

587 Problems

[First installment. Notes: The first two are supposed to be easy, the next two not hard, and the last more interesting. Undefined notation should be in the “handout.” And a possibly useful reminder: $1+x \leq e^x$ is always true (and close to equality if x is small).]

1. Show that there is a fixed C so that the rows of any $n \times n$ real matrix with distinct entries can be permuted to obtain a matrix in which no column contains an increasing subsequence of length at least $C\sqrt{n}$.

[Zhao 1.4; note subsequences needn't be consecutive.]

2. The *trace* of a set family \mathcal{F} on a set A is $\text{Tr}(\mathcal{F}, A) = \{S \cap A : S \in \mathcal{F}\}$. Show that for any n and $k \leq n$, there is $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| = O(k2^k \log n)$ and $\text{Tr}(\mathcal{F}, A) = 2^A$ for every $A \in \binom{[n]}{k}$.

[Zhao 1.8]

3. Let A_1, \dots, A_n be events in a probability space, set $\mu = \sum \mathbb{P}(A_i)$, and let Q_l be the event that some l independent A_i 's occur. Show $\mathbb{P}(Q_l) \leq \mu^l/l!$.

4. A *dominating set* in a graph G is $U \subseteq V(G)$ such that each vertex not in U has a neighbor in U . Show that any n -vertex G with minimum degree $\delta > 1$ has a dominating set of size at most $n[1 + \ln(1 + \delta)]/(\delta + 1)$.

[This is AS Theorem 1.2.2, but a nice exercise, and maybe *slight* preparation for the more challenging Problem 5.]

5. Let $G = (V, E)$ be a graph with $|V| = n$ and minimum degree δ . Show there is a partition $V = A \cup B$ such that each vertex of B has at least one neighbor in each of A, B , and $|A| = O(\frac{n \ln \delta}{\delta})$.

[AS, Problem 1.7.4; note they give it a (*).]

[Second installment.]

6. Recall (or see AS, Sec. 1.3) that a hypergraph \mathcal{H} on V is *2-colorable* (or has *Property B*) if there is a partition $V = R \cup B$ such that

$$A \cap R \neq \emptyset \neq A \cap B \quad \forall A \in \mathcal{H}.$$

Let $m(n)$ be the least size of an n -uniform \mathcal{H} that does *not* have Property B, and let $g(n)$ be the least size of an n -uniform \mathcal{H} , on some V (say, to avoid irrelevancies, with $m := |V|$ even), such that each $S \in \binom{V}{m/2}$ contains a member of \mathcal{H} . (So $m(n) \leq g(n)$, right?) Show

- (a) $g(n) = O(n^2 2^n)$ (so also $m(n) = O(n^2 2^n)$);
- (b) $g(n) = \Omega(n 2^n)$.

[Here (b) is more interesting, but (a) is a good basic exercise. The bound in (a), from Erdős 1964, is still the best known upper bound on $m(n)$; see AS, Thm. 1.3.2 for a precise version (more precise than is needed for (a)).

Estimating $m(n)$ is a classic problem. See AS, Cor. 3.5.2 for the current lower bound (a beautiful argument that was also mentioned in class). To appreciate that the current gap is large, it's natural to consider $m(n) 2^{-(n-1)}$; so we're asking, how large the *expected number of monochromatic edges in a random coloring* (namely $|\mathcal{H}| 2^{-(n-1)}$) must be to guarantee that there's no good coloring, and we only know that the answer is somewhere between $\Omega(\sqrt{n/\log n})$ and $O(n^2)$.

Hints: for (a) choose a random \mathcal{H} ; for (b) use something with deletions to produce a bad S .]

7. Let $X \cup Y$ be a partition of V , with $|X| = |Y| = n$, and let D be the random digraph with edges $vw \in X \times Y$ chosen independently, each with probability $1/2$ (and no other edges).

Let $b = \sqrt{2n \log n}$ (where $\log = \ln$), and set $Y_0 = Y$ and, for $i \geq 1$,

$$Y_i = Y_{i-1} \cup \{v : d(v, Y_{i-1}) > n/2 + b\},$$

where $d(v, Z) = |\{w \in Z : vw \in D\}|$. With $Y_\infty = \cup Y_i$ and $Z = Y_\infty \setminus Y$, show

$$\mathbb{P}(|Z| > b) = o(1).$$

[A standard abuse in this subject—which you may want to use with b —is pretending all large numbers are integers.

