642.581 Problem Set 4 (final)

1. Suppose G is a graph in which every cycle of length at least 4 has a chord. Then $\chi(G) = \omega(G)$.

[A *chord* of a cycle C is an edge not in C with both ends in C.]

2. For a graph G on V and positive integer t, let $\lambda_t(G)$ be the largest $\alpha \in [0,1]$ such that for every $S = (S_v : v \in V)$ with $|S_v| = t\ \forall v$, there is an S-coloring of some $\alpha|V|$ vertices of G. For $t \leq s$ let

$$\lambda_{s,t} = \inf\{\lambda_t(G) : \chi_l(G) = s\}.$$

Conjecture. For every $t \leq s$, $\lambda_{s,t} = t/s$.

[The analogous statement for χ is trivial (right?), as is $\lambda_{s,t} \leq t/s$ (why?). You might convince yourself that the conjecture is true if $t|s$.]

Show that for any G and t,

$$\lambda_t(G) \geq 1 - (1 - 1/\chi(G))^t.$$

3. For a graph G write MIS(G) for the collection of *maximal* independent sets (MIS's) in G and let $\text{mis}(G) = |\text{MIS}(G)|$.

(a) For d-regular bipartite G, $\text{mis}(G) < 2^{(1+o_d(1))n/4}$.

(b) For any d there is a d-regular bipartite G with $\text{mis}(G) > 2^{n/4}$.

[As usual n is the default for $|V(G)|$ and $o_d(1) \to 0$ as $d \to \infty$. As an easy warmup—not to be handed in—you could try showing that

$$\text{if } G \text{ is bipartite then } \text{mis}(G) \leq 2^{n/2} \tag{1}$$

and that this is best possible for even n.]

4. Show that $\chi_\ell(K_2^r) = r$, where K_s^r is the complete r-partite graph in which each color class has size s. (Say the color classes are $\{x_i, y_i\}$, $i \in [r]$. You can skip the trivial lower bound; the problem is really to show "$\leq r$.")

[Hint: induction plus something from our past.]

5. For a simple planar graph G and $W \subseteq V = V(G)$ of size $k \geq 3$, $\sum_{x \in W} d(x) \leq 2n + 6k - 16$.

6. If G is a simple plane graph with edges colored R and B, then there is a vertex x such that there are at most two color changes in the clockwise ordering of the edges at x.

[If it helps you may use the fact (mentioned in class, I think) that any maximal plane graph on a given vertex set is a triangulation.]