L 25

perspective: $k = k_0 - 3$, $X = |\{\text{indept } k\text{-sets in } G\}|$

$\mu := \mathbb{E}X = f(k) \gtrsim \left(\frac{n}{\log n}\right)^3 \longrightarrow$

- 2nd m.m.? (SILLY)
- $\approx$ indept? $[\rightarrow \mathbb{P}(X=0) \approx e^{-\mu}$ NONSENSE (why?) $]$
- Azuma w edge exposure? again N.G. (why?)

NICE IDEA:

$X = \max\{t : G \text{ contains } t \text{ pair-disjt ind } k\text{-sets}\}$

$\longrightarrow \quad X < n^2/k^2 \quad (\asymp n^2/\log^2 n)$

$X \sim \mathrm{Lip}(1)$ wrt edge exp. $\longrightarrow$ "conc in $O(n)$" $\longrightarrow$

for MP ETS:

Claim $\mathbb{E}X = \Omega(n^2/\log^4 n)$

Pf sketch (Ex to fill in and/or cf. AS L7.3.1)

aux gph Γ: $V = \{\text{ind } k\text{-sets}\}$

E: share ≥ 2 verts

typ: $|V| \approx f(k) \quad (\gtrsim (n/\log n)^3)$

$\text{deg's} \approx \binom{k}{2}\binom{n-k}{k-2} 2^{-\binom{k}{2}+1} \asymp \frac{k^4}{n^2} f(k)$

larger n's contrib. less

($\rightarrow$ use $\alpha(\Gamma) \gtrsim |V|/\bar{d}$) "$\square$"

Rödl nibble & Pippenger's Thm

Molloy-Reed
AS § 4.7

$\mathcal{H}$: hypergraph on V, $|V| = n$

r-unif — r fixed (e.g. 3)

D-reg. — D large

"nearly-disjt" (a.k.a. simple): $|A \cap B| \leq 1 \quad \forall A \neq B \in \mathcal{H}$

(codegrees $o(D)$ also okay)

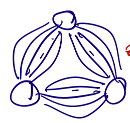
matching: set of disjt edges (members of $\mathcal{H}$)

matching #: $\nu(\mathcal{H}) \underset{\text{triv}}{\leq} n/r$

Pippenger's Thm (~85) $\quad \nu \sim n/r \quad$ (as $D \to \infty$)

equiv: $\forall \delta > 0 \quad \nu(\mathcal{H}) > (1-\delta)\, n/r \quad$ if $D > D_\delta$

Example (& origin): Erdős-Hanani Conj. '63 (see A-S)

- nonsense w/o small codegrees:  $\leftarrow D/2$

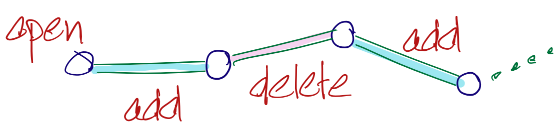
- perspective:

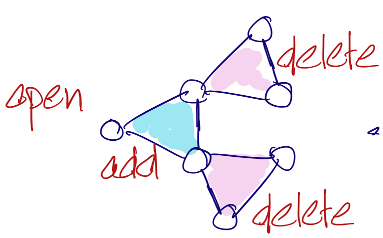
$r = 2$: PT $\leq$ Vizing: $\chi'(\mathcal{H}) \leq D+1$

$\nu \geq |\mathcal{H}|/\chi' \geq \dfrac{nD/r}{D+1} \ldots$

$\longleftrightarrow$ usual dichotomy: $r = 2$ vs. $r > 2$

e.g. ν: try alternating chains:

$r=2$:  $\exists$ hope ...

$r=3$:  ... ??

⇝ how to attack ?

- natural: random greedy (as we've said):

  $\mathcal{H}_0 = \mathcal{H}$ & for $i = 1, \ldots$:

  A_i unif $\in \mathcal{H}_{i-1} \to \mathcal{H}_i = \mathcal{H}_{i-1} \setminus \{\underbrace{\text{edges mtg. } A_i}_{\text{includes } A_i}\}$

  until $\mathcal{H}_{i-1} = \phi \to$ matching $= \{A_1, \ldots, A_{i-1}\}$

  — hard to analyze →

▶ "Rödl nibble"

- following AKS '81, Rödl '85, Frankl–Rödl '85
- now basis of many major results
- details us. (always?) take some work — we just sketch

▶ "nibble": like RG but choose edges in bunches:

- $\varepsilon = \varepsilon_\delta \ll \delta$
- step ("increment") chooses enough (random) edges to cover $\approx \varepsilon$-frac. of {remaining verts} ⟶
- # steps $\approx \frac{1}{\varepsilon} \log \frac{1}{\delta}$

  ($= O(1)$; helpful ∵ estimates deteriorate)

▶ easier to analyze ∵ large increments are more predictable

(⟵ assoc'd r.v.s are concentrated)

Step 1: $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{A}_1 = (\mathcal{H}_0)_{\varepsilon/D}$

$\mathcal{M}_1 = \{ A \in \mathcal{A}_1 : A \cap B = \emptyset \ \ \forall A \neq B \in \mathcal{A}_1 \}$ ← matching

$\mathcal{H}_1$: $V_1 = V \setminus \boxed{\cup \mathcal{A}_1} := \{A : A \in \mathcal{A}_1\}$

$\mathcal{H}_1 = \mathcal{H}_0[V_1]$ $(= \mathcal{H}_0 \setminus \{\text{edges mtg } \cup \mathcal{A}_1\})$

why not delete only $\mathcal{M}_1$?

– maybe could, but harder to analyze and waste is affordable (as we'll see; a MP)

L26

ROUGH: $|\mathcal{A}_1| \approx \frac{\varepsilon}{D} |\mathcal{H}| = \varepsilon n/r$ $\quad$ ε-frac. of goal

$$\mathbb{P}(A \in \mathcal{M}_1) = \frac{\varepsilon}{D}\left(1 - \frac{\varepsilon}{D}\right)^{r(D-1)} \begin{cases} \sim \frac{\varepsilon}{D} e^{-\varepsilon r} \\ = \frac{\varepsilon}{D}\left(1 - \varepsilon r + O(\varepsilon^2)\right) \end{cases}$$

implied const doesn't depend on ε

→ typ: $|\mathcal{A}_1 \setminus \mathcal{M}_1| \approx \varepsilon^2 r |\mathcal{H}|/D = \varepsilon^2 n$ $\quad$ ε-frac of gain

– & hope to iterate, e.g.

Step 2: $\mathcal{A}_2 = (\mathcal{A}_1)_{\varepsilon/D_1}$ $\quad$ $(\mathcal{M}_2 = \dots)$

$D_1 = e^{-\varepsilon(r-1)} D$

MP: whp $\mathcal{H}_1$ approx D_1-reg.
TBD

Whence? $d_1(x) := d_{\mathcal{H}_1}(x)$ (r.v.) $\quad$ $x \in V$

$\mathbb{E}\, d_1(x)$? $\quad$ wrong question → (right q.:)