The proof I have in mind here is a union bound ... but *over what?*]

8. An important though trivial fact (and AS, Theorem 2.2.1): for any graph G on vertex set V , there is a partition $X \cup Y$ of V with

$$|\nabla_G(X, Y)| \geq |G|/2$$

(where $\nabla_G(X, Y) = \{e \in G : e \cap X \neq \emptyset \neq e \cap Y\}$ and $|G| = |E(G)|$).

Show that if G and H are graphs on V and $\min\{|G|, |H|\}$ is sufficiently large, then there is a partition $X \cup Y$ of V with

$$|\nabla_G(X, Y)| \geq .49|G| \quad \text{and} \quad |\nabla_H(X, Y)| \geq .49|H|.$$

9. For $i \in [n]$, let $v_i = (x_i, y_i) \in \mathbb{Z}^2$ with each of $|x_i|, |y_i|$ at most $2^{n/2}/(100\sqrt{n})$. Show that there are disjoint $I, J \subseteq [n]$ with $\sum_{i \in I} v_i = \sum_{i \in J} v_i$.

[AS, 4.8.5]

10. Show that there is a positive constant c for which the following holds. If $a_1, \dots, a_n \in \mathbf{R}$ satisfy $\sum a_i^2 = 1$, and $\varepsilon_1, \dots, \varepsilon_n$ are chosen uniformly and independently from $\{\pm 1\}$, then $\mathbb{P}(|\sum \varepsilon_i a_i| \leq 1) \geq c$.

[AS, 4.8.2 (with a (*)). The old conjecture that one can take $c = 1/2$ (this is best possible, right?) was proved by Keller and Klein in 2020 (in 76 pages).]

[Third installment.]

11. Let A_i be independent events, $X_i = \mathbf{1}_{A_i}$, $X = \sum X_i$, and $\mathbb{P}(A_i) = p_i$. Show that, for a fixed positive μ , $X \xrightarrow{d} \text{Po}(\mu)$ iff

$$\sum p_i \rightarrow \mu \text{ and } \max p_i \rightarrow 0.$$

[As usual, A_i is really $A_i^{(n)}$ and similarly for X_i , X and p_i . A proof—at least for “if”—isn’t necessarily easy to find, but should be short once found.]

12. For $p = n^{-1} \ln n$, show

$$\mathbb{P}(G_{n,p} \text{ has no isolated vertices and is not connected}) \rightarrow 0.$$

13. For events A_1, \dots, A_n in a probability space, with $\mu = \sum \mathbb{P}(A_i)$,

$$\begin{aligned} \mathbb{P}(\text{some } \mu + t \text{ independent } A_i\text{'s occur}) &\leq \exp[-\mu\varphi(t/\mu)] \\ &\leq \exp[-t^2/(2(\mu + t/3))], \end{aligned}$$

where $\varphi(x) = (1+x)\ln(1+x) - x$ for $x \geq -1$ (so $\varphi(-1) = 1$).

[Cf. Problem 3. What’s interesting here is the first inequality; the second bound (a little calculus exercise, or see p. 27 of [JLR]) is included to make sense of the first.

Hint: consider the number of sequences of k (TBA) independent events that occur, and use Markov. You may want to bound some sum by an integral.]

14. Another model: Fix k ; let $D = D(n, k\text{-out})$ be the random digraph on $V = [n]$ gotten by choosing, for each $v \in V$, $\nabla^+(v)$ ($= \{\text{edges out of } v\}$) uniformly from the k -subsets of $\{vw : w \in V \setminus \{v\}\}$ (independently for different v); and let $G = G(n, k\text{-out})$ be the (simple) graph underlying D (so we erase arrows and ignore multiple edges).

Like $G_{n,d}$, this is a model where we build in a lower bound on degrees, so may hope for perfect matchings (p.m.s) and Hamilton cycles (HCs) with $|G|$ much smaller than $n \log n$ ($\asymp |G_{n,p}|$ for $p \asymp \log n/n$); and as it turns out, we a.s. have p.m.s in $G(n, 2\text{-out})$ and HCs in $G(n, 3\text{-out})$ (due respectively to Frieze 1986 and Bohman-Frieze 2009; see Frieze-Karoński, *Introduction to Random Graphs* for more).

Give convincing (even to yourself) reasons why $G(n, 1\text{-out})$ and $G(n, 2\text{-out})$ a.s. do *not* contain p.m.s and HC’s (resp.). (You could also try proving.)

[Hint: consider $W := \{v : \nabla^-(v) = \emptyset\}$.]