Step 1: $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{A}_1 = (\mathcal{H}_0)_{\varepsilon/D}$

$\mathcal{M}_1 = \{A \in \mathcal{A}_1 : A \cap B = \emptyset \ \ \forall A \neq B \in \mathcal{A}_1\}$ ← matching

$\mathcal{H}_1$: $V_1 = V \setminus \boxed{\cup \mathcal{A}_1} := \{A : A \in \mathcal{A}_1\}$

$\mathcal{H}_1 = \mathcal{H}_0[V_1]$ $(= \mathcal{H}_0 \setminus \{\text{edges mtg } \cup\mathcal{A}_1\})$

why not delete only $\mathcal{M}_1$?

- maybe could, but harder to analyze and waste is affordable (as we'll see; a MP)

L26

ROUGH: $|\mathcal{A}_1| \approx \frac{\varepsilon}{D}|\mathcal{H}| = \varepsilon n/r$ ε-frac. of goal

$$\mathbb{P}(A \in \mathcal{M}_1) = \frac{\varepsilon}{D}\left(1 - \frac{\varepsilon}{D}\right)^{r(D-1)} \begin{cases} \sim \frac{\varepsilon}{D} e^{-\varepsilon r} \\ = \frac{\varepsilon}{D}(1 - \varepsilon r + O(\varepsilon^2)) \end{cases}$$

implied const doesn't depend on ε

→ typ: $|\mathcal{A}_1 \setminus \mathcal{M}_1| \approx \varepsilon^2 r |\mathcal{H}|/D = \varepsilon^2 n$ $O(\varepsilon)$-frac of gain

- & hope to iterate, e.g.

Step 2: $\mathcal{A}_2 = (\mathcal{A}_1)_{\varepsilon/D_1}$ $(\mathcal{M}_2 = \dots)$

$D_1 = e^{-\varepsilon(r-1)} D$

MP: whp $\mathcal{H}_1$ approx D_1-reg.

TBD

Whence? $d_1(x) := d_{\mathcal{H}_1}(x)$ (r.v.) $x \in V$

$\mathbb{E}\, d_1(x)$?   wrong question ⟶ (right q.:)

$\mathbb{E}[d_1(x) \mid \underline{x \in V_1}] = (1-\varepsilon/D)^{(r-1)(D-1)} D \sim e^{-\varepsilon(r-1)} D = D_1$

why?

▷ concentration? fix x & condition on $\{x \in V_1\}$

$X_A = \mathbb{1}_{\{A \in \mathcal{H}_1\}}$ $(\to \mathbb{E} X_A \sim e^{-\varepsilon(r-1)}$ for $A \ni x)$

$$X = \sum_{A \ni x} X_A = d_1(x)$$

$$A \neq B \ni x \Rightarrow \mathbb{E} X_A X_B = (1-\varepsilon/D)^{2(r-1)(D-1)-b}$$

$$b = |\{C \not\ni x : C \cap A \neq \emptyset \neq C \cap B\}| \overset{*}{\leq} (r-1)^2$$

[* : $\mathcal{H}$ simple (if codeg's $o(D)$ then $b = o(D)$)]

$\xrightarrow[\text{(EX)}]{\text{Cheb}}$ (on $\{x \in V_1\}$) $\underset{\underset{X}{\|}}{d_1(x)} \sim D_1$ whp

... and iterate: $\approx \frac{1}{\varepsilon} \log \frac{1}{\delta}$ steps

deg's in $\mathcal{H}_i$ shd be $\approx \underbrace{e^{-\varepsilon(r-1)i}}_{\approx \delta^{r-1}} D =: D_i$

$= \underline{\text{still const}}$

But errors accumulate $\to$ can we hang on?

▷ one point: deg's only mostly good; e.g. after Step 1

there will (typ.) be vertices w deg $\not\approx D_1$

serious problem for F-R

Actual P.T. $\forall\, r, \delta > 0$ & K $\exists\, \gamma > 0$ & D_0 s.t.

if $D > D_0$ & $\mathcal{H} \models$

(0) r-unif, n-d (or codeg's ...)

(1) $|\{x : d_{\mathcal{H}}(x) \neq (1 \pm \gamma) D\}| < \gamma n$

(2) $\Delta_{\mathcal{H}} < KD$

Then $\nu(\mathcal{H}) > (1-\delta) n / r$

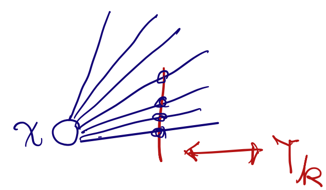
- false if $\gamma \approx 1/K$ (why?) [in fact (EX) need $\gamma \lesssim \delta / K$]
- again, main issue is deterioration of estimates
  — not obv. this is okay (e.g. AS: §19 (!))

alternative to "ac. Pipp." ($X = d_1(x)$, cond on $\{x \in V_1\}$)

Claim: $\mathbb{P}(|X - \mathbb{E}X| > \lambda) < \exp\left[- \dfrac{\lambda^2}{O(\varepsilon D)} \right]$ (why?)

think γD

[edge exposure mart:



- $|Z_k| \leq r$
- length? $< rD^2 = O(D^2)$ — too big → ?

Y_k takes 2 val's, one $\bar{w}$ prob ε / D

→ "var" $\lesssim rD^2 \cdot \varepsilon / D \cdot r^2 = O(\varepsilon D)$]

$\Longrightarrow \exists\, a_1 \ \bar{w}$ no bad verts (why?)

deg $\neq (1 \pm \gamma) D_1$

[Pf: $\mathbb{P}(x \text{ bad}) = \exp[-\Omega(\gamma^2 D / \varepsilon)]$ + LLL

(what are the events?)]

cmts

① codeg's $o(D)$: doesn't quite support this arg.

(gives (Ex) $\mathbb{P}(|X - \mathbb{E}X| > \lambda) = \exp\left[\frac{-\lambda^2}{o(D^2)}\right]$

— so $\mathbb{P}(x \text{ bad}) = o(1)$ but not enough for LLL)

but martingales are crucial for more advanced appl's

② first ver: we just watch & show gd behav.

(e.g. most deg's $\approx D_1$) is likely

(N.B. A-S version doesn't quite do this)

second ver: we interfere: show $\exists$ gd increment &

choose one. (& continue)

– Cf. next development (after P's thm):

Pippenger-Spencer '89: P's hypotheses $\Rightarrow$ $\chi'(\mathcal{H}) \sim D$:

- use Pippenger arg to gen. random matchings (M)

  $\models$ (e.g.) $\mathbb{P}(A \in M) \sim 1/D$ $\forall A \in \mathcal{H}$

- increment ("nibble"): εD of these (indept'ly)

Here if we just let the process run (as in PT)

then there will typically (for large n) be underused verts

— & can't catch up

$\longrightarrow$ must interfere ($\sim$ mart + LLL)