642.581 Problem Set 3 (final)

[Following Diestel we use $d(\cdot,\cdot)$ for distance (though $d(\cdot)$ is degree).]

1. Let x, y be vertices of $G = (V, E)$ with $d(x, y) = d$, and suppose that for any $F \subseteq E$ of size at most $k - 1$, $d_{G-F}(x, y)$ is still d. Then G contains k edge-disjoint $\{x, y\}$-paths of length d.

2. If G is an n-vertex graph with

$$d(x) + d(y) \geq n - 1 \quad \text{for all nonadjacent } x, y \in V(G),$$

then $\delta(G) \leq \lambda(G)$ (so in fact $\delta(G) = \lambda(G)$).

3. Let $G = (V, E)$ be k-connected, $k \geq 2$, and $X = \{x_1, \ldots, x_k\} \subseteq V$. Show G contains a cycle whose vertex set contains X.

4. If $\chi(G) = k$, what's the least size of a collection of r-colorable graphs whose union is G?

[Recall *r-colorable* means $\chi \leq r$.]

5.(a) For $n \geq 2$, if $G \not\succ K_n$ then $\chi(G) \leq 2^{n-2}$.

(b) Improve the bound when $n = 4$: if $G \not\succ K_4$ then $\chi(G) \leq 3$.

[Possibly helpful for (b): a *topological* H is a graph obtained from H by replacing each edge by a path with at least 1 edge. We'll write $G \sqsupset H$ if G contains a topological H, *noting* that this implies $G \succ H$.
(Recall $G \succ H$ means G contains H as a minor. For both $\succ$ and $\sqsupset$, I'm allowing $H = G$, so should probably write $\succeq$ and $\sqsupseteq$.)

As we'll discuss at some point, possibly the most famous open problem in graph theory is *Hadwiger's Conjecture*: if $G \not\succ K_n$ then $\chi(G) \leq n - 1$.]

6. Let G be a connected bigraph with bipartition $X \cup Y$ and $x, y \in X$. Let $\mathcal{C}$ be the set of proper vertex colorings of G using colors from $\{R, B, G\}$ and $\mathcal{A} = \{\sigma \in \mathcal{C} : \sigma(x) = R, \sigma(y) = B\}$, $\mathcal{B} = \{\sigma \in \mathcal{C} : \sigma(x) = \sigma(y) = R\}$. Show $|\mathcal{A}| < |\mathcal{B}|$.