Back to "thresholds" (& containers)

"recent" theme: "sparse random" versions of classical results

Ex. 1 $\quad t_r(G) = \max\{|F| : F \subseteq G,\ F \not\supseteq K_r\}$

Turán (slightly relaxed): $\quad t_r(K_n) \leq \left(1 - \frac{1}{r-1} + o(1)\right)\binom{n}{2}$

[ $o(1) \asymp 1/n$ ; $t_r(K_n) \leq \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}$ is true

(& exact if $(r-1) | n$) but $\binom{n}{2}$ will become $|G|$ ]

$G$ is $(r,\delta)$-Turán if $\quad t_r(G) \leq \left(1 - \frac{1}{r-1} + \delta\right)|G|$

Ex. 2 $\quad G \to (H)_q$ if $\forall\, \varphi : E(G) \longrightarrow [q]$ $\quad \exists$ monochr. $H$

Ramsey: $\forall\, H, q\ \ \exists\, n$ s.t. $K_n \to (H)_q$ $\quad$ (today just $H = K_r$)

Sparse random: $\quad$ "When" does $G = G_{n,p}$ satisfy:

① $G$ $(r,\delta)$-Turán ?

② $G \to (K_r)_q$ ?

N.B. $(r,\delta)$-Turán not increasing ($\to$ "threshold") $\quad$ — why?

Thm D (Conlon-Gowers '16, Schacht '16)

$\forall\, \delta > 0$ & $r$ $\ \exists\, K$ s.t. for $p > K n^{-2/(r+1)}$

$G_{n,p}$ is $(r,\delta)$-Turán whp.

Thm RR (Rödl-Ruciński 95)

$\forall\, q$ & $r$ $\ \exists\, K$ s.t. for $p > K n^{-2/(r+1)}$

$G_{n,p} \to (K_r)_q$ whp.

▶ why $2/(r+1)$ ? $\qquad$ e.g. $r = 2 \longrightarrow$

Thm $D_3$: easy: $p < cn^{-1/2}$ typ. $\Rightarrow$

$t_3(G) \gtrsim |\{\text{edges not in } \Delta\text{'s}\}| \sim (1-c^2)|G|$

(even: $t_3(G) \gtrsim (1 - c^2 + c^2/2)|G|$ — why?)

Thm $RR_3$ (even $q=2$:)

Obs: $G \not\to (K_3)_2 \Rightarrow$

$\exists E \subseteq G$ s.t. every $e \in E(G)$ is in $\geq 2$ $\Delta$'s in $E$ (why?)

... which shd be unlikely when $p < cn^{-1/2}$, but still not so easy (see Nenadov-Steger)

Plan:
- Container Thm for triangles (statement)
- Pfs of thms $D_3$ & $RR_3$
- more on containers, as possible

| |
|---|
| one ref: Morris: The method of hypergraph containers<br>Container Thm: Balogh-Morris-Samotij '15, Saxton-Thomason '15 |

Container context: $\mathcal{I}(\mathcal{H}) = \{\text{indept sets of } \mathcal{H}\}$, where:

$\mathcal{H}$ = hypergraph on $V$, indept := contain no edges

us. $k$-unif w $k$ fixed

E.g. $\mathcal{H}_n = \{$(edge sets of) $\Delta$'s of $K_n\}$ $(V(\mathcal{H}_n) = E(K_n))$

$\longrightarrow$ $\mathcal{I}(\mathcal{H}_n) = \{\Delta\text{-free subgraphs of } K_n\}$

notation: $F \subseteq E(K_n)$ $(= V(\mathcal{H}_n)) \Rightarrow$

$\tau_r(F) = |\{K_r\text{'s in } F\}|$ & $\tau = \tau_3$

**Thm 1** $\forall \varepsilon>0\ \exists B=B_\varepsilon$ s.t.

$\forall n\ \exists \mathcal{T} \subseteq \mathcal{I}(\mathcal{H}_n)$ & $C:\mathcal{T} \to \mathcal{C} \subseteq 2^{V(\mathcal{H}_n)}$ $\models$

