CAMERAS

RECORD

$\boxed{L5}$ (RECAP)

Baranyai's thm:

$r_1, \dots, r_k \in \mathbb{N}$ (repeats okay)

$\mathcal{H}_i$: copy of $\binom{[n]}{r_i}$

$\alpha_{ij} \in \mathbb{N} \quad i \in [k],\ j \in [\ell]$

(i) $\sum_j \alpha_{ij} = \binom{n}{r_i} \quad \forall i$

(ii) $\sum_i \alpha_{ij} r_i = n \quad \forall j$

$\Rightarrow$

$\exists$ partn $\bigcup \mathcal{H}_i = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_\ell$ w/

$\mathcal{F}_j$ = p.m. & $\boxed{|\mathcal{H}_i \cap \mathcal{F}_j| = \alpha_{ij}}$

Pf by ind on n

Claim $\boxed{\text{ETS}}$ $\exists$ $\alpha'_{ij} \in \{0,1\}$ s.t.

(a) $\sum_j \alpha'_{ij} = \binom{n-1}{\boxed{r_i - 1}}$ r_i' $\forall i$

(b) $\sum_i \alpha'_{ij} = 1 \quad \forall j$

(c) $\alpha_{ij} = 0 \Rightarrow \alpha'_{ij} = 0$

$\square$

$$\begin{array}{c|ccccc|} & \mathcal{F}_1 & \cdots & \mathcal{F}_j & & \mathcal{F}_\ell \\ \hline \mathcal{H}_1 & & & & & \\ \mathcal{H}_i & & & \alpha_{ij} & & \\ \mathcal{H}_k & & & & & \\ \hline \end{array}$$

Existence? (punchline)

$$\boxed{x_{ij} := \frac{r_i}{n}\,\alpha_{ij}} \quad ⊛ \longrightarrow$$

(a) $\sum_j x_{ij} = \frac{r_i}{n} \sum_j \alpha_{ij} = \frac{r_i}{n}\binom{n}{r_i} = \binom{n-1}{r_i-1}$ ✓

(b) $\sum_i x_{ij} = \frac{r_i}{n} \sum_i \alpha_{ij} = 1$ ✓

$\longrightarrow \sum_i r_i \alpha_{ij} = n$

(c) $\alpha_{ij} = 0 \Rightarrow x_{ij} = 0$ ✓

→ (d) $\underline{x_{ij} \in [0,1]}$ (by (b) or ⊛) ←

⊛ v unif $\in [n] \to x_{ij} = \mathbb{P}($the $A \in \mathcal{F}_j$ cont'g v is from $\mathcal{H}_i)$

$\boxed{\text{so?}}$

x

Lemma (a_{ij}) $m \times n$ $\boxed{\mathbb{R}}$-mat w integer line sums r_i, c_j

$\Rightarrow \exists$ $m \times n$ $\boxed{\mathbb{Z}}$-mat. (b_{ij}) w

① same line sums

② $|b_{ij} - a_{ij}| < 1 \quad \forall i, j$

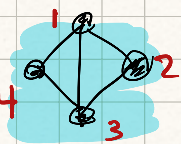
∎

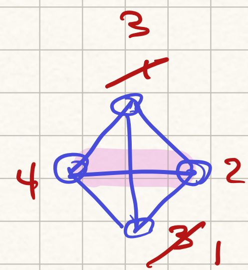
Thresholds & 2nd moment method

E.g. H fixed graph, $G = G_{n,p}$ ($p = p(n)$ TBA) →

▷ when is G likely to contain (a copy of) H?

↳ $p = p(n)$

e.g. $H =$  (then gen'l)



K_4

X = # of H's in G

↳ unlab. (unimp.)

how many H's?

$\longrightarrow \mathbb{P}(\underbrace{X \neq 0}_{G \supseteq H})$ does what?

start with $\mathbb{E}$:

$H_1, \dots, H_m$ copies of H in K_n $\quad n$ =

$$m = \frac{(n)_4}{4} = \Theta(n^4) \qquad 4 \to |\mathrm{Aut}(H)|$$

$(n)_4 \leftrightarrow$ lab copies

$X_\alpha = \mathbb{1}_{\{H_\alpha \subseteq G\}}$ $\quad (X = \sum X_\alpha)$

i.e. $\Theta(1)$

$$\mathbb{E}X = \sum \underbrace{\mathbb{E}X_\alpha}_{p^5} = mp^5 \begin{cases} \asymp 1 & \text{if } \boxed{p \asymp n^{-4/5}} \\ \to 0 & p \ll n^{-4/5} \quad ① \\ \to \infty & p \gg n^{-4/5} \quad ② \end{cases}$$

