claim" instance of:

$Y_1, \dots, Y_m$ indept (not nec. identical, e.g. "vertex exp.")

$Y = (Y_1, \dots, Y_m)$, y: poss. val. of Y

y_k: — — Y_k

$y \sim_k y'$ if $y_j = y'_j \Leftrightarrow j \neq k$

$y \sim y'$ if $y \sim_k y'$ for some k ("Hamming graph")

(X_k, X, Z_k as usual)

Prop If $y \sim_k y' \Rightarrow |X(y) - X(y')| \leq c$

As T7.4.1

then $|Z_k| \leq c$

Cor $[y \sim_k y' \Rightarrow |X(y) - X(y')| \leq c_k \ \forall k] \Longrightarrow$

hypotheses (& conclusion) of Azuma

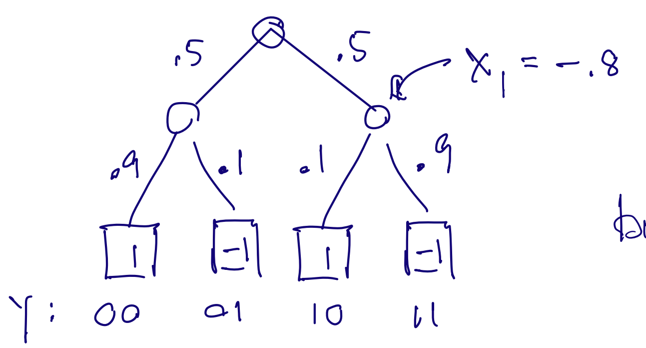
Frequent sp. case: X is Lip(c) (has Lipschitz const c)

if $y \sim y' \Rightarrow |X(y) - X(y')| \leq c$

e.g. $X = \chi(G_{n,1/2})$ is Lip(1) (wrt vert. exp.) $\longrightarrow$ Claim

Silly q: why not edge exposure? (Ans: too long)

Ex. (perspective)



$y \sim_1 y' \Rightarrow X(y) = X(y')$

$(\Leftrightarrow X = X(Y_2))$

but (e.g.) $Z_1(1) = -.8$

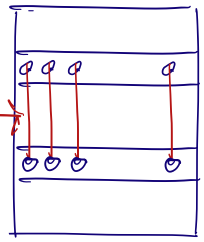
Pf of Prop (easy, annoying to write)

MP $\quad \forall\, y_1,\dots,y_k, w$

$$(*) \quad \Big|\underbrace{\mathbb{E}[X \mid y_1,\dots,y_k]}_{\text{avg on } B} - \underbrace{\mathbb{E}[X \mid y_1,\dots,y_{k-1}, w]}_{\text{avg on } B'}\Big| \le c \qquad \leftarrow \text{see pic.}$$

What's going on:

differ only in k $\rightarrow$



$\Downarrow$

X-val's differ by $\le c$

$B \in \mathcal{a}_k$ $\;(\leftrightarrow y_1,\dots,y_k)$

$B' \in \mathcal{a}_k$ $\;(\leftrightarrow y_1,\dots,y_{k-1},w)$

$A \in \mathcal{a}_{k-1}$ $\;(\leftrightarrow y_1,\dots,y_{k-1})$

▶ and Y_i's indept $\Longrightarrow$ distrib's on B, B' agree

$\hookrightarrow$ of $Y_{k+1},\dots,Y_m$

formal: $u = (y_{k+1},\dots,y_m) \longrightarrow$

$$\text{l.h.s. of } (*) = \Big|\sum_u \mathbb{P}(u)\big[X(y_1,\dots,y_k,u) - X(y_1,\dots,y_{k-1},w,u)\big]\Big|$$

$$\le \sum_u \mathbb{P}(u) \underbrace{\big|X(y_1,\dots,y_k,u) - X(y_1,\dots,y_{k-1},w,u)\big|}_{\le c} \qquad \square$$

End: $$|Z_k(y)| = |\underbrace{X_k(y)}_{\text{avg on } B} - \underbrace{X_{k-1}(y)}_{\text{avg on } A}|$$

$$\le \sum_w \mathbb{P}(Y_k = w)\, \big|\mathbb{E}[X \mid (y_1,\dots,y_k)] - \mathbb{E}[X \mid (y_1,\dots,y_{k-1})]\big| \qquad \square$$

Rmk Often know more ab. dist of $Z_k \mid Y_1,\dots,Y_k$ $\rightsquigarrow$ better "abs" $\rightsquigarrow$ improved conc. (more below)

Remk (minor) Pf of Prop. ac. shows: $W := Z_k \mid Y_1,\dots,Y_{k-1}$ $\models$

"range" $\rightarrow$ $R(W) = \max(W) - \min(W) \le c$

(Toy) example: G on $V=[n]$

$$X = \chi(G[V_p]) = X(\Upsilon_1,\dots,\Upsilon_n) \qquad (\Upsilon_k = \mathbb{1}_{\{k\in V_p\}})$$

— Lip(1) but now W (as above) takes 2 vals, one $\bar{w}$ prob. p.

$\to$ $\mathbb{E}e^{\delta W}$ max when $W = \begin{cases} 1-p & \bar{w} \text{ prob } p \\ -p & \bar{w} \text{ prob } 1-p \end{cases}$

— gets fussy (cf. AS L A.1.8) but roughly:

$\mathrm{Var}(W) \le p(1-p) \approx p$ (thinking p small) sugg

$\mathbb{P}(Z > t) \lesssim \exp[-t^2/np]$ (which turns out to be right)

<u>Conc. of $\chi(G_{n,p})$</u>

<u>Def</u> integer $\boxed{t=t(n)}$ (often fixed), int-val'd $X = X^{(n)}$ is

conc. on t val's if $\exists$ $I = I^{(n)}$ s.t. $X \in I$ whp

e.g. (saw) $\chi(G_{n,1/2})$ conc on $\begin{cases} 2 \text{ vals} \\ 1 \text{ val. for "most" } n \end{cases}$

(status for $p=1/2$ below)

<u>Thm 1</u> (SS '87) $\alpha > 1/2$ fixed, $p < n^{-\alpha}$ $\Rightarrow$

$\chi(G_{n,p})$ conc. on $C = C_\alpha$ vals

<u>Thm 2</u> (Łuczak '91) $\alpha > 5/6$ $\longrightarrow$ $C=2$.

[SS get 5; we'll get 4 (as in A-S), then 2]

<u>Thm 3</u> (Alon-Krivelevich '97) $\alpha > 1/2$ $\longrightarrow$ $C=2$

what about $p = n^{-1/2}$ (e.g.) ?