L28

Pf of Thm RR$_3$ (Nenadov-Steger '14)

FACT 2 $\forall r, q \ \exists \alpha > 0$ s.t.

$$E(K_n) = \bigcup_{i=0}^{q} R_i \implies \exists > \alpha n^r \text{ monochr } K_r\text{'s}$$

Pf: EX (sim. to Erdős-Simonovits, starting from Ramsey)

T RR$_3$: α as in FACT 2 ($\overline{w}$ $r=3$)

$\varepsilon = \alpha/(2q) \longrightarrow \mathcal{T}$ etc as in Thm 1

Show unlikely: $\exists \bigcup_{i=1}^{q} E_i = G$ $\overline{w}$ E_i Δ-free

If $\exists$: $T_1, \dots, T_q \in \mathcal{T}$ $\overline{w}$ $T_i \subseteq E_i \subseteq C(T_i) =: C_i$

$C_0 := E(K_n) \setminus \bigcup_{i=1}^{q} C_i$

FACT 2 + $\tau(C_i) < \varepsilon n^3 \implies \tau(C_0) > \alpha n^3/2$

$\implies |C_0| > \alpha n^2$ (why?)

[$\tau(C_0) < |C_0|(n-2)/3$; EX: $|C_0| \gtrsim \alpha^{2/3} n^2$ ($\approx$ tight)]

$\triangleright$ $G \cap C_0 = \emptyset \longrightarrow$ prob. $< \exp[-\alpha n^2 p]$ $\circledast$

As for Turán:

$$|\{(T_1, \dots, T_q)\text{'s}\}| < \binom{\binom{n}{2}}{B n^{3/2}}^q \approx \exp[q B n^{3/2} \log n]$$

too big for $\circledast$ $\longrightarrow$ use $G \supseteq \bigcup T_i =: T$ but how?

$$\left(\sum_{t_1, \dots, t_q} \prod \binom{\binom{n}{2}}{t_i} p^{??} \qquad ? \right)$$

<u>Ans</u> (not ex. as in N-S, Morris): choose:

① $T := \bigcup T_\lambda$, $|T| < qBn^{3/2}$

② $T_1, \dots, T_q \subseteq T$ (not using $\left\{ \begin{array}{l} |T_i| < \dots \\ \bigcup T_\lambda = T \end{array} \right.$)

$$\longrightarrow \sum_{t < qBn^{3/2}} \underbrace{\binom{\binom{n}{2}}{t} \boxed{2^{tq}} p^{t}}_{\text{incr. in } t \text{ (check)}} \cdot \exp[-\alpha n^2 p/2]$$

⊛ ⟷ ② (minor)

(etc.)

∎

Regarding pf of Thm 1 (needs generalization:)

Container Lemma for 3-gphs:

Version 1: $\forall c\ \exists \delta > 0$ s.t.

$\mathcal{H}$: 3-unif, avg deg d,

$\Delta_{\mathcal{H}} < cd$, $\Delta_2 < c\sqrt{d}$ ($\Delta_2 =$ max codeg)

$\Rightarrow \exists\, \mathcal{T} \subseteq \mathcal{I}(\mathcal{H})$ & $C : \mathcal{T} \to \mathcal{C} \subseteq 2^{V(\mathcal{H})}$ s.t.

① $\forall\ I \in \mathcal{I}(\mathcal{H})\ \exists\ T \in \mathcal{T}$ w $T \subseteq I \subseteq C(T)$;

② $T \in \mathcal{T} \Rightarrow |T| < v_{\mathcal{H}}/\sqrt{d}$;

③ $C \in \mathcal{C} \Rightarrow |C| < (1-\delta)v_{\mathcal{H}}$

e.g. $\mathcal{H} = \mathcal{H}_n$: $v_{\mathcal{H}} = \binom{n}{2}$; d-reg w $d = (n-2)$

$\Delta_2 = 1$ $\quad \deg(\ ) = 1$, $\deg(\ ) = 0$

($\to$ ② improves Thm 1, but ③?? $\longrightarrow$)

Ver. 2: $\forall c\ \exists \delta > 0$ s.t. $\forall c$ & $\varepsilon > 0$ $\exists B$ s.t.

$\mathcal{H}$: 3-unif, avg deg d,

$\Delta_{\mathcal{H}} < cd$, n-d (can relax to codeg's $\lesssim \sqrt{d}$)

$\Rightarrow \exists\, \mathcal{T} \subseteq \mathcal{I}(\mathcal{H})$ & $C : \mathcal{T} \to \mathcal{C} \subseteq 2^{V(\mathcal{H})}$ s.t.

