L 24 $[G = G_{n,p}]$

Technical Lemma (for pf. of Łuczak; source of 5/6)

$\exists\, \delta = \delta_\alpha > 0$ s.t. whp

$M \subseteq V$, $|M| \le n^{1/2+\delta} =: m \Longrightarrow$

(i) $|G[M]| < (3/2 - \delta)|M|$

(ii) $\chi(G[M]) \le 3$

(A)

Pf indication

▷ (i) $\longrightarrow$ (ii) std obs (& culture):

$\chi(H) \le \boxed{\max_{K \subseteq H} \delta_K + 1} =: \mathrm{col}(H)$ ("coloring #")

Pf: EX (neat, easy once found)

Pf of (i): EX (U bd; AS L 7.3.4)

NICE IDEA (Alon (?), improving SS):

$\varepsilon = \varepsilon(n) := 1/\log n$ [lots of room, need $\varepsilon \to 0$ fairly slowly]

▷▷ $k = k(n) = \min\{\ell : \mathbb{P}(\chi(G) \le \ell) \ge \varepsilon\}$

$\longrightarrow \chi(G) \ge k$ whp.

SHOW: $\boxed{\chi(G) \le \left\{ \begin{matrix} k+3 \\ k+1 \end{matrix} \right\} \text{ whp}}$

▷▷▷ $X = X(G) = \min\{|S| : S \subseteq V,\ \chi(G-S) \le k\} \rightarrow$

$\mu = \mathbb{E}X$

- $\mathbb{P}(X=0) \geq \varepsilon$
- X Lip(1) (w.r.t. vert. exp.) $\xrightarrow{A_2}$ $\forall \lambda > 0$

$$\left.\begin{array}{l}\mathbb{P}(X-\mu > \lambda) \\ \mathbb{P}(X-\mu < -\lambda)\end{array}\right\} < \exp[-\lambda^2/2n] \Longrightarrow$$

or $n-1$

$$\varepsilon \leq \mathbb{P}(X=0) \leq e^{-\mu^2/2n} \Longrightarrow$$

- $\mu \leq \sqrt{2n \log(1/\varepsilon)} =: \lambda$
- whp $X \leq \mu + \lambda \leq 2\lambda$, i.e.

($\because$ $\mathbb{P}(X \geq \mu+\lambda) \leq \exp[-\lambda^2/(2n)] = \varepsilon$)

whp $\boxed{\exists S \subseteq V,\ |S| \leq 2\lambda \text{ s.t. } \chi(G-S) \leq k}$ (B)

$\longrightarrow$ BTS <u>Claim 1</u> (A) + (B) $\Longrightarrow \chi(G) \leq k+3$

<u>Claim 1</u> (A) + (B) $\Longrightarrow \chi(G) \leq k+1$

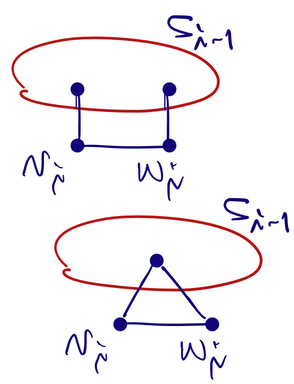
<u>Remk</u> this is now <u>deterministic</u>

<u>Pf of Cl. 1.</u> S as in (B) $\xrightarrow{(A)}$ $\chi(G) \leq k + \chi(G[S]) \leq k+3$ ⊠

<u>Pf of Cl. 3</u> $S_0 = S$ (as in (B)) & do for $i \geq 1$ (until can't):

$S_i = S_{i-1} \cup \{v_i, w_i\}$ w/

$v_i, w_i \in V \setminus S_{i-1}$, $v_i \sim w_i$ & $v_i, w_i \sim S_{i-1}$



end at S_b

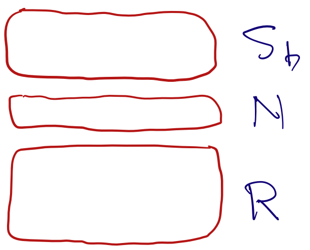
Obs $D < b_0 := \sqrt{n}\log n$ (want $n^{1/2+o(1)} > b_0 \gg \lambda$)

rough: each step adds 2 v's, 3 e's $\longrightarrow$
if continue too long, violate Ⓐ(i)

precise: if reach b_0 then $|S_{b_0}| \leq 2\lambda + 2b_0$ $(< m)$

& $|G[S_{b_0}]| \geq 3b_0 > (\frac{3}{2} - o(1))|S_{b_0}|$ via Ⓐ(i) ⊠

End: $\chi(G[S_b]) \leq 3$ (by Ⓐ(ii))



$N := \{v \in V \setminus S_b : v \sim S_b\}$

$R = V \setminus (S_b \cup N)$ $\longrightarrow$ ③

N indept $\longrightarrow$ color:

R by $[k]$, N by $k+1$, S_b by $[3]$ ⊠ ?