① $\forall\ I \in \mathcal{I}(\mathcal{H}_n)\ \exists\ T \in \mathcal{T}$ w/ $T \subseteq I \subseteq C(T)$ → "containers"

② $|T| < B n^{3/2}\quad \forall T \in \mathcal{T}$

$\longrightarrow |\mathcal{C}| \lesssim \binom{\binom{n}{2}}{< Bn^{3/2}} \lesssim \exp[B n^{3/2} \log n^{3/2}]$

③ $\tau(C) < \varepsilon n^3\quad \forall C \in \mathcal{C}$

**Remk:** $n^{3/2}$ best poss. (we'll see — is it obv?)

**Pf of thm D3**

$\ulcorner\ p = K n^{-1/2}$ ($K > K_\delta$ TBA) [$p =$ unimp. but conv.]

$\longrightarrow$ whp $t_3(G) < (1+\delta) n^2 p/4$ (note whp $|G| \sim n^2 p/2$) $\lrcorner$

**FACT 1** $\forall\ \alpha > 0$ & $r$ $\exists\ \beta > 0$ s.t.

$F \subseteq K_n,\ |F| > \left(1 - \frac{1}{r-1} + \alpha\right)\binom{n}{2} \Rightarrow \tau_r(G) > \beta n^r$

**Pf:** EX (cons. w/ m TBA $G[M]$'s, $M \in \binom{V}{m}$; cf. HW Prob 22) ∎

- Choose $\varepsilon > 0$ s.t.

  $F \subseteq K_n,\ \tau(F) < \varepsilon n^3 \Longrightarrow |F| < \left(\frac{1}{2} + \frac{\delta}{2}\right)\binom{n}{2}$

- apply Thm 1 (w/ $\varepsilon$) $\to B = B_\varepsilon$, $\mathcal{T}$ etc as in Thm

- Show unlikely: $\exists\ F \subseteq G$ $\Delta$-f, $|F| > (1+\delta) n^2 p/4$

If $\exists F$ then $\exists\, T \in \mathcal{T}$, $T \subseteq F \subseteq C = C(T)$

OTOH $C \in \mathcal{C} \Rightarrow |C| < \left(\frac{1}{2} + \frac{\delta}{2}\right)\binom{n}{2} \Longrightarrow$

$$\mathbb{P}\left(\boxed{|G \cap C| > \left(\tfrac{1}{2} + \delta\right) n^2 p/4}\right) < \exp\left[-c\,\delta^2 \underbrace{K n^{3/2}}_{\leq n^2 p}\right]$$

$\circledast$ universal

$$\to \quad \mathbb{P}(\exists F) \leq \mathbb{P}(\exists C \in \mathcal{C} \text{ w/ } \circledast)$$

$$\lesssim \exp\left[B n^{3/2} \log n^{3/2} - c\,\delta^2 K n^{3/2}\right] \qquad \text{(N.G.)}$$

Q: what are we missing?

A: pay for $T \subseteq G$:

$$\mathbb{P}(\exists F) \leq \sum_{T \in \mathcal{T}} \mathbb{P}(G \supseteq T \ \& \ C(T) \models \circledast)$$

$$\leq \boxed{\sum_{t < B n^{3/2}} \binom{\binom{n}{2}}{t} p^t} \exp\left[-c\,\delta^2 K n^{3/2}\right]$$

$$\lesssim \left(\frac{eK}{2B}\right)^{B n^{3/2}}$$

$$\lesssim \exp\left[B n^{3/2} \log K - c\,\delta^2 K n^{3/2}\right]$$

$$= o(1) \text{ for large enough } K$$

∎

▶ Remk if bd. in Thm 1 is $|T| < \theta\binom{n}{2}$ then (EX)

pf here works w/ $p = K\theta$

$\longrightarrow$ can't do better than $n^{3/2}$ (as men'd above)