① $\forall\ I \in \mathcal{I}(\mathcal{H})\ \exists\ T \in \mathcal{T}$ w $T \subseteq I \subseteq C(T)$;

② $T \in \mathcal{T} \Rightarrow |T| < v_{\mathcal{H}}/\sqrt{d}$; $|T| < B\, v_{\mathcal{H}}/\sqrt{d}$;

③ $C \in \mathcal{C} \Rightarrow |C| < (1-\delta)v_{\mathcal{H}}$ $\quad |\mathcal{H}[C]| < \varepsilon|\mathcal{H}|$

$\rightsquigarrow$ Thm 1

Ver 1 $\Longrightarrow$ Ver 2 (sketch): I given $\to$

c as in Ver 1 & iterate ($\bar{w}$ evolving c):

if $\boxed{|\mathcal{H}[C]| > \varepsilon |\mathcal{H}|}$ — ⊛

then apply Ver 1 to $\mathcal{H}[C]$

check ⊛ $\Longrightarrow$

(i) $d(\mathcal{H}[C]) > \varepsilon d$ ($\to$ use Ver 1 $\bar{w}$ $c \leftarrow c/\varepsilon$)

(ii) $|C| > \varepsilon N_{\mathcal{H}}/c$

$\rightsquigarrow$ reach $\neg$⊛ in $B = B(\varepsilon, c)$ iterations

$\hookrightarrow$ bd in ② of Ver 2 "☒"

Sketch of pf of Ver 1

Graph Container Lemma: $\forall c \; \exists \delta > 0$ s.t.

G (simple) gph, avg deg d, $\Delta_G < cd$ $\Longrightarrow$

$\exists \mathcal{T} \subseteq \mathcal{I}(\mathcal{H})$ & $C: \mathcal{T} \to \mathcal{C} \subseteq 2^{V(\mathcal{H})}$ s.t.

① $\forall\; I \in \mathcal{I}(\mathcal{H}) \;\exists\; T \in \mathcal{T}$ $\bar{w}$ $T \subseteq I \subseteq C(T)$;

② $T \in \mathcal{T} \Longrightarrow |T| < v_G/d$;

③ $C \in \mathcal{C} \Longrightarrow |C| < (1-\delta) v_G$

[BTW: why do we need $\Delta < cd$?]

algorithm (input $I \in \mathcal{I}(\mathcal{H})$)

think of reducing to GCL (though won't always):

alg. produces G on $V(\mathcal{H})$ w/ $I \in \mathcal{I}(G)$ →

done by GCL if $|G| \gtrsim v_{\mathcal{H}} \cdot \sqrt{d}$

maintain: "fingerprint" S (first part of T)

$\mathcal{Q}$ ("avail." edges of $\mathcal{H}$)

G (for GCL; $V(G) = V(\mathcal{H})$ always)

notation: given $\mathcal{Q}$: $N(u) = \{vw : uvw \in \mathcal{Q}\}$ (a.k.a. link)

recall ~~gph alg~~ (MAIN):

choose $u \in I$ w/ big (current) deg &

① move u to T, ② delete nbrs ($\notin I$)

(big $d(u)$ → small T)

NOW (MAIN): $u \in I$ w/ biggest $d_{\mathcal{Q}}(u)$ →

⓪ delete (from $\mathcal{Q}$) verts w/ bigger deg (can't be in I)

① move u to S (from $V(\mathcal{Q})$; stays in $V(G)$)

② add $N(u)$ to G (NOTE I indept in G) →

$\boxed{\text{big } d_{\mathcal{Q}}(u) \implies \text{big add'n to } G}$ (A)

and delete from $\mathcal{Q}$:

③ edges cont'g edges of G (to maintain (A))

④ vert's w/ high deg in G (for $\Delta_G < c d_G$)

$> c\sqrt{d}$

STOP when $|S| < v_{\mathcal{H}} \cdot \sqrt{d}$

Claim either $|G| \gtrsim v_{\mathcal{H}} \cdot \sqrt{d}$ (a)

or S identifies $C \supseteq I$ w/ $|C| < (1-\delta) v_{\mathcal{H}}$ (b)

} done either way

Because (rough & qualitative):

Obs: the only deleted verts that might be in I are those in (4) (call these $V_{(4)}$) — so $\boxed{V(Q) \cup V_{(4)} \supseteq I}$

Suppose (a) fails. Then:

(i) final Q is small rel. to $\mathcal{H}$ (why?)

(ii) few vertices were deleted in (4) (why?)

By (i), alg. deleted $\approx v_{\mathcal{H}} d$ edges — where?:

- edges containing verts deleted in (4)
  - few such edges by (ii)
- edges containing edges of G
  - few of these since (a) fails & $\Delta_2 < c\sqrt{d}$

— so there must be many in the last category,

- edges mtg $V(\mathcal{H}) \setminus V(Q)$,

implying $v_{\mathcal{H}} - v_Q$ is "large" (eventually meaning $\gtrsim \delta v_{\mathcal{H}}$).

But then, by Obs & (ii), $C := V(Q) \cup V_{(4)}$ satisfies (b).

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