↑ lin of $\mathbb{E}$ $\qquad (n \to \infty)$ $\qquad p = \omega(n^{-4/5})$

① $\Longrightarrow$ $\underbrace{\mathbb{P}(X \neq 0)}_{G \supseteq H} \to 0$ if $p \ll n^{-4/5}$

$\mathbb{P}(X \neq 0) \leq \mathbb{E}X$ $\quad X \in \mathbb{N}$ r.v.

$$\boxed{\triangleright \ \underbrace{\mathbb{P}(X \neq 0) \to 1}_{G \supseteq H \ \left\{ \begin{array}{l} \text{w.h.p.} \\ \text{a.s.} \\ \text{a.a.s.} \end{array} \right.} \quad \text{if } p \gg n^{-4/5}} \quad ②$$

e.g. $\left.\begin{array}{l} \mathbb{P}(Y=0) = 1 - \frac{1}{M} \\ \mathbb{P}(Y = M^2) = \frac{1}{M} \end{array}\right\}$ M big

[More gen'l for a while, then back to ②]

- X fin. set
  sorry

"p-biased" product meas

- μ_p: prob. measure on 2^X: $\underline{\mu_p(A) = p^{|A|}(1-p)^{|X - A|}}$

  $\to$ random $\underline{A =: X_p}$ ✓

  $\triangleright$ e.g. $X = \binom{[n]}{2} = E(K_n) \to X_p = G_{n,p}$

- $\mathcal{F} \subseteq 2^X$: "fam," "prop" ✓

  e.g. $X = \binom{[n]}{2}$: $\mathcal{F}$ graph. prop. if iso. invar

  e.g. {cont H}, {conn. (sp.)}, {planar} $\Longleftarrow$

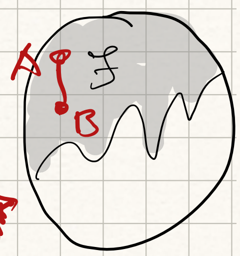
$X = \binom{[n]}{2}$ $\quad$ $[n]$

1 2

on

$\mathcal{F}$ = {cont. 12} NOT gph prop

$\mathcal{F}$ incr (a.k.a. "mono") if $A \supseteq B \in \mathcal{F} \Rightarrow A \in \mathcal{F}$



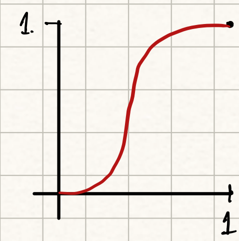
e.g. $\{\text{cont. } x\}$, $\{|A| \geq k\}$ $\quad x \in X$

e.g. gph prop's above are incr, incr, decr

Obs $\mathcal{F}$ incr $\Rightarrow$ $\mu_p(\mathcal{F})$ incr.

& strict if $\mathcal{F} \neq \emptyset, 2^X$

$\mu_p(\mathcal{F}) \uparrow$



$p \rightarrow$

— "obv" but why? (temp. EX)

seen: n "hidden" param

thresholds: X_n fin sets

$\mathcal{F}_n \subseteq 2^{X_n}$ incr, $\mathcal{F} = \{\mathcal{F}_n\}$

JŁR: $\mathcal{F} \leftarrow \mathcal{Q}$

E-R 60: $p_0 = p_0(n)$ a $\left\{\begin{matrix}\text{th.} \\ \text{th. fn.}\end{matrix}\right\}$ for $\mathcal{F}$ if

Rényi

$$\mu_p(\mathcal{F}_n) \longrightarrow \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases} \qquad [p = p(n)]$$

e.g. $\mathbb{P}(G_{n,p} \in \mathcal{F})$

E.g. $\mathbb{P}(G \supseteq H) \longrightarrow 1$ if $p \gg n^{-4/5}$?

is $n^{-4/5}$ a threshold for $\mathcal{F} = \{\text{contain}\ \}$?

us. $\mathcal{F}$ "nat" (e.g. doesn't men. n) but not nec'ly

ER'60 (e.g.) $n^{-4/5}$ is a th. for {cont. H = ◇}

[a.k.a. $\mathbb{P}(G \supseteq H) \longrightarrow 1$ if $p \gg n^{-4/5}$]

Pf: "2nd m.m." (can do better here, but ...)