Small glitch: what if $k = 2$? ($k = 1$ silly)

Sketch/Ex: $k = 2$ $\Rightarrow$ $\mathbb{P}(\nexists$ odd cycle$) \geq \varepsilon$ $\Rightarrow$
$p < (1+o(1))/n$ $\Rightarrow$ $\mathrm{col}(G) \leq 3$ whp. ⊠

Back to $\chi(G = G_{n,1/2})$

start w/ $\alpha(G)$ (men'd in Lec. 6)

- $f(k) \; (= f_n(k)) = \binom{n}{k} 2^{-\binom{k}{2}}$ $\quad (= \mathbb{E}\,|\{\text{ind } k\text{-sets}\}|)$

- $k_0 = k_0(n) = \max\{k : f(k) \geq 1\}$

$$= 2\log n - 2\log\log n + O(1)$$

$\boxed{\log = \log_2}$

- $k = k_0 \pm O(1) \xrightarrow{\text{"Ex"}} \frac{f(k+1)}{f(k)} \asymp \frac{k}{n}$ $\quad \left(\asymp \frac{\log n}{n}\right)$

- 2^{nd} mm $\to$

a.s. $\alpha(G) \in \{k_0 - 1, k_0, k_0 + 1\}$ $\quad$ [& more: AS Cor 4.5.2

$\Longrightarrow$ a.s. $\chi(G) > \frac{n}{2\log n}$

Th Bollobás 88: $\chi(G) \sim \frac{n}{2\log n}$ a.s.

$\left[\text{i.e. } \forall \varepsilon > 0 \quad \mathbb{P}\left(\chi \neq (1 \pm \varepsilon)\frac{n}{2\log n}\right) \to 0\right]$

naive: $\quad m = \left\lfloor \frac{n}{\log^2 n} \right\rfloor$ $\quad$ [just need: ① $m \ll n/\log n$
② $\log m \sim \log n$]

- iterate: whp. $\exists\, I \in \mathcal{I}(G)$, $|I| \sim 2\log n$

$\to$ assign (new) color & delete

- STOP when $|V| \leq m$ $\quad (|V| > m \Rightarrow \log|V| \sim \log n)$

& color rest w/ m new colors $\longrightarrow$

$\chi(G) < (1 + o(1))\frac{n}{2\log n} + m \sim \frac{n}{2\log n}$ $\quad$ ☒ ?

— NO! (why not?) $\quad$ ($\because$ $G - I \nsim G_{[n]\setminus I,\, 1/2}$)

▷ OTOH: greedy ind. sets (equiv: gr. coloring) $\rightarrow$

$$\chi(G) < (1+o(1))\frac{n}{\log n} \quad \text{whp.}$$

[Ex to see why shd work — why is it different?]

MP for Ball: $\mathbb{P}(\alpha(G) < k_0 - 3) = \exp[-\Omega(n^2/\log^8 n)]$

▷ Janson Ineq's (~ 90 mot'd by Ball. but gen'l machine):

$\log^8 \leftarrow \log^4$

▷ conj $\log^2$ (EX: wd be tight)

} irrelevant for Ball.

▷ Then easy via "naive": (m as before)

$$MP \overset{\circledast}{\Longrightarrow} \text{whp.} \quad \forall M \in \binom{[n]}{m}$$

$$\alpha(G[M]) > \underbrace{k_0(m) - 3}_{\sim 2\log n} \qquad (m \approx \text{[illegible]})$$

$\circledast$ $\mathbb{P}(\text{fail}) < \binom{n}{m}\exp\left[-\Omega\left(\frac{m^2}{\log^8 m}\right)\right]$ (& no contest)

• Remk: role of MP analogous to that of "Tech. lemma" in Łuczak