642.581 Problem Set 2 (final)

[Please don't assume the later problems are harder.]

1. What's the maximum possible number of edges in a graph on $2n$ vertices with exactly one perfect matching?

2. A deck of mn cards with m values and n suits (consisting of one card of each value in each suit) is dealt into an $n \times m$ array.

(a) There is a subset of m cards in distinct columns having distinct values.

(b) There is a sequence of switches of pairs of cards with equal values that yields an arrangement with each column containing one card from each suit.

[Remark: the rows have nothing to do with it.]

3. (Notation: $\Phi(G)$ is the number of perfect matchings in G.) Show that if G is bipartite with bipartition $A \cup B$, $d(a) \geq d \ \forall a \in A$, and $\Phi(G) > 0$, then $\Phi(G) \geq d!$

[This should be the hardest of the above. Please don't go looking up the solution; it will appear here next week. There's a *small* hint on page 3 (so you won't see it by accident if you like your challenges more challenging).]

4. For a *connected* bigraph $G = (A \cup B, E)$, give necessary and sufficient conditions along the lines of Hall's Condition for the existence of a (*positive*) weight function $w : E \to (0, 1]$ satisfying $\sum\{w(e) : x \in e \in E\} = 1 \ \ \forall x \in V$.

5. Derive the Marriage Theorem from Tutte's Theorem.

[Recall that a bigraph G on $A \cup B$ is *balanced* if $|A| = |B|$ and the "Marriage Theorem" is Hall's Theorem for balanced bigraphs. As in our proof of Tutte, it *may* help to consider a *maximal* violator of Tutte's condition.]

6. Let $G = (V, E)$ be a graph and $k : V \to \mathbb{N}$ $(= \{0, 1, \ldots\})$. Show that G has an orientation with $d^+(v) \leq k(v)$ $\forall v \in V$ provided

$$|E(W)| \ \leq \ \sum\{k(v) : v \in W\} \qquad \forall W \subseteq V.$$

(To be understood but not written: this is trivially necessary.)

[An *orientation* of G is a digraph gotten by *orienting* (or *directing*) each edge from one of its ends to the other (so, violating our usual convention, the edge $\{x, y\}$ becomes either xy or yx; you can call an orientation σ and use d^+_σ, ∇^+_σ and so on.) You can do this problem using Hall, but it takes a while; for something quick try choosing a σ that's optimal in some sense.]

7. For any permutation $a_1, \ldots, a_n$ of $[n]$ there exists $f : [n] \to \{\pm 1\}$ such that $|\sum_{i=1}^{t} f(i)| \leq 1\ \forall t$ and $|\sum_{i=1}^{t} f(a_i)| \leq 1\ \forall t$.

[Minor hint: you can assume n is even, say $n = 2k$. And comment: a nice example of easy once found but maybe not so easy *to* find.]

8. Suppose that $M = (m_{ij})$ is an $m \times n$ $\mathbb{N}$-matrix with row sums a and column sums b, $\delta \in [0, 1]$, and $a' := \delta a$ and $b' := \delta b$ are integers. Then there is an $m \times n$ $\mathbb{N}$-matrix $M' = (m'_{ij})$ with row sums a', column sums b', and $m'_{ij} \leq m_{ij}\ \ \forall i, j$.

[Here $\mathbb{N} = \{$non-negative integers$\}$.]

[For the next one you may want to wait for Tuesday's lecture.]

9. For a graph G on n vertices, let

$$\pi(G) = \min_{\mathcal{C}} \sum_{C \in \mathcal{C}} |V(C)|,$$

the minimum over $\mathcal{C}$ a collection of cliques whose union is G. Show

$$\pi(G) \leq 2\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil.$$

[Equality holds iff $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, but you don't have to show this.

Hint for Problem 3: Use induction (on whatever) and consider: are there edges not contained in any perfect matchings?