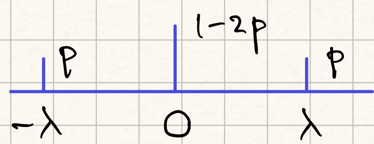
↳ use $\mathbb{E}X^2$ (2nd moment)

▷ Cheb: $\underline{\underline{\forall}}$ r.v. X & $\lambda > 0$

$$\mathbb{P}(\underbrace{|X-\mu| \geq \lambda}) \leq \frac{\sigma^2}{\lambda^2}$$

$\left[\begin{array}{l}\mu = \mu_X \\ \sigma^2 = \sigma_X^2\end{array}\right.$

- makes sense
- sharp:



$\sigma^2 = 2p\lambda^2 \quad (\mu = 0)$

$\mathbb{P}(|X| \geq \lambda) = 2p$

Pf $\mathbb{P}(|X-\mu| \geq \lambda) = \boxed{\mathbb{P}(\underbrace{(X-\mu)^2}_{\geq 0} \geq \lambda^2)} \leq \lambda^{-2}\, \underbrace{\mathbb{E}(X-\mu)^2}_{\sigma_X^2}$

↑ Markov

Rmks
- Cheb ⟷ conc. of meas (...)
- weak for "nice" X (as here) but cf. Chernoff $\mathbb{E}e^{sX}$
  - always true & rel. easy
  - often enough

} ⟶ useful esp. for tougher X's

↓

some spectacular successes

[For me: e.g.
- Robinson-W
- Achlioptas & Co.
- Riordan]

- "2nd m.m" = ?

[back to E-R]

$$\sigma^2 = \mathbb{E}X^2 - \mu^2$$

$$\mathbb{P}(X=0) \le \mathbb{P}(|X-\mu| \ge \mu) \le \boxed{\frac{\sigma^2}{\mu^2}} = \boxed{\frac{\mathbb{E}X^2}{\mu^2}} - 1$$

hope for: $\sigma^2 = o(\mu^2)$ } equiv

$\mathbb{E}X^2 \sim \mu^2$

note hidden n

– in which case $X \sim \mu$ a.s. i.e.

$$\forall \varepsilon > 0 \quad \mathbb{P}\left(\frac{X}{\mu} \in (1-\varepsilon, 1+\varepsilon)\right) \to 1$$

$$\mathbb{P}(\text{false}) \underset{\text{Cheb}}{\le} \frac{\sigma^2}{\varepsilon^2\mu^2} \to 0$$

[Recall: $H_1, \dots, H_m$ copies of H in K_n $(m = \Theta(n^4))$

$X_\alpha = \mathbb{1}_{\{H_\alpha \subseteq G\}}$ $(G = G_{n,p})$

$\boxed{(X = \sum X_\alpha}$, $\mu = mp^5$)]

Aim for: $(\sigma^2 =)$ $\mathbb{E}X^2 - \mu^2 \ll \mu^2$ (*)

$$\mathbb{E}X^2 = \mathbb{E}\sum_\alpha\sum_\beta X_\alpha X_\beta = \sum\sum \boxed{\mathbb{E} X_\alpha X_\beta}$$

$$\mu^2 = \sum\sum \boxed{\mathbb{E}X_\alpha \boxed{\mathbb{E}} X_\beta} = m^2 p^{10}$$

$(\mu = \sum \mathbb{E}X_\alpha)$

heur: pretend X_α's ind $\longrightarrow$

$$\mathbb{E}X_\alpha X_\beta = \mathbb{E}X_\alpha \, \mathbb{E}X_\beta \quad \forall \alpha \neq \beta \longrightarrow$$

$$\mathbb{E}X^2 - \mu^2 = \boxed{\sum\left(\underbrace{\mathbb{E}X_\alpha}_{=\mathbb{E}X_\alpha^2} - \mathbb{E}^2 X_\alpha\right)} \le \boxed{\mu \ll \mu^2}$$

$\mathbb{E}X = mp^5 \gg 1$

$\Longrightarrow$ (*) if $\mu \to \infty$ (which we have)

IDEA/HOPE: this is $\approx$ right

Now just calc's

edges

$$\mathbb{E}X_\alpha X_\beta = p^{|H_\alpha \cup H_\beta|} = p^{10-|H_\alpha \cap H_\beta|}$$

e.g. X_α, X_β ind. $\Longleftrightarrow H_\alpha \cap H_\beta = \emptyset$

notation: $\alpha \sim \beta$: $H_\alpha \cap H_\beta \neq \emptyset$

$$\mathbb{E}X^2 = \sum\sum \mathbb{E}X_\alpha X_\beta = m \sum_\beta \mathbb{E}X_1 X_\beta$$

$$\mathbb{E}X^2 - \mu^2 = m \sum_{\beta \sim 1} (\mathbb{E}X_1 X_\beta - \underbrace{\mathbb{E}X_1 \mathbb{E}X_\beta}_{p^{10}})$$

$$\le m \sum_{\beta \sim 1} \mathbb{E}X_1 X_\beta = mp^{10} \sum_{\beta \sim 1} p^{-|H_1 \cap H_\beta|}$$